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Lacunary sets on transformation groups

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The classical F. and M. Riesz theorem asserts that a measure μ on the circle group $\boldsymbol{T}$, such that

$$\forall n<0, \quad \hat{\mu}(n)=\int_{T} e^{-int}\, d\mu(t)=0$$

is absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{T}$. Many extensions of this result have been obtained. Helson-Lowdenslager [15, Theorem 8.2.3] and De Leeuw-Glicksberg [5] extended the theorem to compact abelian groups with certain ordered duals. Forelli extended it to transformation groups such that $\boldsymbol{R}$ acts on a locally compact Hausdorff space [8]. Recently, Yamaguchi got the compact analogue of Forelli's result [18, 19, 20]. He proved:

PROPOSITION [19]. *Let (G, X) be a transformation group with G a compact abelian group which acts on a locally compact Hausdorff space X. Let σ be a positive Radon measure on X which is quasi-invariant, and let Λ be a Riesz set in $\hat{G}$. Let μ be a measure in $\mathscr{M}(X)$ with* spec μ *contained in Λ. Then* spec μ_a *and* spec μ_s *are both contained in* spec μ, *where $\mu=\mu_a+\mu_s$ is the Lebesgue decomposition of μ with respect to σ.*

By using Yamaguchi's technique, we get in [7] the same result for a nicely placed subset Λ of $\hat{G}$. In this paper, we consider (G, X) as in the proposition and introduce a new notion of lacunarity in the transformation group case. It is what we call "σ-lacunarity" (Definition 1.4). The idea is the following: let Λ be a subset of $\hat{G}$, we consider functions and measures on X with spectrum contained in Λ and we work with a "reference" measure σ on X which is positive and quasi-invariant. We then define the analogue of the usual lacunarity notions. We want to transfer lacunarity properties on $\hat{G}$ to σ-lacunarity properties.

In section 1, we give the necessary preliminaries and notation. Section 2 is devoted to a positive result: a nicely placed (resp. Shapiro) subset of $\hat{G}$ is also σ-nicely placed (resp. σ-Shapiro). This is Theorem 2.1 and

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its corollary. Of course, such a result does not hold for Riesz sets (see Example 1. 3). Moreover σ-Shapiro sets are not necessarily σ-Riesz sets. This leads us to introduce a smaller class contained in the class of σ-Riesz sets. It is what we call the $N(\sigma)$-Riesz sets (Definition 3.14). In the classical case (when $X=G$ and $\sigma=m_G$) the usual class of Riesz sets in $\widehat{G}$ and our class of $N(\sigma)$-Riesz sets both coincide. The first part of section 3 is devoted to the study of the set $N(\sigma)$. Some examples are given. The second part of section 3 is devoted to the study of identity approximations in our context. We then get the implication: σ-Shapiro $\Rightarrow$ $N(\sigma)$-Riesz (Theorem 3. 13). We also extend Shapiro's lemma [16]. In section 4, we develop in our context the localization technique introduced by Meyer [14]. We show that the classes of σ-Riesz sets, $N(\sigma)$-Riesz sets, σ-nicely placed and σ-Shapiro sets are localizable. This leads us to another transfer theorem: every Riesz subset of $\widehat{G}$ is $N(\sigma)$-Riesz (Theorem 4. 10). In section 5, we use techniques of infinite dimensionnal Banach space theory and we consider on transformation groups Lust-Piquard's result [13] and Bachelis and Ebenstein's result [2].

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1-Preliminaries and notation

Let G be a compact abelian group and X be a locally compact Hausdorff space. We say that (G, X) is a transformation group such that G acts on X if there exists a continuous map from $G\times X$ onto $X:(g, x)\mapsto g.x$ such that

$$\forall g\in G, \quad \forall h\in G, \quad \forall x\in X, \quad e.x=x, \quad g.(h.x)=(g.h).x.$$

Let us give a few examples. The following pairs (G, X) are transformation groups:

a) if G is a compact abelian subgroup of a locally compact group X.

b) let G be a compact abelian group and H be a subgroup of G. Take for X the homogeneous space G/H.

c) let G be a metrizable compact abelian group and μ be a probability measure on G. Then take for X the Poisson space Π_μ of μ [1].

A Borel measure σ on X is called quasi-invariant if

$$\forall F\subseteq X, \quad F \text{ Borel}, \quad (|\sigma|(F)=0) \Longrightarrow (\forall g\in G, \quad |\sigma|(gF)=0).$$

For example, let $X=G$ be the circle group. Then to say that σ is quasi-invariant means that every rotation carries the collection of σ-null sets onto itself.

We denote by $\mathscr{K}(X)$ the space of continuous functions on X with compact support and by $\mathscr{M}(X)$ the space of regular bounded measures on X. By dg or m_G, we denote the Haar measure on G normalized to total mass one. We denote by $\hat{G}$ the dual of G. We denote by (G, X) a transformation group and σ a positive quasi-invariant Radon measure on X. We also denote by I_E the characteristic function of a set E. $L^1(G, dg)=L^1(G)$, $L^1(X, \sigma)=L^1(\sigma)$ and $\|f\|_1$ have their usual meaning. If $\mu\in\mathscr{M}(X)$ we write $\mu\ll\sigma$ for "μ absolutely continuous with respect to σ". We identify $k\in L^1(\sigma)$ with the element of $\mathscr{M}(X)$ absolutely continuous with respect to σ which k defines. Then for a Borel set Y of X, we write: $k(Y)=\int_Y k(x)\, d\sigma(x)$. For a Banach space Y, we denote by $B(Y)$ the unit ball of Y.

The usual notion of convolution can be generalized in the following way [18]. For μ in $\mathscr{M}(X)$ and λ in $\mathscr{M}(G)$, the convolution is defined by

$$(\lambda*\mu)(f)=\int_X\int_G f(g\cdot x)\, d\lambda(g)\, d\mu(x) \quad \text{for} \quad f\in\mathscr{K}(X).$$

Then $\lambda*\mu$ is an element of $\mathscr{M}(X)$ and satisfies $\|\lambda*\mu\|\leq\|\lambda\| . \|\mu\|$. Furthermore, if $\nu\in\mathscr{M}(X)$ and $\lambda, \mu\in\mathscr{M}(G)$, then $(\lambda*\mu)*\nu=\lambda*(\mu*\nu)$.

We also have the following property:

LEMMA 1.1. *Let γ be in $\hat{G}$, ν be in $\mathscr{M}(G)$ and μ be in $\mathscr{M}(X)$. Then $(\gamma m_G)*\nu*\mu=\hat{\nu}(\gamma)\cdot(\gamma m_G*\mu)$.*

PROOF.

$$\begin{aligned}(\gamma m_G)*\nu*\mu&=((\gamma m_G)*\nu)*\mu\\ &=(\hat{\nu}(\gamma)\gamma m_G)*\mu\\ &=\hat{\nu}(\gamma)\,(\gamma m_G)*\mu.\end{aligned}$$

We now give a characterization of quasi-invariant measures on X. It is the compact analogue of De Leeuw and Glicksberg's result [5], (see also [18]).

