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Invertibility of some singular integral operators and a lifting theorem

Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

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Abstract. Let P_+ be an analytic projection and let P_- be a co-analytic projection. Let $L^2(W)$ be the usual weighted Lebesgue space on the unit circle. For some weight W, P_+ is not continuous in the norm of $L^2(W)$. We shall define the Hilbert space $L^2((W))$ such that for any weight W, P_+ is continuous from $L^2((W))$ to $L^2(W)$. For an essentially bounded function ϕ, we shall consider a singular integral operator $\phi P_+ + P_-$ as a densely defined continuous operator from $L^2((W))$ to $L^2(W)$. Then $S_{\phi,(W)}$ denotes the bounded extension of $\phi P_+ + P_-$. Necessary and sufficient conditions for the (left) invertibility of $S_{\phi,(W)}$ are given as applications of the Cotlar-Sadosky's lifting theorem. Our results involve the Helson-Szegö theorem and the Widom-Devinatz-Rochberg theorem.

§ 1. Introduction.

Let $C(\mathrm{T})$ be an algebra of all continuous functions f on the unit circle T, and let A be a disc algebra of all functions f in $C(\mathrm{T})$ whose negative Fourier coefficients vanish. For $1 \leqq p \leqq \infty$, let $L^p = L^p(\mathrm{T})$ denote the L^p space of T with respect to the normalized Lebesgue measure m on T. Put $A_0 = \{f ; f \text{ is in } A, \text{ and } \int_{\mathrm{T}} f\, dm = 0\}$, and put $\bar{A}_0 = \{\bar{f} ; f \text{ is in } A_0\}$. By $\bar{f}$ we denote the complex conjugate function of f. Let H^p be the subspace of L^p consisting of functions whose negative Fourier coefficients vanish. Put $H_0^p = \{f ; f \text{ is in } H^p, \text{ and } \int_{\mathrm{T}} f\, dm = 0\}$, and put $\bar{H}_0^p = \{\bar{f} ; f \text{ is in } H_0^p\}$. For an f in L^1, its harmonic conjugate function $\tilde{f}$ is defined by

$$\tilde{f}(e^{i\theta}) = \int_{\mathrm{T}} \cot\left[\frac{\theta - t}{2}\right] f(e^{it})\, dm(e^{it}),$$

the integral being a Cauchy principal value. A function Q in H^∞ is an inner function if $|Q| = 1$. A function h is an outer function if there exists a real function V in L^1 and a real constant c such that $h = e^{V + i\tilde{V} + ic}$. Let

$H^{1/2}$ denote the subspace of functions of the form Qh^2 where Q is an inner function and h is an outer function in H^1. The singular integral operator S is defined by

$$Sf(\zeta)=\frac{1}{\pi i}\int_{\mathrm{T}}\frac{f(\eta)}{\eta-\zeta}d\eta,\ (\zeta\in\mathrm{T})$$

(cf. [2, p. 38]). The analytic projection P_+ and the coanalytic projection P_- is defined by $P_+=(I+S)/2$ and $P_-=(I-S)/2$. Then,

$$Sf(\zeta)=(P_+-P_-)f(\zeta)=i\tilde{f}(\zeta)+\int_{\mathrm{T}}f\ dm.$$

For a ϕ in L^∞, the singular integral operator $\phi P_+ + P_-$ is denoted by S_ϕ for short. In this paper, a positive function W in L^1 is said to be a weight. For a weight W, $L^p(W)(1<p<\infty)$ is a space of m-measurable functions equipped with the norm

$$\|f\|_{p,\,W}=\left\{\int_{\mathrm{T}}|f|^p Wdm\right\}^{1/p}<\infty.$$

The weighted Hardy space $H^p(W)$ (resp. $\bar{H}_0^p(W)$) is the norm closure of A (resp. $\bar{A}_0$) in $L^p(W)$. When we consider S_ϕ as an densely defined operator in $L^p(W)$, we wright $S_\phi=S_{\phi,p,W}$. $S_{\phi,p,W}$ may not be continuous. In this paper, we shall consider the case $p=2$, and remain entirely in Hilbert spaces. $L^2(W)$ is a Hilbert space equipped with the inner product

$$(f,g)_W=\int_{\mathrm{T}}f\bar{g}\ Wdm.$$

We shall wright $S_{\phi,2,W}$ as $S_{\phi,W}$, and $\|\cdot\|_{2,W}$ as $\|\cdot\|_W$ for short. For an f in the algebraic sum $A+\bar{A}_0$, we shall define the inner product

$$(f,g)_{(W)}=(P_+f,P_+g)_W+(P_-f,P_-g)_W.$$

Then $A+\bar{A}_0$ becomes a pre-Hilbert space. $L^2((W))$ denotes the completion of $A+\bar{A}_0$ with norm $\|\cdot\|_{(W)}$ defined by

$$\|f\|_{(W)}=(f,f)_{(W)}{}^{1/2}.$$

Then $L^2((W))$ is a Hilbert space, and P_+ is a contraction operator from $L^2((W))$ to $L^2(W)$, since for all f in $A+\bar{A}_0$,

$$\|P_+f\|_W\leqq\|f\|_{(W)}.$$

We shall define the Helson-Szegö class (HS) as follows (cf. [12]).

$$(HS)=\{e^{u+\tilde{v}};\ u \text{ and } v \text{ in } L^\infty,\ \|v\|_\infty<\pi/2\}.$$

If W is in (HS), then W^{-1} is also in (HS), and hence W^{-1} is in L^1. If W is in (HS), then $\|\cdot\|_W$ and $\|\cdot\|_{(W)}$ are equivalent norms. If W is not in (HS), then $S_{\phi,W}$ may not be continuous. For a general weight W, S_ϕ is a continuous operator from $L^2((W))$ to $L^2(W)$, that is, there exists a constant c such that for all f in $A+\bar{A}_0$,

$$\|S_\phi f\|_W \leqq c\|f\|_{(W)}.$$

In fact, we can take $c=2^{1/2}\max\{\|\phi\|_\infty, 1\}$. Let $S_{\phi,(W)}$ denote the bounded extension of S_ϕ. Hence $S_{\phi,(W)}$ is a bounded operator from $L^2((W))$ to $L^2(W)$ satisfying $S_{\phi,(W)}\, f=S_\phi f$ for all f in $A+\bar{A}_0$. We shall study the (left) invertibility of $S_{\phi,(W)}$ using Hilbert space methods and the following Cotlar-Sadosky's lifting theorem (cf. [1], [5], [15], [22]).