LEMMA 1.2. *Let λ be in $\mathscr{M}(X)$. Then the following are equivalent:*
(1) *λ is quasi-invariant on X.*
(2) *$|\lambda|$ and $m_G*|\lambda|$ are equivalent.*

PROOF. (1) implies (2). Let E be a λ-null Borel set. Then for all g

in G, $|\lambda|(g^{-1}.E)=0$ and $(m_G*|\lambda|)(E)=\int_G|\lambda|(g^{-1}.E)dg=0$. And $m_G*|\lambda|$ is absolutely continuous with respect to $|\lambda|$. On the other hand, let E be a Borel subset of X such that $(m_G*|\lambda|)(E)=0$, then for almost all g in G, $|\lambda|(g^{-1}.E)=0$ and $|\lambda|(E)=0$.

(2) implies (1). Let E be such that $(m_G*|\lambda|)(E)=0$. It follows that $|\lambda|(E)=0$. And $\int_G|\lambda|(g^{-1}.(s.E))dg=(m_G*|\lambda|)(s.E)=0$ for all s in G. But $|\lambda|$ and $m_G*|\lambda|$ are equivalent. So $|\lambda|$ is quasi-invariant.

Let σ be a positive quasi-invariant Radon measure on X. If $\mu\ll\sigma$, then for $\lambda\in\mathscr{M}(G)$, one has $\lambda*\mu\ll\sigma$. And we have the following inclusion: $\mathscr{M}(G)*L^1(\sigma)\subset L^1(\sigma)$. More precisely, Gulick, Liu and Van Rooij proved that $\mathscr{M}(G)*L^1(\sigma)=L^1(\sigma)$ [10]. But usually, $L^1(G)*\mathscr{M}(X)\nsubseteq L^1(\sigma)$ [10]. They proved the existence of a modular function $\mathscr{T}$ such that $\mathscr{T}$ is positive, locally integrable, defined on $G\times X$ and $\int_{G\times X}F(g,x)d(m_G\otimes\sigma)(g,x)=\int_{G\times X}F(g,g.x)\mathscr{T}(g,x)d(m_G\otimes\sigma)(g,x)$ for F in $L^1(G\times X)$. Then, for f in $L^1(G)$ and k in $L^1(\sigma)$, we have:

(a) for almost all x in X

$$(f*k)(x)=\int_G f(g)k(g^{-1}.x)\mathscr{T}(g^{-1},x)dg\,;$$

(b) for all g in G

$$\int_X k(x)d\sigma(x)=\int_X k(g.x)\mathscr{T}(g,x)d\sigma(x).$$

We can now define the spectrum of a measure μ in $\mathscr{M}(X)$ [18]. Let $J(\mu)$ be the set of all f in $L^1(G)$ with $f*\mu=0$. The <u>spectrum</u> of μ, denoted by spec μ or $\mathrm{spec}_G\mu$, is the closed subset of $\hat{G}$ where all the Fourier transforms of functions in $J(\mu)$ vanish. We have that $s\in$ spec μ if and only if $(sm_G)*\mu\neq0$ [18]. Of course, when $X=G$ and $\sigma=m_G$, spec μ is just the support of $\hat{\mu}$. By Lemma 1.1, it follows that if ν is in $L^1(G)$ and μ in $\mathscr{M}(X)$, then $\mathrm{spec}(\nu*\mu)\subseteq\mathrm{supp}(\hat{\nu})\cap\mathrm{spec}(\mu)$. Let us give an example:

EXAMPLE 1.3. Let G be the circle group $\boldsymbol{T}$, $X=G\times G$ and $\sigma=m_G\otimes m_G$. For g in G and (x,y) in X, the action of G on X is given by the application: $(g,(x,y))\mapsto(gx,y)$. For μ in $\mathscr{M}(X)$, the spectrum $\mathrm{spec}_T\mu$ is exactly the projection of supp $\hat{\mu}$ on the first coordinate.

Let Λ be a subset of $\hat{G}$, we will denote by $\mathscr{M}_\Lambda(X)$ (resp, $L^1_\Lambda(\sigma)$) the subspace of $\mathscr{M}(X)$ (resp. $L^1(\sigma)$) of measures (resp. functions) with spectrum in Λ. We are now ready to give our definition of lacunary sets.

DEFINITION 1.4. Let Λ be a subset of $\hat{G}$:

a) Λ is *σ-Riesz* if every measure in $\mathscr{M}_\Lambda(X)$ is absolutely continuous with respect to σ.

b) Λ is *σ-nicely placed* if the unit ball of $L^1_\Lambda(\sigma)$ is closed in $L^p(\sigma)$, $(0<p<1)$.

c) Λ is *σ-Shapiro* if every subset of Λ is σ-nicely placed.

Of course, similar definitions can be given for Λ(p)-sets, Sidon sets, see [12], [15]. This paper is devoted to the study of these sets.

REMARK 1.5. As the space $(L^p(\sigma), \|\cdot\|_p)$, $0<p<1$, is metrizable, the unit ball of $L^1_\Lambda(\sigma)$ is closed in $L^p(\sigma)$ if and only if it is sequentially closed in $L^p(\sigma)$.

2-Transference of nicely placed and Shapiro sets

THEOREM 2.1. *Let (G, X) be a tranformation group with G metrizable. If Λ is a nicely placed subset of $\hat{G}$, then Λ is σ-nicely placed.*

PROOF. Let (f_n) be a sequence in $B(L^1_\Lambda(\sigma))$, which converges to f in $\|.\|_p$. Then, up to a subsequence, we may assume that (f_n) converges to f almost everywhere. Let s be in $\hat{G}\backslash\Lambda$; we have to prove that, $s*f=0$. We have that, for all n, $s*f_n=0$. By equality (a) in section 1, we have that for almost all x in X:

$$(s*f_n)(x)=\int_G s(g)f_n(g^{-1}.x)\mathscr{T}(g^{-1}, x)dg.$$

We consider the following functions defined on G: for almost all x in X,

$$k_{n,x}(g)=f_n(g.x)\mathscr{T}(g, x),$$
$$k_x(g)=f(g.x)\mathscr{T}(g, x).$$

We will show the following assertions: for almost all x in X,

(1) supp $\widehat{k_{n,x}}\subseteq\Lambda$,

(2) $k_{n,x}\in L^1(G)$,

(3) $k_{n,x}$ converges to k_x almost everywhere,

(4) $\underline{\lim}\,\|k_{n,x}\|_1$ is finite.

(1) One has, for almost all x in X:

$$(s*f_n)(x)=\int_G s(g)k_{n,x}(g^{-1})dg$$

$$=\int_G s(g^{-1})k_{n,x}(g)dg$$
$$=\widehat{k_{n,x}}(s).$$

Therefore, for almost all x in X, supp $\widehat{k_{n,x}}\subseteq\Lambda$.

(2)

$$\|f_n\|_1=\int_X|f_n(x)|d\sigma(x)$$
$$=\int_G\int_X|f_n(x)|d\sigma(x)dg.$$

By equality (b) of section 1, one has

$$\|f_n\|_1=\int_G\Big[\int_X|f_n(g.x)|\mathscr{T}(g,x)d\sigma(x)\Big]dg$$
$$=\int_X\Big[\int_G|f_n(g.x)|\mathscr{T}(g,x)dg\Big]d\sigma(x).$$

Then, for almost all x in X, $\int_G|f_n(g.x)|\mathscr{T}(g,x)dg$ is finite. This proves assertion (2).