THEOREM(Cotlar-Sadosky). *Suppose W_1, W_2, W_3 are in L^1, and W_1, W_2 are real functions. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *For all f_1 in A and f_2 in $\bar{A}_0$,*

$$\int_{\mathrm{T}}\{|f_1|^2 W_1+|f_2|^2 W_2+2\mathrm{Re}(f_1 \bar{f}_2 W_3)\}dm \geqq 0.$$

(2) $W_1 \geqq 0$, $W_2 \geqq 0$ *and there exists a k in H^1 such that*

$$|W_3-k|^2 \leqq W_1 W_2.$$

When $W=1$, Doninguez [7] studied the invertibility of systems of Toeplitz operators using the Cotlar-Sadosky's type lifting theorem. When W is in (HS), Rochberg [18] defined the Toeplitz operator $T_{\phi,p,W}$ on $H^p(W)$ by $T_{\phi,p,W}\, f=P_+(\phi f)$ for all f in $H^p(W)$, and got the necessary and sufficient condition for the invertibility of $T_{\phi,p,W}$ (cf. [2, p. 216], [3]). If P_+ is continuous in the norm of $L^p(W)$, then $T_{\phi,p,W}$ is (left) invertible if and only if $S_{\phi,p,W}$ is (left) invertible (cf. [9, p. 124], [17, p. 393]). When $W=1$, Widom [21] and Devinatz [6] considered the left invertibility and the invertibility of T_ϕ and S_ϕ (cf. [8, p. 187], [17, p. 371]). Shinbrot [20] considered the invertibility of S_ϕ on L^2 and derived the method for finding the inverse operator of S_ϕ. Many generalizations of these results have been considered (cf. [2], [3], [4], [10], [11], [14]). For functions α and β in L^∞, the continuity of $\alpha P_+ + \beta P_-$ in the norm of $L^2(W)$ was considered in our preceding paper [15]. For a general weight W, $\alpha P_+ + \beta P_-$ has a bounded extension which is (left) invertible as an operator from $L^2((W))$ to $L^2(W)$ if and only if α^{-1}, β^{-1} are in L^∞ and $S_{\alpha/\beta,(W)}$ is (left) invertible. When W is in (HS), we can give a simple necessary and sufficient condition for the (left) invertibility of $S_{\phi,W}$. But when W is not in (HS), we can not give a simple condition.

In Section 2, by the Hilbert space methods and the Cotlar-Sadosky's lifting theorem, we shall give necessary and sufficient conditions for the left invertibility of $S_{\phi,(W)}$. Theorems 1 and 2 are main theorems. In Section 3, Theorem 3 is the main theorem. We shall give necessary and sufficient conditions for the invertibility of $S_{\phi,(W)}$ using the results of Section 2.

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§ 2. Left invertibility.

In Theorem 1 and Theorem 2, we shall give necessary and sufficient conditions for the left invertibility of the singular integral operator $S_{\phi,(W)}$. When $S_{\phi,(W)}$ is bounded and has a bounded left inverse operator, we shall say $S_{\phi,(W)}$ is left invertible. Since $S_{\phi,(W)}$ is always bounded, $S_{\phi,(W)}$ is left invertible if and only if $S_{\phi,(W)}$ is bounded below. When W is not in (HS), we have been unable to give a simple necessary and sufficient condition for the left invertibility of $S_{\phi,(W)}$. For the left invertibility of $S_{\phi,(W)}$, Prof. T. Nakazi suggested the simple condition (2) in Theorem 2, and the equivalence of (2) and (5). We use the Cotlar-Sadosky's lifting theorem to prove Theorem 1. We use Theorem 1 to prove Theorem 2. Each Theorem involves the Helson-Szegö theorem (cf. [12]). We shall consider weighted norm inequalities.

THEOREM 1. *Suppose* $|\phi|=1$, W *is a weight*, δ *is a constant satisfying* $0<\delta\leqq 1$, *and put*

$$r=\delta(2-\delta^2)^{1/2}.$$

Then the following conditions (1) *and* (2) *are equivalent.*

(1) *For all* f *in* $A+\bar{A}_0$,

$$\delta\|f\|_{(W)}\leqq\|S_\phi f\|_W.$$

(2) *There exists an inner function* Q, *a real function* V *in* L^1, u *and* v *in* L^∞ *such that*

$$\phi=Qe^{-i\tilde{V}},\quad We^V=e^{u+\tilde{v}},$$

$$\|v\|_\infty\leqq\cos^{-1}r<\pi/2,\ |u|\leqq\cosh^{-1}\{(\cos v)/r\}.$$

PROOF. We shall use the idea of Rochberg (cf. [18]) and the idea of Arocena, Cotlar and Sadosky (cf. [1], [5]). We shall show that (1) implies (2). By (1),

$$\delta^2 \int_{\mathrm{T}} (|f_1|^2 + |f_2|^2) W \; dm \leqq \int_{\mathrm{T}} |\phi f_1 + f_2|^2 \; W \; dm,$$

for all f_1 in A and f_2 in $\bar{A}_0$. Hence

$$\int_{\mathrm{T}} \{(1-\delta^2)(|f_1|^2 + |f_2|^2) + 2\mathrm{Re}(\phi f_1 \bar{f}_2)\} W \; dm \geqq 0.$$

By the Cotlar-Sadosky's lifting theorem, $\delta \leqq 1$ and there exists a k in H^1 such that

$$|\phi w - k| \leqq (1-\delta^2) W.$$

Since $|\phi| = 1$,

$$| W - |k|| \leqq |\phi W - k| \leqq (1-\delta^2) \; W \leqq W.$$

Hence

$$\delta^2 W \leqq |k| \leqq 2 W.$$

Since W is a non-zero function, k is a non-zero function in H^1. Hence $\log W$ is in L^1. Put $g = ke^{-\log W - i(\log W)^{\sim}}$. Since $|\phi| = 1$,

$$|1 - \bar{\phi} g e^{i(\log W)^{\sim}}| \leqq 1 - \delta^2.$$

Since $0 < \delta \leqq 1$ and $r = \delta(2-\delta^2)^{1/2}$, $0 < r \leqq 1$. Then there exists a real function v in L^∞ such that

$$\bar{\phi} g e^{i(\log W)^{\sim}} = |g| e^{-iv},$$
$$\|v\|_\infty \leqq \cos^{-1} r < \pi/2$$

(cf. [15, Lemma 2]). Hence $(\cos v)/r \geqq 1$. Since $\delta^2 \leqq |g| \leqq 2$, g is in H^∞, and there exists an inner function Q_0 and a real function u in L^∞ such that $g = rQ_0 e^{-u-i\tilde{u}}$. Since $\bar{\phi} g e^{i(\log W)^{\sim}} = re^{-u-iv}$, we have

$$|1 - re^{-u-iv}| \leqq 1 - \delta^2.$$

Since r is a positive constant and

$$|1 - re^{-u-iv}|^2 - (1-\delta^2)^2 = r^2\{e^{-2u} - 2\{\cos v)/r\} e^{-u} + 1\},$$

we have

$$e^{-2u} - 2\{(\cos \; v)/r\} e^{-u} + 1 \leqq 0.$$

Since $(\cos \; v)/r \geqq 1$, we have

$$|u| \leqq \cosh^{-1}\{(\cos \; v)/r\}.$$

Put $V = u + \tilde{v} - \log \; W$, then V is in L^1, and there exists a real constant c

such that $\tilde{V}=\tilde{u}-v-(\log W)^{\sim}+c$. Put $Q=Q_0e^{ic}$, then

$$\phi=ge^{i(v+(\log W)^{\sim})}/|g|=Q_0e^{i(v-\tilde{u}+(\log W)^{\sim})}=Qe^{-i\tilde{V}}.$$

We shall show that (2) implies (1). By (2),

$$e^{-2u}-2\{(\cos v)/r\}e^{-u}+1\leqq 0.$$

By the calculation,

$$|1-re^{-u-iv}|\leqq 1-\delta^2.$$

Put $k=\phi Wre^{-u-iv}$, then k is in H^1, since

$$k=rQe^{-i\tilde{V}+(u+\tilde{v}-V)-u-iv}=rQe^{-V-i\tilde{V}}e^{\tilde{v}-iv}.$$

Hence

$$|\phi W-k|=|\phi W(1-re^{-u-iv})|\leqq(1-\delta^2)W.$$

By the Cotlar-Sadosky's lifting theorem, for all f_1 in A and f_2 in $\bar{A}_0$,

$$\int_{\mathrm{T}}\{(1-\delta^2)(|f_1|^2+|f_2|^2)+2\mathrm{Re}(\phi f_1\bar{f}_2)\}Wdm\geqq 0.$$

Since $|\phi|=1$,

$$\delta^2\int_{\mathrm{T}}(|f_1|^2+|f_2|^2)Wdm\leqq\int_{\mathrm{T}}|\phi f_1+f_2|^2\ Wdm.$$

This implies (1). This completes the proof.

COROLLARY 1. *Suppose ϕ is in L^∞ and W is a weight. Then the following conditions are mutually equivalent.*
(1) *$S_{\phi,(W)}$ is an isometry, that is, for all f in $A+\bar{A}_0$,*

$$\|S_\phi f\|_W=\|f\|_{(W)}.$$

(2) *$|\phi|=1$, and for all f in $A+\bar{A}_0$,*

$$\|f\|_{(W)}\leqq\|S_\phi f\|_W.$$

(3) *There exists an inner function Q and a real function V in L^1 such that $\phi=Qe^{-i\tilde{V}}$ and $We^V=1$.*

PROOF. By (1), for all f_1 in A,

$$\int_{\mathrm{T}}(|\phi|^2-1)|f_1|^2\ Wdm=0.$$

This implies $|\phi|=1$. Hence (1) implies (2). By Theorem 1, (2) and (3) are equivalent. By (3), $\phi W=Qe^{-V-i\tilde{V}}$ and ϕW is in L^1. This implies

ϕW is in H^1, and hence for all f_1 in A and f_2 in $\bar{A}_0$,

$$\int_{\mathrm{T}} |\phi f_1 + f_2|^2 W dm - \int_{\mathrm{T}} (|f_1|^2 + |f_2|^2) W dm$$
$$= 2\mathrm{Re}\int_{\mathrm{T}} \phi f_1 \bar{f}_2 W dm = 0.$$

This implies (1). This completes the proof.

Let $H^2(W) \oplus \bar{H}_0^2(W)$ denote the algebraic direct sum of $H^2(W)$ and $\bar{H}_0^2(W)$ (cf. [8, p. 78]). Then $H^2(W) \oplus \bar{H}_0^2(W)$ is the Hilbert space equipped with the inner product

$$(\langle f_1, f_2 \rangle, \langle g_1, g_2 \rangle)_{\langle W \rangle} = (f_1, g_1)_W + (f_2, g_2)_W,$$

and the norm

$$\|\langle f_1, f_2 \rangle\|_{\langle W \rangle} = (\langle f_1, f_2 \rangle, \langle f_1, f_2 \rangle)_{\langle W \rangle}.$$

For any f in $L^2((W))$, there exists a sequence f_{1n} in A and a sequence f_{2n} in $\bar{A}_0$ such that $f_{1n} + f_{2n}$ converges to f in the norm of $L^2((W))$. Then there exists an f_1 in $H^2(W)$ and an f_2 in $\bar{H}_0^2(W)$ such that $\langle f_{1n}, f_{2n} \rangle$ converges to $\langle f_1, f_2 \rangle$ in the norm of $H^2(W) \oplus \bar{H}_0^2(W)$. Let J denote the isometry from $L^2((W))$ onto $H^2(W) \oplus \bar{H}_0^2(W)$ defined by

$$Jf = \langle f_1, f_2 \rangle.$$

This definition is correct in the sense that it does not depend on the particular choice of the Cauchy sequence which defines f_1 and f_2. Let $R_{\phi, W}$ denote the operator from $H^2(W) \oplus \bar{H}_0^2(W)$ to $L^2(W)$ defined by

$$R_{\phi, W} \langle f_1, f_2 \rangle = \phi f_1 + f_2.$$

LEMMA 1. *Suppose ϕ is in L^∞, and W is a weight. Then $R_{\phi, W}$ is a bounded operator from $H^2(W) \oplus \bar{H}_0^2(W)$ to $L^2(W)$. $R_{\phi, W}$ is (left) invertible if and only if $S_{\phi, (W)}$ is (left) invertible.*

PROOF. $R_{\phi, W}$ is bounded, since for all $\langle f_1, f_2 \rangle$ in $H^2(W) \oplus \bar{H}_0^2(W)$,

$$\|R_{\phi, W} \langle f_1, f_2 \rangle\|_W \leqq \max\{\|\phi\|_\infty, 1\}(\|f_1\|_W + \|f_2\|_W)$$
$$\leqq 2^{1/2} \max\{\|\phi\|_\infty, 1\} \|\langle f_1, f_2 \rangle\|_{\langle W \rangle}.$$

Since $S_{\phi, (W)} = R_{\phi, W} J$, $R_{\phi, W}$ is (left) invertible if and only if $S_{\phi, (W)}$ is (left) invertible.