(3) This assertion will follow directly from the following lemma:

LEMMA 2.2. *Let E be a Borel subset of X. Then the following assertions are equivalent*:

(i) $\sigma(E)=0$.

(ii) $m_G(\{g\in G: g.x\in E\})=0$, *for almost all x in X.*

PROOF. The proof follows from the equality

$$\forall x\in X,\quad m_G(\{g\in G: g.x\in E\})=\int_G I_E(g.x)dg$$

and from the quasi-invariance of σ.

(4) We will follow the proof of Lemma 2.8 of [9].

For each l and n, we consider the set $A_{n,l}=\{x\in X: \|k_{n,x}\|_1\geq l\}$. One has

$$l\sigma(A_{n,l})\leq\int_X\|k_{n,x}\|_1\ d\sigma(x)=\|f_n\|_1.$$

And, $\forall n$, $\forall l$, $\sigma(A_{n,l})\leq l^{-1}$. Let us consider $B_l=\{x\in X, \exists$ infinitely many n s.t. $x\notin A_{n,l}\}$. One has $X\backslash B_l=\bigcup_j\bigcap_{n\geq j}A_{n,l}$. And $\sigma(\bigcap_{n\geq j}A_{n,l})\leq l^{-1}$ for every j. Therefore, for all l, $\sigma(X\backslash B_l)\leq l^{-1}$ and $\sigma(X\backslash\bigcup_{l\geq 1}B_l)=0$. If x does not belong to this set, the condition (4) is satisfied.

Since Λ is nicely placed in $\widehat{G}$, then for almost all x in X, k_x is in $L^1_\Lambda(G)$. That is, for almost all x in X,

$$\forall s \notin \Lambda, \quad \hat{k}_x(s) = \int_G s(g^{-1}) f(g.x) \mathscr{T}(g.x) dg$$
$$= (s*f)(x)$$
$$= 0.$$

This proves the theorem.

COROLLARY 2.3. *Let (G, X) be a transformation group with G metrizable. If Λ is a Shapiro subset of $\hat{G}$, then Λ is a σ-Shapiro set.*

Let us come back to Example 1.3. We know that $\boldsymbol{N}$ is a Shapiro subset of $\boldsymbol{Z}$ [9], then by the corollary, $\boldsymbol{N}$ is a σ-Shapiro set. If we only suppose Λ to be a Riesz subset of $\hat{G}$, there is no reason for Λ to be a σ-Riesz set as we can see with our example. Indeed, there are many Riesz sets in $\hat{G}=\boldsymbol{Z}$ which are not σ-Riesz. For example, consider Λ a subset of $\hat{G}$ containing 0. Let f be in $L^1_\Lambda(G)$ such that $\hat{f}(0)\neq 0$ and μ be a measure in $\mathscr{M}_s^+(G)$. Then it is easy to see that the measure $f\otimes\mu$ in $\mathscr{M}_\Lambda(X)$ is not absolutely continuous with respect to σ. This shows also that the empty set is the only σ-Riesz set. Therefore, $\boldsymbol{N}$ is not a σ-Riesz set. This gives an example of a σ-Shapiro set which is not a σ-Riesz set. This situation is different from the usual one where every Shapiro set of $\hat{G}$ is a Riesz set [9]. As seen in this example, the notion of σ-Riesz set is much too strong. It is why we will introduce a new notion. It is what we call $N(\sigma)$-Riesz set (see Definition 3.14). And we will get the following implications:

$$\sigma\text{-Shapiro set} \Longrightarrow N(\sigma)\text{-Riesz set (Theorem 3.13)} \quad (1)$$
$$\text{Riesz set} \Longrightarrow N(\sigma)\text{-Riesz set (Theorem 4.10)} \quad (2)$$

The proof of implication (1) is based on Godefroy's ideas [9]. We need to know more about the different kinds of convergence of convolutions between identity approximations on G and measures on X. It is the object of the next section. The proof of implication (2) uses localization techniques and Yamaguchi's result [18, 19]. This is postponed to section 4.

3. The set $N(\sigma)$ and identity approximations

When $X=G$, it is known [3] that there exists an identity approximation $(f_V)_{V\in\mathscr{F}}$ in $B(L^1(G))$ such that

(1) if $f\in L^1(G)$, then $\lim_{\mathscr{F}} f_V*f=f$ in $L^1(G)$, and

(2) if $\mu\in\mathscr{M}_s(G)$, then $\lim_{\mathscr{F}} f_V*\mu=0$ in Haar measure.

In the second part of this section, we will extend this result to the transformation group case. Let $\mathscr{F}$ be a filter of symmetric neighborhoods of e in G. We let $f_V = m_G(V)^{-1} I_V$. It is easy to check that if $f \in L^1(\sigma)$, then $\lim_{\mathscr{F}} f_V * f = f$ in $L^1(\sigma)$. The problem in proving the second assertion for $\mu \in \mathscr{M}_s(X)$ is that the measure $f_V * \mu$ on X is not necessarily absolutely continuous with respect to σ. This leads us to consider a set $N(\sigma)$ introduced by Gulick, Liu and Van Rooij [10], see also [11].

a-The set $N(\sigma)$

$$N(\sigma) = \{\mu \in \mathscr{M}(X) : \forall f \in L^1(G) \quad f * \mu \ll \sigma\}.$$

In this section, we will recall some facts on this set [10], [11], and give some examples. We have the following inclusions: $L^1(\sigma) \subseteq N(\sigma) \subseteq \mathscr{M}(X)$. If $X = G$ and $\sigma = m_G$ then $N(\sigma) = \mathscr{M}(X)$. On the other hand, if the action is given by $(g, x) \to x$ for all g in G and x in X, then $N(\sigma) = L^1(\sigma)$. This is also the case when G is discrete. Let us come back to Example 1.3: $N(\sigma) \neq \mathscr{M}(X)$ and $N(\sigma) \neq L^1(\sigma)$ [10]. In that example, we can say more about $N(\sigma)$. We get

PROPOSITION 3.1. *Let (G, X) be as in Example 1.3. Let μ_1, μ_2 be two non zero measures on G and $\mu = \mu_1 \otimes \mu_2$. Then μ is in $N(\sigma)$ if and only if μ_2 is absolutely continuous with respect to m_G.*

PROOF. The proof follows from the fact that $f * (\mu_1 \otimes \mu_2) = (f * \mu_1) \otimes \mu_2$ for f in $L^1(G)$.

Let us now consider other examples.

EXAMPLE 3.2. Let $G = \boldsymbol{T}$ and $X = \boldsymbol{T}$ and m_T be the Haar measure on $\boldsymbol{T}$. Define for a Borel subset Y of X, $\sigma(Y) = m_T(Y)$. The action of G on X is given by

$$\pi(e^{it}, e^{ix}) = e^{i(t+x)}.$$

It is easy to see that σ is quasi-invariant and $N(\sigma) = \mathscr{M}(X)$.

EXAMPLE 3.3. Let G be a compact abelian group and H be a closed subgroup of G. Then the action of G on G/H is given by $\pi(g, \tilde{x}) = \widetilde{g+x}$ where $\tilde{x} = x + H$ for some x in G. Let $\sigma = m_{G/H}$. Then $N(\sigma) = \mathscr{M}(X)$.