LEMMA 2. *Suppose ϕ, ϕ^{-1} are in L^∞ and W is a weight. If there exists an inner function Q, outer functions α, β such that $|\alpha|^2 W$, $|\beta|^2 W$ are in (HS), and $\phi = Q\bar{\beta}/\alpha$, then $R_{\phi, W}$ and $S_{\phi, (W)}$ are left invertible. If*

T is defined by

$$Tf=\langle \alpha P_+(\bar{Q}f/\bar{\beta}),\ Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta})\rangle,$$

for all f in $L^2(W)$, then T is the left inverse to $R_{\phi,W}$, and $J^{-1}T$ is the left inverse to $S_{\phi,(W)}$. Then

$$J^{-1}Tg=(\alpha P_+ + Q\bar{\beta}P_-)(\bar{Q}g/\bar{\beta}),$$

for all g in $\phi A+\bar{A}_0$.

PROOF. Since $|\alpha|^2W$, $|\beta|^2W$ are in (HS), $(|\alpha|^2W)^{-1}$, $(|\beta|^2W)^{-1}$ are also in (HS). Hence $(|\alpha|^2W)^{-1}$, $(|\beta|^2W)^{-1}$ are in L^1. For all f in $L^2(W)$, by the Schwarz inequality, $f/\bar{\beta}$ is in L^1. By the Helson-Szegö theorem (cf. [12]), there exist constants γ, γ' such that

$$\begin{aligned}\|Tf\|_{\langle W\rangle^2}&=\int_{\mathrm{T}}|\alpha P_+(\bar{Q}f/\bar{\beta})|^2W\,dm+\int_{\mathrm{T}}|Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta}|^2\ W\,dm\\ &\leqq\gamma\int_{\mathrm{T}}|\bar{Q}f/\bar{\beta}|^2|\alpha|^2W\,dm+\gamma'\int_{\mathrm{T}}|\bar{Q}f/\bar{\beta}|^2|\beta|^2W\,dm\\ &\leqq(\gamma\|\phi^{-1}\|_\infty^2+\gamma')\int_{\mathrm{T}}|f|^2W\,dm.\end{aligned}$$

For all f_1 in $H^2(W)$ and f_2 in $\bar{H}_0^2(W)$, by the Schwarz inequality, f_1/α is in H_1 and $\bar{Q}f_2/\bar{\beta}$ is in $\bar{H}_0^1$. Hence

$$\begin{aligned}&\alpha P_+(\bar{Q}(\phi f_1+f_2)/\bar{\beta})=\alpha P_+(f_1/\alpha+\bar{Q}f_2/\bar{\beta})=\alpha P_+(f_1/\alpha)=f_1,\\ &Q\bar{\beta}P_-(\bar{Q}(\phi f_1+f_2)/\bar{\beta})=Q\bar{\beta}P_-(f_1/\alpha+\bar{Q}f_2/\bar{\beta})=Q\bar{\beta}P_-(\bar{Q}f_2/\bar{\beta})=f_2.\end{aligned}$$

This implies $\alpha P_+(\bar{Q}f/\bar{\beta})$ is in $H^2(W)$ and $Q\bar{\beta}P_-(\bar{Q}f/\bar{\beta})$ is in $\bar{H}_0^2(W)$. Hence

$$TR_{\phi,W}\langle f_1, f_2\rangle=T(\phi f_1+f_2)=\langle f_1, f_2\rangle.$$

Hence T is the left inverse to $R_{\phi,W}$. By Lemma 1, $J^{-1}T$ is the left inverse to $S_{\phi,(W)}$. For any g in $\phi A+\bar{A}_0$, there exists a g_1 in A and a g_2 in $\bar{A}_0$ such that $g=\phi g_1+g_2$. By the calculation, $\alpha P_+(\bar{Q}g/\bar{\beta})=g_1$, and $Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})=g_2$. Hence $\alpha P_+(\bar{Q}g/\bar{\beta})$ is in A, and $Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})$ is in $\bar{A}_0$. Hence

$$\begin{aligned}J^{-1}Tg&=J^{-1}\langle\alpha P_+(\bar{Q}g/\bar{\beta}),\ Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})\rangle\\ &=\alpha P_+(\bar{Q}g/\bar{\beta})+Q\bar{\beta}P_-(\bar{Q}g/\bar{\beta})=(\alpha P_+ + Q\bar{\beta}P_-)(\bar{Q}g/\bar{\beta}).\end{aligned}$$

This completes the proof.

THEOREM 2. *Suppose ϕ is in L^∞ and W is a weight. Then the following conditions on ϕ and W are mutually equivalent.*

(1) *$S_{\phi,(W)}$ is left invertible.*

(2) *ϕ^{-1} is in L^∞, and there exists an inner function Q and a real function V in L^1 such that We^V is in (HS), and*

$$\phi/|\phi|=Qe^{-i\tilde{V}}.$$

(3) *ϕ^{-1} is in L^∞, and there exists an inner function Q, outer functions α, β such that $|\alpha|^2W$, $|\beta|^2W$ are in (HS), and $\phi=Q\bar{\beta}/\alpha$,*
(4) *ϕ^{-1} is in L^∞, and there exists a k in H^1 such that*

$$\|1-k/(\phi W)\|_\infty<1.$$

(5) *There exists a positive constant δ such that for all f in $A+\bar{A}_0$.*

$$\delta\|f\|_W\leqq\min\{\|S_\phi f\|_W,\ \|S_{-\phi}f\|_W\}.$$

PROOF. We shall show that (1) implies (4) and (2). By (1), there exists a positive constant δ such that

$$\delta\|f\|_{(W)}\leqq\|S_\phi f\|_W,$$

for all f in $A+\bar{A}_0$. Hence

$$\int_T\{(|\phi|^2-\delta^2)|f_1|^2+(1-\delta^2)|f_2|^2+2\mathrm{Re}(\phi f_1\bar{f}_2)\}\,Wdm\geqq0,$$

for all f_1 in A and f_2 in $\bar{A}_0$. By the Cotlar-Sadosky's lifting theorem, $0<\delta\leqq1$, $\delta\leqq|\phi|$ and there exists a k in H^1 such that

$$|\phi W-k|\leqq(1-\delta^2)^{1/2}(|\phi|^2-\delta^2)^{1/2}W\leqq(1-\delta^2)^{1/2}|\phi|\,W.$$

This implies 4). Put $\phi_0=\phi e^{-\log|\phi|-i(\log|\phi|)^\sim}$, $k_0=ke^{-\log|\phi|-i(\log|\phi|)^\sim}$, and $\delta_0=\{1-(1-\delta^2)^{1/2}\}^{1/2}$. Then $|\phi_0|=1$, $0<\delta_0\leqq1$, k_0 is in H^1 and

$$|\phi_0W-k_0|\leqq(1-\delta_0^2)\,W.$$

By the Cotlar-Sadosky's lifting theorem,

$$\int_T\{(1-\delta_0^2)(|f_1|^2+|f_2|^2)+2\mathrm{Re}(\phi_0f_1\bar{f}_2)\}\,Wdm\geqq0.$$

Hence, for all f in $A+\bar{A}_0$,

$$\delta_0\|f\|_{(W)}\leqq\|S_{\phi_0}f\|_W.$$

By Theorem 1, there exists an inner function Q, a real function V in L^1, and u, v in L^∞ such that

$$\phi_0=Qe^{-i\tilde{V}},\ \ We^V=e^{u+\tilde{v}},$$
$$\|v\|_\infty\leqq\cos^{-1}\delta<\pi/2,\ |v|\leqq\cosh^{-1}\{(\cos v)/\delta\},$$

since $\delta_0(2-\delta_0^2)^{1/2}=\delta$. Hence

$$\phi/|\phi|=\phi_0 e^{i(\log|\phi|)^\sim}=Qe^{-i(V-\log|\phi|)^\sim},$$
$$We^{V-\log|\phi|}=e^{u-\log|\phi|+\tilde{v}}.$$

Since $\delta\leqq|\phi|$, $\log|\phi|$ is in L^∞. This implies $u-\log|\phi|$ is in L^∞, and hence $We^{V-\log|\phi|}$ is in (HS). We shall show that (2) implies (3). Put $U=\log|\phi|$, then U is in L^∞. Put

$$\alpha=e^{\frac{1}{2}(V-U+i(V-U)^\sim)},\quad \beta=e^{\frac{1}{2}(V+U+i(V+U)^\sim)},$$

then α, β are outer functions, and $\phi=Q\bar{\beta}/\alpha$. Since We^V is in (HS), $|\alpha|^2 W$ and $|\beta|^2 W$ are in (HS). This implies (3). By Lemma 2, (3) implies (1). We shall show that (4) implies (1). By (4), there exists a constant δ and a k in H^1 such that $0<\delta\leqq 1$, $\delta\leqq|\phi|^2$, and $|\phi W-k|\leqq(1-\delta)|\phi|W$. Then

$$\begin{aligned}&(1-\delta^2)(|\phi|^2-\delta^2)-(1-\delta)^2|\phi|^2\\&\quad=\delta(1-\delta)\{2|\phi|^2-\delta(1+\delta)\}\\&\quad\geqq 2\delta(1-\delta)(|\phi|^2-\delta)\geqq 0.\end{aligned}$$

Hence

$$|\phi W-k|^2\leqq(1-\delta^2)(|\phi|^2-\delta^2)W^2.$$

By the Cotlar-Sadosky's lifting theorem, for all f_1 in A and f_2 in $\bar{A}_0$,

$$\int_T\{(|\phi|^2-\delta^2)|f_1|^2+(1-\delta^2)|f_2|^2+2\mathrm{Re}(\phi f_1\bar{f}_2)\}Wdm\geqq 0.$$

This implies (1). Since

$$\begin{aligned}\|f\|_W^2+\|Sf\|_W^2&=\|P_+f+P_-f\|_W^2+\|P_+f-P_-f\|_W^2\\&=2(\|P_+f\|_W^2+\|P_-f\|_W^2)=2\|f\|_{(W)}^2,\end{aligned}$$

we have

$$2^{1/2}\|f\|_{(W)}\leqq\|f\|_W+\|Sf\|_W\leqq 2\|f\|_{(W)}.$$

Hence, $S_{\phi,(W)}$ is left invertible if and only if there exists a positive constant δ such that for all f in $A+\bar{A}_0$,

$$\delta(\|f\|_W+\|Sf\|_W)\leqq\|S_\phi f\|_W.$$

Since $S^2f=f$ and $S_\phi Sf=S_\phi(P_+-P_-)f=\phi P_+f-P_-f=-S_{-\phi}f$, we have

$$S_\phi f=S_\phi S^2 f=-S_{-\phi}Sf.$$

Hence

$$\delta\|f\|_W\leqq\|S_\phi f\|_W,\quad \delta\|Sf\|_W\leqq\|S_{-\phi}Sf\|_W.$$

Since f is in $A+\bar{A}_0$ if and only if Sf is in $A+\bar{A}_0$, (1) and (5) are equivalent. This completes the proof.

REMARK. (*a*) *If* $S_{\phi,(W)}$ *is left invertible, then* $\log W$ *is in* L^1, *and there exists an inner function* Q, *real functions* u, v *in* L^∞ *such that* $\|v\|_\infty < \pi/2$ *and*

$$\phi/|\phi| = Qe^{i\{v-(u-\log W)^\sim\}}.$$

(*b*) *By condition* (2), $S_{\phi,(W)}$ *is left invertible if and only if* ϕ^{-1} *is in* L^∞ *and* $S_{\phi/|\phi|,(W)}$ *is left invertible.*

The equivalence of conditions (3) and (4) in Corollary 2 is the Helson -Szegö theorem (cf. [12]). Since $\|f\|_W^2 + \|Sf\|_W^2 = 2\|f\|_{(W)}^2$, we have

$$\|f\|_W \leqq 2^{1/2}\|f\|_{(W)}.$$

COROLLARY 2. *For a weight* W, *the following conditions are mutually equivalent.*

(1) $\|S_{1,(W)}\| < 2^{1/2}$. *That is, there exists a positive constant* ε *such that for all* f *in* $A + \bar{A}_0$,

$$\|f\|_W \leqq (2^{1/2} - \varepsilon)\|f\|_{(W)}.$$

(2) $S_{1,(W)}$ *is left invertible. That is, there exists a positive constant* δ *such that for all* f *in* $A + \bar{A}_0$,

$$\delta\|f\|_{(W)} \leqq \|f\|_W.$$

(3) *There exists a positive constant* γ *such that for all* f *in* $A + \bar{A}_0$,

$$\|P_+ f\|_W \leqq \gamma\|f\|_W.$$

(4) W *is in* (HS).