EXAMPLE 3.4. Let $G = \boldsymbol{T}$, $X = \boldsymbol{T} \times \boldsymbol{T}$ and $\sigma = m_T \otimes m_T$. Let a be a real number. The action of G on X is defined by

$$\pi_a(e^{ir}, (e^{ix}, e^{iy})) = (e^{i(x+r)}, e^{i(y+ar)}).$$

It is also easy to see that σ is quasi-invariant and that for all u, $\sigma(G.u) = 0$, thus $N(\sigma) \neq \mathscr{M}(X)$.

REMARKS 3.5.

(1) Let us recall the nice description of $N(\sigma)$ that Liu, Van Rooji and Wang got in [11]. Let G_0 be an open set which is also a countable union of compact sets and a subgroup of G. Let I_σ be the σ-ideal in the σ-algebra of all Borel subsets defined by

$$I_\sigma=\{Y: Y\text{Borel} \quad \exists B\subset Y \quad \text{Borel invariant under } G_0 \ \sigma(B)=0\}.$$

Then a measure belongs to $N(\sigma)$ if and only if it vanishes on I_σ. And it follows that every measure in $\mathscr{M}(X)$ which is absolutely continuous with respect to a measure in $N(\sigma)$ is also in $N(\sigma)$.

(2) Let μ be another positive quasi-invariant Radon measure on X. If μ belongs to $N(\sigma)$, then it is easy to see that μ is absolutely continuous with respect to σ. Then, when $N(\sigma)=\mathscr{M}(X)$, every positive quasi-invariant Radon measure on X is absolutely continuous with respect to σ (the converse is false). In this situation, every μ-Riesz set (μ a positive quasi -invariant Radon measure on X) is a σ-Riesz set.

(3) Gulick, Liu and Van Rooij studied the case: $N(\sigma)=\mathscr{M}(X)$ (for G an abelian locally compact group) [10]. Let us say a few words on the case: $L^1(\sigma)=N(\sigma)$. It is easy to get:

PROPOSITION 3.6. *The following assertions are equivalent*:

1) $L^1(\sigma)=N(\sigma)$.

2) $L^1(G)*N(\sigma)=N(\sigma)$.

3) $\forall \mu\in N(\sigma), \quad \exists g\in G, \quad \delta_g*\mu\in L^1(\sigma)$.

4) $\forall \mu\in N(\sigma), \quad \forall g\in G, \quad \delta_g*\mu\in L^1(\sigma)$.

We close the first part of this section by giving the descriptive complexity of the set $N(\sigma)$ when X is σ-compact metrizable and G is metrizable. Let $\mathscr{M}^1(X)$ denote the unit ball of $\mathscr{M}(X)$. It is a polish space when equipped with the vague topology. Let $N^1(\sigma)=N(\sigma)\cap\mathscr{M}^1(X)$. Then, we have the following result:

PROPOSITION 3.7. *$N^1(\sigma)$ is a $F_{\sigma\delta}$ set in $\mathscr{M}^1(X)$.*

PROOF. Let (f_n) be an identity approximation in $L^1(G)$. It is easy to see that

$$N^1(\sigma)=\{\mu\in\mathscr{M}^1(X): \forall n, \quad f_n*\mu\ll\sigma\}.$$

Let

$$\begin{aligned}\psi_n: \quad \mathscr{M}^1(X) &\longrightarrow \mathscr{M}^1(X).\\ \mu &\longmapsto f_n*\mu\end{aligned}$$

Then ψ_n is continuous for the vague topology [4]. But $N^1(\sigma)=$

$\bigcap_n \psi_n^{-1}(L^1(\sigma)\cap \mathscr{M}^1(X))$. Let us compute the complexity of the set $L^1(\sigma)\cap \mathscr{M}^1(X)$:

$$\mu \ll \sigma \Longleftrightarrow |\mu| \ll \sigma \ ;$$
$$\mu \ll \sigma \Longleftrightarrow \forall \varepsilon>0 \quad \exists \eta>0 \quad \forall B\subset X,\ B \text{ Borel} \quad \sigma(B)<\eta \Rightarrow |\mu|(B)\leq \varepsilon;$$
$$\mu \ll \sigma \Longleftrightarrow \forall k>0 \quad \exists p>0 \quad \forall B\subset X,\ B \text{ Borel} \quad \sigma(B)<1/p \Rightarrow |\mu|(B)\leq 1/k.$$

It is enough to check the last implication on finite unions of basis open sets. Moreover these unions form a countable set since X is σ-compact metrizable. Let us denote by (U_l) the members of this last set. Then we get

$$\mu \ll \sigma \Longleftrightarrow \forall k>0 \quad \exists p>0 \quad \forall l \quad \sigma(U_l)<1/p \Rightarrow |\mu|\ (U_l)\leq 1/k.$$

Thus

$$L^1(\sigma)\cap \mathscr{M}(X)=\bigcap_k \bigcup_p \bigcap_{\{l:\sigma(U_l)<p^{-1}\}}\{\mu : |\mu|(U_l)\leq k^{-1}\}.$$

Let $\mathscr{K}_U^+(X)$ be the set of positive continuous functions with compact support. It is easy to see that for any open set U, we have $|\mu|\ (U)=\sup_{\mathscr{K}_U^+(X)}|\mu|\ (f)$ where

$$\mathscr{K}_U^+(X)=\{f\in \mathscr{K}^+(X) : \forall x\in X \quad f(x)\leq I_U(x)\}.$$

So

$$|\mu|\ (U_l)\leq 1/k \Longleftrightarrow \forall f\in \mathscr{K}_{U_l}^+(X) \quad |\mu|\ (f)\leq 1/k.$$

It follows that

$$L^1(\sigma)\cap \mathscr{M}^1(X)=\bigcap_k \bigcup_p \bigcap_{\{l:\sigma(U_l)<p^{-1}\}} \bigcap_{f\in \mathscr{K}_{U_l}^+(X)} \{\mu : |\mu|(f)\leq k^{-1}\}.$$

For f in $\mathscr{K}_U^+(X)$ let

$$\psi_f \ : \ \mathscr{M}^1(X) \to \boldsymbol{C}.$$
$$\mu \mapsto |\mu|(f)$$

Then ψ_f is l.s.c. for the vague topology [4] and $\{\mu : |\mu|(f)\leq k^{-1}\}$ is closed. Thus $L^1(\sigma)\cap \mathscr{M}^1(X)$ is a $F_{\sigma\delta}$ set and $N^1(\sigma)$ is also a $F_{\sigma\delta}$ set since for any n, ψ_n is continuous.

b-Identity approximations

The main result of this section is the following

THEOREM 3.8. *Suppose σ is in $\mathscr{M}^+(X)$. Then there exists a net of functions $\{f_\alpha\}$ in $B(L^1(G))$ such that for μ in $\mathscr{M}_s(X)\cap N(\sigma)$, the net $\{f_\alpha * \mu\}$ converges in σ-measure to zero.*

PROOF. The proof follows the one of Shapiro in the group case [16]. Consider $\mathscr{U}$ a basis of symmetric neighborhoods of e in G. We direct the net in the usual way: $U \geq V$ if $U \subseteq V$. For $V \in \mathscr{U}$, let $f_V = m_G(V)^{-1} I_V$. Since $|f_V * \mu| \leq f_V * |\mu|$, we may suppose without loss of generality that μ is a positive measure. Let $\varepsilon > 0$ and $a > 0$. Since μ is a regular and singular measure on X, there exists a compact set H and an open set O such that $H \subset O \subset X$ and

$$\mu(O) = \mu(X) = \|\mu\|,$$
$$\mu(O \backslash H) < \frac{\varepsilon a}{2},$$
$$\sigma(O) < \frac{\varepsilon}{2}.$$