(5) *There exists a* k *in* H^1 *such that*

$$\|1 - k/W\|_\infty < 1.$$

PROOF. We shall show that (1) implies (2). By (1), there exists a positive constant δ such that

$$\|f_1 - f_2\|_W^2 \leqq (2 - \delta^2)(\|f_1\|_W^2 + \|f_2\|_W^2),$$

for all f_1 in A and f_2 *in* $\bar{A}_0$. Hence

$$(1 - \delta^2)\|f_1\|_W^2 + (1 - \delta^2)\|f_2\|_W^2 + 2\mathrm{Re}(f_1, f_2)_W \geqq 0.$$

Hence

$$\delta(\|f_1\|_W^2 + \|f_2\|_W^2)^{1/2} \leqq \|f_1 + f_2\|_W.$$

This implies (2). This proof is reversible. Since $\|P_+f\|_W \leqq \|f\|_{(W)}$, (2) implies (3). We shall show that (3) implies (2). By (3),

$$\|P_-f\|_W \leqq \|P_+f\|_W + \|f\|_W \leqq \gamma' \|f\|_W,$$

for some constant γ'. Hence

$$\|f\|_{(W)}^2 = \|P_+f\|_W^2 + \|P_-f\|_W^2 \leqq (\gamma^2 + \gamma'^2)\|f\|_W^2.$$

This implies (2). We shall show that (2) implies (4). By Theorem 2, there exists an inner function Q and a real function V in L^1 such that We^V is in (HS) and $Qe^{-i\tilde{V}}=1$. Since $1/(We^V)$ is also in (HS), $1/(We^V)$ is in L^1. By the Schwarz inequality, $e^{-V/2}$ is in L^1. Since $Qe^{-V-i\tilde{V}}=e^{-V}$, a positive function e^{-V} is in $H^{1/2}$. By the Neuwirth-Newman theorem (cf. [16]), V is a constant. Hence W is in (HS). Conversely when W is in (HS), we can choose $Q=1$, $V=0$, and $\phi=1$ in the condition (2) of Theorem 2. Hence (4) implies (2). When $\phi=1$, by Theorem 2, $S_{1,(W)}$ is left invertible if and only if there exists a k in H^1 such that $\|1-k/W\|_\infty<1$. Hence (2) and (5) are equivalent. This completes the proof.

Put $W(e^{i\theta})=|1-e^{i\theta}|^2$, $\phi(e^{i\theta})=e^{i\theta}$ and $k(e^{i\theta})=(1-e^{i\theta})^2$, then k is in H^1 and $\phi W+k=0$. By Theorem 2, this implies $S_{\phi,(W)}$ is left invertible. Since W^{-1} is not in L^1, W is not in (HS). Then by Theorem 2, there exists a positive constant δ such that for all f in $A+\bar{A}_0$,

$$\delta\|f\|_W \leqq \|S_\phi f\|_W.$$

By Corollary 2, the converse is not true. If W is not in (HS), then $S_{1,(W)}$ is not left invertible, and $S_{1,W}$ is an isometry. But we have the following result.

COROLLARY 3. *Suppose ϕ and $(\phi-1)^{-1}$ are in L^∞, and W is a weight. If there exists a positive constant δ such that for all f in $A+\bar{A}_0$,*

$$\delta\|f\|_W \leqq \|S_\phi f\|_W,$$

then $S_{\phi-\varepsilon,(W)}$ is left invertible for any constant ε satisfying $0<\varepsilon\leqq\delta^2$.

PROOF. Since $\varepsilon\leqq\delta^2$, for all f in $A+\bar{A}_0$,

$$\int_T \{(|\phi|^2-\varepsilon)|f_1|^2+(1-\varepsilon)|f_2|^2+2\mathrm{Re}((\phi-\varepsilon)f_1\bar{f}_2)\}\,W dm \geqq 0.$$

By the Cotlar-Sadosky's lifting theorem, $\varepsilon\leqq|\phi|^2$, $\varepsilon\leqq 1$ and there exists a k in H^1 such that

$$|(\phi-\varepsilon)W-k|^2 \leqq (1-\varepsilon)(|\phi|^2-\varepsilon)W^2$$
$$=\{1-\varepsilon(|\phi-1|/|\phi-\varepsilon|)^2\}|(\phi-\varepsilon)W|^2.$$

Since ϕ and $(\phi-1)^{-1}$ are in L^∞, there exists a constant ρ, $0\leqq\rho<1$ such that

$$\varepsilon(|\phi-1|/|\phi-\varepsilon|)^2 \geqq 1-\rho^2.$$

Hence

$$|(\phi-\varepsilon)W-k| \leqq \rho|\phi-\varepsilon|W.$$

Since

$$|\phi-\varepsilon| \geqq |\phi|-\varepsilon \geqq \varepsilon^{1/2}(1-\varepsilon^{1/2})>0,$$

$(\phi-\varepsilon)^{-1}$ is in L^∞, and

$$\|1-k/\{(\phi-\varepsilon)W\}\|_\infty \leqq \rho<1.$$

By Theorem 2, this implies $S_{\phi-\varepsilon,(W)}$ is left invertible.

COROLLARY 4. *Suppose ϕ is in L^∞ and W is a weight. If there existes a real function s in L^1 such that $\phi=e^{is}|\phi|$, and $We^{\tilde{s}}$ is in L^1, then the following conditions* (1) *and* (2) *are equivalent.*

(1) *$S_{\phi,(W)}$ is left invertible.*

(2) *ϕ^{-1} is in L^∞, and $We^{\tilde{s}}$ is in (HS).*

PROOF. By Theorem 2, (1) implies ϕ^{-1} is in L^∞ and there exists a k in H^1 such that $\|1-k/(\phi W)\|_\infty<1$. Hence

$$\|1-(ke^{\tilde{s}-is})/(|\phi|We^{\tilde{s}})\|_\infty<1.$$

Since $|\phi|We^{\tilde{s}}$ is in L^1, $ke^{\tilde{s}-is}$ is in H^1. By Corollary 2, $|\phi|We^{\tilde{s}}$ is in (HS) and hence $We^{\tilde{s}}$ is in (HS). Conversely, (2) implies $|\phi|We^{\tilde{s}}$ is in (HS). By Corollary 2, there exists a k in H^1 such that $\|1-k/(|\phi|We^{\tilde{s}})\|_\infty<1$. Hence

$$\|1-ke^{is-\tilde{s}}/(\phi W)\|_\infty<1.$$

By Theorem 2, this implies (1). This completes the proof.

COROLLARY 5. *Suppose ϕ is in L^∞ and W is a weight. Suppose the argument of ϕ is in L^1 and it's harmonic conjugate function is in L^∞. (This condition is satisfied if ϕ is invertible in H^∞, or the argument of ϕ is Dini continuous.) Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *$S_{\phi,(W)}$ is left invertible.*

(2) *ϕ^{-1} is in L^∞, and W is in (HS).*

PROOF. There exists a real function s in L^1 such that $\phi=e^{is}|\phi|$ and $\tilde{s}$ is in L^∞. Hence $We^{\tilde{s}}$ is in L^1. By Corollary 4, ϕ and W satisfy (1) if and only if ϕ^{-1} is in L^∞, and $We^{\tilde{s}}$ is in (HS). Since $e^{\tilde{s}}$ is invertible in L^∞, $We^{\tilde{s}}$ is in (HS) if and only if W is in (HS). This completes the proof.

§ 3. Invertibility.

When P_+ is continuous in the norm of $L^p(W)$, Rochberg [18] solved the invertibility problem of the Toeplitz operator on the weighted Hardy space $H^p(W)$. When $S_{\phi,(W)}$ has a bounded inverse operator, we shall say $S_{\phi,(W)}$ is invertible.

Prof. T. Nakazi privately communicated me the equivalence of simple conditions (1) and (2) in Theorem 3. We shall prove Theorem 3 using Theorem 2. In Theorem 3, we shall give the form of the inverse to $S_{\phi,(W)}$.

THEOREM 3. *Suppose ϕ is in L^∞ and W is a weight. Then the following conditions on ϕ and W are mutually equivalent.*
(1) *$S_{\phi,(W)}$ is invertible.*
(2) *ϕ^{-1} is in L^∞, and there exists a real constant c and a real function V in L^1 such that We^V is in (HS), and*

$$\phi/|\phi|=e^{i(c-\tilde{V})}.$$

(3) *ϕ^{-1} is in L^∞, and there exist outer functions α, β such that $|\alpha|^2W$, $|\beta|^2W$ are in (HS), and $\phi=\bar{\beta}/\alpha$.*
(4) *ϕ^{-1} is in L^∞, and there exists an outer function k in H^1 such that*

$$\|1-k/(\phi W)\|_\infty<1.$$

Suppose $S_{\phi,(W)}$ is invertible. Let T be the operator defined in Lemma 2 with $Q=1$. Then $S_{\phi,(W)}{}^{-1}=J^{-1}T$. For all g in $\phi A+\bar{A}_0$,

$$S_{\phi,(W)}{}^{-1}g=(\alpha P_+ + \bar{\beta}P_-)(g/\bar{\beta}).$$

PROOF. We shall show that (1) implies (2). Since $S_{\phi,(W)}$ is invertible, by Theorem 2, there exists an inner function Q and a real function V in L^1 such that We^V is in (HS), and $\phi/|\phi|=Qe^{-i\tilde{V}}$. Since $S_{\phi,(W)}$ is invertible, there exists an f in $L^2((W))$ such that $S_{\phi,(W)}f=1$. Hence there exists an f_1 in $H^2(W)$ and an f_2 in $\bar{H}_0^2(W)$ such that $\phi f_1+f_2=1$. Then,

$$Qf_1(1-\bar{f}_2)e^{-i\tilde{V}-V}=|1-f_2|^2W/(|\phi|We^V)\geqq 0.$$

Since ϕ is invertible in L^∞ and We^V is in (HS), $(|\phi|We^V)^{-1}$ is in L^1. Since f_2 in $\bar{H}_0^2(W)$, $|1-f_2|^2W$ is in L^1. Hence the left hand side is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem, $Q=e^{ic}$ for some real constant c. Hence $\phi/|\phi|=e^{i(c-\tilde{V})}$. This implies (2). By Theorem 2 and it's proof with $Q=e^{ic}$, (2) implies (3). We shall show that (3) implies (1). By Lemma 1 and Lemma 2, it is sufficient to show that $R_{\phi,W}$ is right invertible. Let T be the operator defined in Lemma 2 with $Q=1$. By (3), $\log W$ is in L^1. Hence there exists an outer function h in H^2 such that $W=|h|^2$. Since $|\beta|^2W$ is in (HS), $(|\beta|^2W)^p$ is also in (HS) for some p, $p>1$. Hence $(|\beta|^2W)^{-p}$ is in L^1. For all f in $L^2(W)$,

$$\int_T |f/\bar{\beta}|^{2p/(p+1)}\,dm$$
$$\leqq\{\int_T |f|^2\,W\,dm\}^{p/(p+1)}\{\int_T (|\beta|^2W)^{-p}\,dm\}^{1/(p+1)}<\infty.$$

Since $2p/(p+1)>1$, by the Riesz theorem (cf. [13, p. 132]), $P_+(f/\bar{\beta})$ is in $H^{2p/(p+1)}$. Since $|\alpha|^2W$ is in (HS), by the Helson-Szegö theorem, there exists a constant γ such that for all f in $L^2(W)$,

$$\int_T |\alpha hP_+(f/\bar{\beta})|^2\,dm=\int_T |P_+(f/\bar{\beta})|^2|\alpha|^2\,W\,dm$$
$$\leqq\gamma\int_T |f/\bar{\beta}|^2|\alpha|^2\,W\,dm\leqq\gamma\|\phi^{-1}\|_\infty^2\int_T |f|^2\,W\,dm<\infty.$$