Define λ in $\mathscr{M}(X)$ as follows: $\lambda(B) = \mu(B \cap H)$ for B a Borel subset of X. Then $\mu = \lambda + \theta$ where $\theta(X) < \frac{\varepsilon a}{2}$. There exists a neighborhood W in $\mathscr{U}$ such that $W.H \subseteq O$. Since μ is in $N(\sigma)$, $f_W * \mu \ll \sigma$ and $f_W * \lambda \ll \sigma$. As σ and $f_W * \mu$ are finite measures on X, one has:

$$\begin{aligned}(I_W * \lambda)(X \backslash O) &= \int_G I_W(g) \lambda(g^{-1}.(X \backslash O)) dg \\ &= \int_G I_W(g) \mu((g^{-1}.(X \backslash O)) \cap H) dg \\ &= 0\end{aligned}$$

since the set $(g^{-1}.(X \backslash O)) \cap H$ is empty, when $g \in W$.

For $V \subset W$,

$$\begin{aligned}(f_V * \mu)(X \backslash O) &= \int_{X \backslash O} (f_V * \mu)(x) d\sigma(x) \\ &= \int_{X \backslash O} (f_V * \theta)(x) d\sigma(x) \\ &\leq (f_V * \theta)(X) \\ &\leq \|f_V\|_1 \theta(X) \\ &\leq \frac{\varepsilon a}{2}.\end{aligned}$$

Let $A = \{x \in X \backslash O : (f_V * \mu)(x) \geq a\}$, then $\sigma(A) \leq \frac{\varepsilon}{2}$ and $\sigma(\{f_V * \mu \geq a\}) \leq \frac{\varepsilon}{2} + \sigma(O) < \varepsilon$. This proves the theorem.

COROLLARY 3.9. *Suppose that σ is in $\mathscr{M}^+(X)$. Then there exists a net of functions $\{f_\alpha\}$ in $B(L^1(G))$ such that*

(1) *for $f \in L^1(\sigma)$, the net $\{f_\alpha * f\}$ converges in L^1-norm to f,*

(2) *for $\mu \in \mathscr{M}_s(X) \cap N(\sigma)$, the net $\{f_\alpha * \mu\}$ converges in σ-measure to*

zero.

COROLLARY 3.10. *Let (G, X) be a transformation group with G metrizable. Let Λ be a σ-nicely placed subset of $\hat{G}$ and μ be in $N_{\Lambda}(\sigma)$. Then* spec μ_a *and* spec μ_s *are both contained in* Λ.

PROOF. Since μ is bounded and regular on X, there exists a σ-compact open set X_0 in X with $G.X_0=X_0$ and a quasi-invariant measure σ' in $\mathscr{M}^+(X)$ such that μ is concentrated on X_0 and $\sigma'|_{X_0}\sim\sigma|_{X_0}$. Hence $\mu=\mu_a+\mu_s$ is also the Lebesgue decomposition of μ with respect to σ'. Moreover $\mu\in N(\sigma)$ implies $\mu\in N(\sigma')$. Thus, we may assume that σ is a measure in $\mathscr{M}^+(X)$ that is quasi-invariant. Let (f_n) be an identity approximation satisfying Corollary 3.9. Since spec $(f_n*\mu)\subset$spec μ, the functions $(f_n*\mu)$are in $L^1_{\Lambda}(\sigma)$. And by Corollary 3.9, the sequence $(f_n*\mu)$ converges in σ-measure to μ_a. Since $(f_n*\mu)$ is bounded in $L^1(\sigma)$ and σ is finite, $(f_n*\mu)$ also converges in $L^p(\sigma)$ $(0<p<1)$. Then spec μ_a is contained in Λ and spec μ_s is also contained in Λ.

By Theorem 2.1, it follows

COROLLARY 3.11. *Let (G, X) be a transformation group with G metrizable. If Λ is a nicely placed subset of $\hat{G}$ and μ is in $N_{\Lambda}(\sigma)$, then both* spec μ_a *and* spec μ_s *are contained in* Λ.

Let us mention that we got the same result (without the restriction for μ to be in $N(\sigma)$) by using Yamaguchi's technique [7].

Let $\mathscr{C}\subset\mathscr{P}(\hat{G})$ be a family of subsets of $\hat{G}$. We denote by $\mathscr{C}^0$ the biggest hereditary class contained in $\mathscr{C}$, that is

$$\mathscr{C}^0=\{\Lambda\subset\hat{G}:\quad \forall\Lambda'\subset\Lambda,\quad \Lambda'\in\mathscr{C}\}.$$

LEMMA 3.12 (cf. [9, Lemma 1.1]). *Let $\mathscr{C}\subset\mathscr{P}(\hat{G})$ be a family of subsets of $\hat{G}$. Suppose that every $\Lambda\in\mathscr{C}$ satisfies the condition*

$$(*)\quad \mu\in N_{\Lambda}(\sigma)\quad \textit{implies}\quad \mu_s\in\mathscr{M}_{\Lambda}(X).$$

Then every $\Lambda\in\mathscr{C}^0$ satisfies that $\mu\in N_{\Lambda}(\sigma)$ implies $\mu\ll\sigma$.

PROOF. Let $\Lambda\in\mathscr{C}^0$ and $\mu\in N_{\Lambda}(\sigma)$. By hypothesis, $\mu_s\in\mathscr{M}_{\Lambda'}(X)$. Suppose that $\mu_s\neq0$. Then there exists $\alpha\in\Lambda$ such that $\alpha*\mu_s\neq0$. Consider $\Lambda'=\Lambda\backslash\{\alpha\}$ and $\mu'=\mu-\alpha*\mu$. Then Λ' satisfies the condition $(*)$. Since spec $(\mu')\subset\Lambda'$ and $\mu\in N_{\Lambda'}(\sigma)$, we have $\mu_s=(\mu')_s\in\mathscr{M}_{\Lambda'}(X)$. Since $\alpha\notin\Lambda'$, we have $\alpha\star\mu_s=0$. This gives a contradiction.

By Corollary 3.10 and Lemma 3.12, we have

THEOREM 3.13. *Let (G, X) be a transformation group with G metrizable. Let Λ be σ-Shapiro in $\hat{G}$. If μ is in $N_\Lambda(\sigma)$, then μ is absolutely continuous with respect to σ.*

This leads us to introduce a new class of σ-lacunary sets.

DEFINITION 3.14. *A set* Λ in $\hat{G}$ is *$N(\sigma)$-Riesz* if every measure μ in $N_\Lambda(\sigma)$ is absolutely continuous with respect to σ.

Of course when $N(\sigma)=\mathcal{M}(X)$ the notion of $N(\sigma)$-Riesz set coincides with the usual notion of σ-Riesz set. See also Examples 3.2 and 3.3. And we proved (Theorem 3.13) that every σ-Shapiro set is a $N(\sigma)$-Riesz set.