Hence $\alpha hP_+(f/\bar{\beta})$ is in H^2. Similarly, $\bar{\beta}\bar{h}P_-(f/\bar{\beta})$ is in $\bar{H}_0^2$. By the Beurling theorem (cf. [13, p. 110]), there exists a sequence g_n in A such that hg_n converges to $\alpha hP_+(f/\bar{\beta})$ in the norm of L^2. Hence g_n converges to $\alpha P_+(f/\bar{\beta})$ in the norm of $L^2(W)$. This implies $\alpha P_+(f/\bar{\beta})$ is in $H^2(W)$. Similarly, $\bar{\beta}P_-(f/\bar{\beta})$ is in $\bar{H}_0^2(W)$. Hence

$$R_{\phi,W}Tf=R_{\phi,W}\langle\alpha P_+(f/\bar{\beta}),\ \bar{\beta}P_-(f/\bar{\beta})\rangle$$
$$=\phi\alpha P_+(f/\bar{\beta})+\bar{\beta}P_-(f/\bar{\beta})=\bar{\beta}(P_++P_-)(f/\bar{\beta})=f.$$

Hence $T=R_{\phi,W}{}^{-1}$. We shall show that (2) implies (4). By (2), there exist u, v in L^∞ and a real constant c such that $\|v\|_\infty<\pi/2$, $We^V=e^{u+\tilde{v}}$ and $\phi/|\phi|=e^{i(c-\tilde{V})}$. Hence there exists a real constant c' such that

$$\phi/|\phi|=e^{i\{c'+V-(u-\log W)^\sim\}}.$$

Put $k=e^{ic'-(u-\log W)-i(u-\log W)^\sim}$, then k is an outer function. Since $|k|=We^{-u}$, k is in H^1. Put $\varepsilon=(\cos\|v\|_\infty)/\|\phi e^u\|_\infty$, then $\varepsilon>0$, since $\|v\|_\infty<\pi/2$. Put $\gamma=\|(\phi e^u)^{-1}\|_\infty$, then

$$\varepsilon\leqq(\cos v)/(|\phi|e^u)=\mathrm{Re}\{k/(\phi W)\}$$
$$\leqq|k|/|\phi W|=|\phi e^u|^{-1}\leqq\gamma.$$

This implies (let the reader make a diagram)

$$|\gamma^2/\varepsilon - k/(\phi W)| \leqq (\gamma/\varepsilon)(\gamma^2-\varepsilon^2)^{1/2}.$$

Put $k'=(\varepsilon/\gamma^2)k$, then k' is an outer function in H^1 such that

$$|1-k'/(\phi W)| \leqq \{1-(\varepsilon/\gamma)^2\}^{1/2}<1.$$

This implies (4). We shall show that (4) implies (2). By (4), $k/(\phi W)$ is invertible in L^∞. Since $\log|k|$ is in L^1, $\log W$ is in L^1. Since ϕ^{-1} is in L^∞, k/W is invertible in L^∞. Put $g=(k/W)e^{-i(\log W)^\sim}$, then g is invertible in H^∞. Hence there exists a real function u in L^∞ and a real constant c such that $g=e^{u+i(\tilde{u}+c)}$. Since

$$\|1(g/\phi)e^{i(\log W)^\sim}\|_\infty=\|1-k/(\phi W)\|_\infty<1,$$

there exists a real function v in L^∞ such that $\|v\|_\infty<\pi/2$, and $(g/\phi)e^{i(\log W)^\sim}=|g/\phi|e^{-iv}$. Put $V=\tilde{v}-u-\log W$, then $We^V=e^{\tilde{v}-u}$. Hence We^V is in (HS), and there exists a real constant c' such that $\phi/|\phi|=e^{i(c'-\tilde{V})}$. This completes the proof.

REMARK. (*a*) *Rochberg* [18] *showed that if* We^V *and* $We^{V'}$ *are in* (HS), *and* $e^{i(c-\tilde{V})}=e^{i(c'-\tilde{V}')}$, *then* $V-V'$ *is a constant.*
(*b*) *If* $|\alpha|^2W$, $|\beta|^2W$, $|\alpha'|^2W$ *and* $|\beta'|^2W$ *are in* (HS) *and* $\bar{\beta}/\alpha=\bar{\beta}'/\alpha'$, *then there exists a constant* c *such that* $\alpha'=c\alpha$ *and* $\beta'=\bar{c}\beta$, *since* α'/α, β'/β *and their complex conjugate functions are in* H^1, *and hence they are constants.*
(*c*) *If* W^{-1} *is in* L^1 *and* $S_{\phi,(W)}$ *is invertible, then* $S_{\phi,W}$ *and* $S_{-\phi,W}$ *have a dense range, and there exists a positive constant* δ *such that for all* f *in* $A+\bar{A}_0$,

$$\delta\|f\|_W \leqq \min\{\|S_\phi f\|_W,\ \|S_{-\phi}f\|_W\}.$$

COROLLARY 6. *Suppose* ϕ *is in* L^∞ *and* W *is a weight such that* W^{-1} *is in* L^1. *Then,* $S_{\phi,(W)}$ *is invertible if and only if* $S_{\phi,(W)}$ *and* $S_{\bar{\phi},(W^{-1})}$ *are left invertible.*

PROOF. Suppose $S_{\phi,(W)}$ and $S_{\bar{\phi},(W^{-1})}$ are left invertible. By Theorem 2, there exist inner functions Q, Q' and real functions V, V' in L^1 such that We^V, $W^{-1}e^{V'}$ are in (HS), and $\phi/|\phi|=Qe^{-i\tilde{V}}$, $\bar{\phi}/|\phi|=Q'e^{-i\tilde{V}'}$. Hence

$$QQ'e^{-(V+V')-i(V+V')^\sim}=e^{-(V+V')}\geqq 0.$$

Since $W^{-1}e^{-V}$, $We^{-V'}$ are in L^1, $e^{-(V+V')/2}$ is in L^1. By the Neuwirth-Newman theorem, Q and Q' are constants. By Theorem 3, $S_{\phi,(W)}$ is invertible. Suppose $S_{\phi,(W)}$ is invertible. By Theorem 3, there exists a real constant c and a real function V in L^1 such that We^V is in (HS), and $\phi/|\phi|=e^{i(c-\tilde{V})}$. Hence $W^{-1}e^{-V}$ is in (HS), and $\bar{\phi}/|\phi|=e^{i\{-c-(-\tilde{V})\}}$. By Theorem 2, this implies $S_{\bar{\phi},(W^{-1})}$ is left invertible. This completes the proof.

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