4. Localization techniques in transformation groups

A way to construct lacunary sets is the localization technique introduced by Meyer [14] and used by Godefroy [9] and Tardivel [17]. If G is a compact abelian group, we denote by τ the Bohr topology on $\hat{G}$. It is the topology induced on $\hat{G}$ by the pointwise convergence on G. If $\mathscr{C}\subset\mathscr{P}(\hat{G})$ is a family of subsets of $\hat{G}$, we will say that $\mathscr{C}$ is <u>localizable</u> if the following property holds:

$$\Lambda\in\mathscr{C} \iff \forall\alpha\in\hat{G},\ \exists V_\alpha, \text{ a } \tau\text{-neighborhood of } \alpha \text{ in } \hat{G},\ (\Lambda\cap V_\alpha)\in\mathscr{C}$$

and that $\mathscr{C}$ is <u>strongly localizable</u> if

$$\Lambda\in\mathscr{C} \iff \forall E\in\mathscr{C},\ \forall\alpha\in\hat{G}\backslash E,\ \exists V_\alpha, \text{ a } \tau\text{-neighborhood of } \alpha \text{ in } \hat{G},\ (\Lambda\cap V_\alpha)\in\mathscr{C}.$$

Meyer proved that the class of Riesz sets is localizable [14]. Godefroy proved that the classes of nicely placed sets, of Shapiro sets and the class $\mathscr{C}_0=\{\Lambda\subset\hat{G}: \forall\mu\in\mathscr{M}_\Lambda(G),\ \mu_s\in\mathscr{M}_\Lambda(G)\}$ are localizable [9], Tardivel proved that the class of Riesz sets is strongly localizable [17]. We will give other examples.

LEMMA 4.1. *Let ν be a discrete measure on G and μ_s be a singular bounded measure on X with respect to σ. Then $(\nu*\mu)_s=\nu*\mu_s$*

PROOF. Let us write $\nu=\sum_n a_n\delta_{g_n}$ with $a_n\in\boldsymbol{C}$, $\sum_n|a_n|<\infty$ and $g_n\in G$. Since μ_s is singular, there exists two Borel sets B_1 and B_2 in X such that $B_1\cap B_2=\emptyset$, $B_1\cup B_2=X$, $|\mu_s|(B_1)=0$ and $\sigma(B_2)=0$. Consider $B=\bigcup_n g_n.B_2$. It is a Borel set. Let E be a Borel subset of X such that $E\cap B=\emptyset$, then

$\nu*\mu_s(E)=\sum_n a_n\mu_s(g_n^{-1}.E)=0$. This shows that $\nu*\mu_s$ is concentrated in B. On the other hand, $\sigma(B_2)=0$ implies that $\sigma(g_n^{-1}.B_2)=0$ for all n by the quasi-invariance of σ. Thus $\sigma(B)=0$. This proves the lemma.

PROPOSITION 4.2. *Let H be either $\mathscr{M}(X)$ or $N(\sigma)$. The class*

$$\mathscr{C}_H=\{\Lambda\subset\hat{G} : \forall\mu\in H,\ \text{spec}\,\mu\subset\Lambda \Longrightarrow \mu\ll\sigma\}$$

is strongly localizable.

PROOF. Let $\Lambda\subset\hat{G}$ be such that

(1) $$\forall E\in\mathscr{C}_H,\quad \forall\alpha\in\hat{G}\backslash E,\quad \exists V_\alpha,\quad (V_\alpha\cap\Lambda)\in\mathscr{C}_H.$$

Let $E\in\mathscr{C}_H$, $\alpha\in\hat{G}\backslash E$ and $\mu\in H$ such that spec $\mu\subset\Lambda$. There exist a τ-neighborhood V_α of α and a discrete measure ν, on G such that

(2) $$\begin{cases}\hat{\nu}(\alpha)=1 \\ \hat{\nu}(\lambda)=0 \quad \forall\lambda\in\hat{G}\backslash V_\alpha\end{cases}$$

By Lemma 1.1, we have : $\alpha*\nu*\mu=\hat{\nu}(\alpha)(\alpha*\mu)=\alpha*\mu$ and spec $(\nu*\mu)$ is contained in $V_\alpha\cap\Lambda$. By (1), $\nu*\mu\ll\sigma$. We have $\nu*\mu=\nu*(\mu_a+\mu_s)=(\nu*\mu_a)+(\nu*\mu_s)$ and $\nu*\mu_a\ll\sigma$. Therefore $\nu*\mu_s=(\nu*\mu)_s=0$. We have : $\alpha*\mu_s=\alpha*\nu*\mu_s=0$, and $\alpha\notin$ spec μ_s. It follows that spec μ_s is contained in E which belongs to $\mathscr{C}_H$. Thus $\mu_s=0$.

COROLLARY 4.3. *The following classes are strongly localizable:*
1) *The class of σ-Riesz sets.*
2) *The class of $N(\sigma)$-Riesz sets.*

PROPOSITION 4.4. *Suppose $\sigma\in\mathscr{M}^+(X)$. Then the class of σ-nicely placed sets is localizable.*

We need a lemma

LEMMA 4.5. *Suppose $\sigma\in\mathscr{M}^+(X)$. Let ν be a discrete measure on G. Let $f\in L^1(\sigma)$ and let (f_n) be a sequence in $B(L^1(\sigma))$ such that (f_n) converges to f in $L^p(\sigma)$, $0<p<1$. Then there exists a subsequence (f_{n_k}) of (f_n) such that $(\nu*f_{n_k})$ converges to $\nu*f$ in $L^p(\sigma)$, where $\nu*f(x)=\int_G f(g^{-1}.x)\mathscr{T}(g^{-1},x)\,d\nu(g)$.*

PROOF. There exists a subsequence (f_{n_k}) of (f_n) such that (f_{n_k}) converges to f almost everywhere. Put $\nu=\sum_n a_n\delta_{g_n}$, $g_n\in G$ and $\sum_n|a_n|<\infty$. For $\varepsilon>0$, there exists $\nu'=\sum_{p\in J}a_p\delta_{g_p}$, where J is a finite set, such that $\|\nu-\nu'\|<\varepsilon$. Then

$$\|\nu*f_{n_k}-\nu*f\|_p^p\leq\|\nu*f_{n_k}-\nu'*f_{n_k}\|_p^p+\|\nu'*f_{n_k}-\nu'*f\|_p^p+\|\nu'*f-\nu*f\|_p^p$$

where $\|\nu*f\|_p^p=\int_X|\nu*f(x)|^p d\sigma(x)$. Since $\|f_{n_k}\|_1\leq 1$, we have

$$\begin{aligned}\|(\nu-\nu')*f_{n_k}\|_p^p&=\int_X|\int_G f_{n_k}(g^{-1}.x)\mathcal{T}(g^{-1}.x)d(\nu-\nu')(g)|^p d\sigma(x)\\&\leq\Big(\int_X\int_G|f_{n_k}(g^{-1}.x)|\mathcal{T}(g^{-1}.x)d|\nu-\nu'|(g)d\sigma|(x)\Big)^p\sigma(X)^{1-p}\\&\leq\sigma(X)^{1-p}\varepsilon^p.\end{aligned}$$

Similarly, we have

$$\|(\nu-\nu')*f\|_p^p\leq\sigma(X)^{1-p}\varepsilon^p.$$

Since J is finite and σ is quasi-invariant, $\nu'*f_{n_k}$ converges to $\nu'*f$ almost everywhere. Moreover $\|\nu'*f_{n_k}\|_1\leq\|\nu\|$ and $\|\nu'*f\|_1\leq\|\nu\|$. Hence

$$\lim_{k\to\infty}\|\nu'*f_{n_k}-\nu'*f\|_p=0.$$

Thus we have

$$\lim_{k\to\infty}\|\nu*f_{n_k}-\nu*f\|_p=0,$$

which proves the lemma.

PROOF OF PROPOSITION 4.4. Let $\Lambda\subset\hat{G}$ be such that $\forall\alpha\in\hat{G}$, $\exists V_\alpha$, $V_\alpha\cap\Lambda$ is σ-nicely placed. Let (f_n) in $B(L_\Lambda^1(\sigma))$ be such that (f_n) converges to f in $L^p(\sigma)$, $0<p<1$. Let $\alpha\notin\Lambda$. We need to prove that $\alpha*f=0$. There exists a discrete measure ν on G satisfying (2). By Lemma 4.5, there exists a subsequence (f_{n_k}) of (f_n) such that $(\nu*f_{n_k})$ converges to $\nu*f$ in $L^p(\sigma)$. Since spec$(\nu*f_{n_k})$ is contained in $V_\alpha\cap\Lambda$, spec$(\nu*f)$ is also contained in $V_\alpha\cap\Lambda$. Since $\alpha\notin V_\alpha\cap\Lambda$, $\alpha*\nu*f=0$, and so $\hat{\nu}(\alpha)(\alpha*f)=0$. But by (2), $\hat{\nu}(\alpha)=1$. This implies that $\alpha*f=0$. And spec f is contained in Λ.

PPOPOSITION 4.6. *The class* $\mathscr{C}_0=\{\Lambda\subset\hat{G}:\forall\mu\in\mathscr{M}_\Lambda(X),\ \mu_s\in\mathscr{M}_\Lambda(X)\}$ *is localizable.*

PROOF. Let $\Lambda\subset\hat{G}$ be such that $\forall\alpha\in\hat{G}$, $\exists V_\alpha$, $V_\alpha\cap\Lambda\in\mathscr{C}_0$. Let μ be in $\mathscr{M}_\Lambda(X)$. We have to show that spec μ_s is contained in Λ. Let $\alpha\notin\Lambda$. There exists a discrete measure ν on G satisfying (2) and spec $(\nu*\mu)$ is contained in $V_\alpha\cap\Lambda$. Therefore spec $(\nu*\mu)_s$=spec $(\nu*\mu_s)$ is also contained in $V_\alpha\cap\Lambda$. But $\alpha\notin V_\alpha\cap\Lambda$. Then $0=\alpha*\nu*\mu_s=\alpha*\mu_s$. And $\alpha\notin$spec μ_s.

REMARK. Yamaguchi's result [18, 19] says that the class $\mathscr{C}_0$ contains the Riesz subsets of $\hat{G}$. This class $\mathscr{C}_0$ also contains the nicely placed subsets of $\hat{G}$ [7].

Let $\mathscr{C}\subset\mathscr{P}(\hat{G})$ be a family of subsets of $\hat{G}$. Let $\mathscr{C}^0$ be the biggest hereditary class contained in $\mathscr{C}$.

Of course, if $\mathscr{C}$ is (strongly) localizable, then $\mathscr{C}^0$ is also (strongly) localizable.

COROLLARY 4.7. *Suppose $\sigma\in\mathscr{M}^+(X)$. Then the class of σ-Shapiro sets is localizable.*

COROLLARY 4.8. *The class $\mathscr{C}_0^0$ is localizable.*

PROPOSITION 4.9. *Let Λ be in $\mathscr{C}_0^0$, then Λ is a $N(\sigma)$-Riesz set.*

PROOF. We can prove the proposition as we proved Lemma 3.12.

From the remark and Proposition 4.9, we get the following theorem:

THEOREM 4.10. *Every Riesz subset of $\hat{G}$ is a $N(\sigma)$-Riesz set.*

COROLLARY 4.11. *When $N(\sigma)=\mathscr{M}(X)$, every Riesz subset of $\hat{G}$ is a σ-Riesz set.*

Let us come back again to our Example 1.3. In this case, we obtain a nice characterization of Riesz subsets of $\boldsymbol{Z}$.

PROPOSITION 4.12. *Let (G, X) be as in Example 1.3. Let Λ be a subset of $\hat{G}$. Then Λ is Riesz if and only if Λ is $N(\sigma)$-Riesz.*

PROOF. Suppose that $L^1_\Lambda(\sigma)=N_\Lambda(\sigma)$. Let ν be in $\mathscr{M}_\Lambda(\boldsymbol{T})$. Consider μ in $N(\sigma)$ defined for $Y\subset\boldsymbol{T}\times\boldsymbol{T}$ by $\mu(Y)=\int_{\boldsymbol{T}} I_Y(1, g)dg$. The measure $\nu*\mu$ is in $N_\Lambda(\sigma)$. Let K_1 and K_2 be compact subsets of $\boldsymbol{T}$ such that $m_{\boldsymbol{T}}(K_1)=0$ and $m_{\boldsymbol{T}}(K_2)\neq 0$. Consider $K=K_1\times K_2$. Then $\sigma(K)=0$ and $(\nu*\mu)(K)=0$. But $(\nu*\mu)(K)=\nu(K_1)\cdot m_{\boldsymbol{T}}(K_2)$. It follows that $\nu(K_1)=0$. Thus ν is absolutely continuous with respect to $m_{\boldsymbol{T}}$ and Λ is a Riesz set. The converse implication follows from Theorem 4.10.

Let us also mention the following transfer result : when G is metrizable, if $\Lambda\subset\hat{G}$ is τ-closed, then Λ is σ-nicely placed. This result follows from the fact that a τ-closed subset of $\hat{G}$ is nicely placed [9] and from Theorem 2.1.

5-Lacunarity and geometry of Banach spaces

Lust-Piquard got a nice characterization of Riesz sets [13] : let G be a compact abelian group, a subset Λ of $\hat{G}$ is a Riesz set if and only if the space $L^1_\Lambda(G)$ has the Radon-Nikodym property. Finet extended this result to the compact non abelian group case and the hypergroup case [6]. Let us recall that a Banach space Y has the *Radon-Nikodym Property* (R.N.P.) if and only if every linear continuous operator $T: L^1(\Omega, \mathscr{A}, \mu) \to Y$ (where μ is a probability measure) is representable by a Y-valued strongly μ-measurable and bounded function F, that is

$$\forall \varphi \in L^1(\Omega, \mathscr{A}, \mu) \quad T(\varphi) = \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega)$$

In this situation, we get

THEOREM 5.1. *Let (G, X) be a transformation group with X metrizable, Let σ be a quasi-invariant measure in $\mathscr{M}^+(X)$. If $L^1_\Lambda(\sigma)$ has R.N.P., then Λ is $N(\sigma)$-Riesz set.*

PROOF. Let μ be in $N_\Lambda(\sigma)$ and define the operator T_μ on $L^1(G)$ by $T_\mu(f) = f * \mu$. As μ is in $N_\Lambda(\sigma)$, $f*\mu$ is in $L^1_\Lambda(\sigma)$. For f in $L^1(G)$ and Θ in $\mathscr{K}(X)$

$$\begin{aligned}(f*\mu)(\Theta) &= \int_G f(g)\Big[\int_X \Theta(g\cdot x) d\mu(x)\Big] dg \\ &= \int_G f(g)(\delta_g * \mu)(\Theta) dg.\end{aligned}$$

The space $L^1_\Lambda(\sigma)$ has R.N.P., then for almost all g in G, $\delta_g * \mu$ is in $L^1(\sigma)$. Thus for almost all g in G, the measure $\delta_{g^{-1}} * (\delta_g * \mu)$ is in $L^1(\sigma)$ (since for a quasi-invariant positive measure σ, $\mathscr{M}(G) * L^1(\sigma) \subset L^1(\sigma)$). Thus μ is in $L^1(\sigma)$ and Λ is a $N(\sigma)$-Riesz set.

Bachelis and Ebenstein showed that a subset Λ of $\hat{G}$ is $\Lambda(1)$ if and only if $L^1_\Lambda(G)$ is reflexive [2], (see also [6]). A similar proof gives:

PROPOSITION 5.2. *Let σ be a quasi-invariant measure in $\mathscr{M}^+(X)$. If $\Lambda \subset \hat{G}$ is σ-$\Lambda(1)$ then $L^1_\Lambda(\sigma)$ is reflexive.*

COROLLARY 5.3. *Let (G, X) be a transformation group with X metrizable and let σ be quasi-invariant measure in $\mathscr{M}^+(X)$. If $\Lambda \subset \hat{G}$ is σ-$\Lambda(1)$, then Λ is a $N(\sigma)$-Riesz set.*

PROOF. The corollary follows from the fact that a reflexive Banach space has R.N.P.

We are now concerned with the question : what can be said on $L^1_\Lambda(G)$

if $L^1_\Lambda(\sigma)$ has R.N.P.? Let us recall a definition :

DEFINITION 5.4. Let (G, X) be a transformation group. We say that G acts *freely* on X if for any x in X, the map $g \mapsto g.x$ is one-to-one.

A trivial example is the transformation group (G, X), when G is a compact abelian subgroup of a locally compact group X.

We get the following result:

THEOREM 5.5. *Let (G, X) be a transformation group with X metrizable. Suppose that G acts freely on X. Let σ be a quasi-invariant measure in $\mathscr{M}^+(X)$ and Λ be a subset of $\widehat{G}$. If $L^1_\Lambda(\sigma)$ has R.N.P., then $L^1_\Lambda(G)$ also has R.N.P.*

PROOF. It is equivalent to prove that Λ is a Riesz set. Let μ be in $\mathscr{M}_\Lambda(G)$. We define the operator T_μ on $L^1_\Lambda(\sigma)$ by $T_\mu(f)=\mu*f$. We have that $T_\mu(f)$ is in $L^1_\Lambda(\sigma)$. For f in $L^1(\sigma)$ and Θ in $\mathscr{M}(X)$:

$$(\mu*f)(\Theta)=\int_X f(x)\left[\int_G \Theta(g.x)d\mu(g)\right]d\sigma(x)$$

$$=\int_X f(x)(\mu*\delta_x)(\Theta)d\sigma(x)$$

As the space $L^1_\Lambda(\sigma)$ has R.N.P., it follows that for almost all x in X, $\mu*\delta_x$ is in $L^1(\sigma)$. We will show that this implies that μ is in $L^1(G)$. Let K be a Borel subset of G such that $m_G(K)=0$. Then, for all x in X, $m_G(\{g\in G, g.x\in K.x\})=0$. Therefore $\int_X\int_G I_{K.x}(g.x)dg\, d\sigma(x)=0$. And, for almost all g in G, $\int_X I_{K.x}(g.x)d\sigma(x)=0$. But this last integral is exactly $\sigma(g^{-1}K.x)$. Since σ is quasi-invariant, it follows that $\sigma(K.x)=0$ for all x in X. And, for almost all x in $X, (\mu*\delta_x)(K.x)=0$. That is $\int_G I_{K.x}(g.x)d\mu(g)=0$. Hence $\mu(K)=0$ since the map $g\mapsto g.x$ is one-to-one.

COROLLARY 5.6. *Let (G, X) be a transformation group, where G acts freely on X. Let σ be a quasi-invariant measure in $\mathscr{M}^+(X)$. If $\Lambda\subset\widehat{G}$ is a σ-Riesz set, then Λ is a Riesz set.*

References

[1] R. AZENCOTT, Espaces de Poisson des groupes localement compacts, Springer Verlag, Lecture Notes 148.
[2] G.F. BACHELIS and S.E. EBENSTEIN, On $\Lambda(p)$-sets, Pacific J. Math. 5 (1974), 35-38.
[3] J. BOCLÉ, Sur la théorie ergodique, Ann. Inst. Fourier 10 (1960), 1-45.
[4] N. BOURBAKI, Integration Eléments de Mathémathiques Livre VI Ch. 6 Hermann, Paris

1959.
[5] K. DE LEEUW and I. GLICKSBERG, Quasi-invariance and analicity of measures on compact groups, Acta Math. 109 (1963), 179-205.
[6] C. FINET, Lacunary sets for groups and hypergroups, J. Austral. Math. Soc. 54 (1993), 39-60.
[7] C. FINET and V. TARDIVEL-NACHEF, A variant of a Yamaguchi's result, Hokkaido Math. J. 21 (1992) 483-489.
[8] F. FORELLI, Analytic and quasi-invariant measures, Acta Math. 118 (1967), 33-57.
[9] G. GODEFROY, On Riesz subsets of abelian discrete groups, Israel J. Math 61 (1988), 301-331.
[10] S. L. GULICK, T. S. LIU and A. C. VAN ROOIJ, Group algebra modules II, Canad. J. Math. 19 (1967), 151-173.
[11] T. S. LIU, A. VAN ROOIJ and J. K. WANG, Transformation groups and absolutely continuous measures II, Indag. Math. 32 (1970), 57-61.
[12] L. A. LINDHAL and F. POULSEN, Thin sets in Harmonic Analysis, Marcel Decker, New York 1971.
[13] F. LUST-PIQUARD, Ensembles de Rosenthal et Ensembles de Riesz, C.R. Acad. Sci. Paris 282 (1976), 833.
[14] Y. MEYER, Spectres des mesures absolument continues, Studia Math. 30 (1968), 87-89.
[15] W. RUDIN, Fourier Analysis on Groups, Interscience, New York 1962.
[16] J. SHAPIRO, Subspaces of $L^p(G)$ spanned by characters, $0<p<1$, Israel J. Math. 29 (1978), 248-264.
[17] V. TARDIVEL, Ensembles de Riesz, Trans. Amer. Math. Soc. 305 (1988), 167-174.
[18] H. YAMAGUCHI, The F. and M. Riesz Theorem on certain transformation groups, Hokkaido Math. J. 17 (1988) 289-332.
[19] H. YAMAGUCHI, The F. and M. Riesz Theorem on certain transformation groups II, Hokkaido Math. J. 19 (1990) 345-359.
[20] H. YAMAGUCHI, A property of spectrum of measures on certain transformation groups, Hokkaido Math. J. 20 (1991), 109-121.

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