Hokkaido Mathematical Journal Vol. 21(1992) p. 483-489

A variant of a Yamaguchi's result

Catherine FINET and Valérie TARDIVEL-NACHEF
(Received November 5, 1991, Revised May 19, 1992)

Introduction

Yamaguchi extended the classical F. and M. Riesz theorem to a transformation group such that a compact abelian group acts on a locally compact Hausdorff space. In fact, he proved the following theorem.

THEOREM A. [5, Theorem 2. 4] *Let (G, X) be a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Let σ be a positive Radon measure on X that is quasi-invariant and let Λ be a Riesz set in $\hat{G}$. Let μ be a measure in $\mathscr{M}(X)$ with spec $\mu\subset\Lambda$. Then spec μ_a and spec μ_s are both contained in spec μ, where $\mu=\mu_a+\mu_s$ is the Lebesgue decomposition of μ with respect to σ.*

Let us recall that a subset Λ of $\hat{G}$ is a Riesz set if $\mathscr{M}_\Lambda(G)\subset L^1(G)$ (where $\mathscr{M}_\Lambda(G)$ denotes the space of measures in $\mathscr{M}(G)$ whose Fourier transforms vanish off Λ). With this terminology, the classical F. and M. Riesz theorem asserts that $\boldsymbol{N}$ is a Riesz subset of $\boldsymbol{Z}$.

Godefroy introduced and studied the notion of nicely placed and Shapiro sets [2]. Let us recall the definitions.

DEFINITION 1. [2]

1. A subset Λ of $\hat{G}$ is said nicely placed if the unit ball of $L^1_\Lambda(G)$ is closed in measure.

2. A subset Λ of $\hat{G}$ is said Shapiro if every subset of Λ is nicely placed.

The Alexandrov's example shows that there exists a Riesz subset Λ of $\boldsymbol{Z}$ which is not nicely placed [2]: take $\Lambda=\bigcup_{n=0}^{\infty}D_n$, where $D_n=\{k2^n, |k|\leq 2^n, k\neq 0\}$. On the other hand of course $\boldsymbol{Z}$ is nicely placed in $\boldsymbol{Z}$ but not Riesz in $\boldsymbol{Z}$.

But Godefroy proved that every Shapiro set is a Riesz set [2].

Our aim is to show that the conclusion of theorem A also holds for another class of subsets Λ of $\hat{G}$: the nicely placed subsets. More precise-

[0]This work was supported by CGRI-FNRS-CNRS-grant.

ly, we will prove the following theorem:

THEOREM 2. *Let (G, X) be a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space. Let σ be a positive Radon measure on X which is quasi-invariant and let Λ be a nicely placed subset of $\hat{G}$. If μ is in $\mathscr{M}(X)$ with spec μ contained in Λ then spec μ_a and spec μ_s are both contained in Λ (where $\mu=\mu_a+\mu_s$ is the Lebesgue decomposition of μ with respect to σ).*

This result is better than the one of [1].
The proof follows Yamaguchi's ideas [4], [5], see also [6].

Preliminaries and notation

In what follows G will be a compact abelian group and X a locally compact Hausdorff space. We say that (G, X) is a transformation group if there exists a continuous map from $G\times X$ onto $X:(g, x)\to g\cdot x$ such that:

$$e\cdot x=x,\ g\cdot(h\cdot x)=(g\cdot h)\cdot x \tag{1}$$

for g, h in G and x in X.
Let us remark that for $g\in G$, the map $\theta_g: X\to X$ defined by $\theta_g(x)=g\cdot x$ is a homeomorphism (This directly comes from (1)).
A trivial example is (G, X) where G is a compact abelian subgroup of a locally compact group X.

A Borel measure σ on X is called quasi-invariant if $|\sigma|(F)=0$(F is a Borel subset of X) implies $|\sigma|(g\cdot F)=0$, for all g in G.

We will denote by $\mathscr{M}(X)$ the space of regular bounded Borel measures on X and by $K(X)$ the space of continuous functions on X with compact support. We denote by $\hat{\mu}$ the Fourier transform of μ. For a closed subgroup H of G, $H^\perp$ is the annihilator of H. The usual notion of convolution can be generalized in the following way [3], [4]:
For μ in $\mathscr{M}(X)$ and λ in $\mathscr{M}(G)$, we define $\lambda\star\mu$ in $\mathscr{M}(X)$ by:

$$(\lambda\star\mu)(f)=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu(x), \text{ for } f\in K(X).$$

We can now define the spectrum of a measure μ in $\mathscr{M}(X)$ [4]: let $J(\mu)$ be the set of all f in $L^1(G)$ with $f\star\mu=0$.
The spectrum of μ (denoted by spec μ) is $\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0)$.
We have that $s\in$spec μ if and only if $(sm_G)\star\mu\neq0$ [4] (m_G is the Haar measure on G).

Let $\pi : X \to X/G$ be the canonical map.
Yamaguchi introduced in [4] the conditions called (D. I) and (D. II).

(D. I) For any $\mu \in \mathscr{M}^+(X)$, put $\eta = \pi(\mu)$. Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $\mathscr{M}^+(X)$ with the following properties:

1. $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X,
2. $\|\lambda_{\dot{x}}\| = 1$,
3. supp $(\lambda_{\dot{x}}) \subset \pi^{-1}(\dot{x})$,
4. $\mu(f) = \int_{X/G} \lambda_{\dot{x}}(f) d\eta(\dot{x})$, for each bounded Baire function f on X.

(D. II) Let $\nu \in \mathscr{M}^+(X/G)$. Suppose $\{\lambda^i_{\dot{x}}\}_{\dot{x} \in X/G}$ $(i=1, 2)$ are families of measures in $\mathscr{M}(X)$ with the following properties:

1. $\dot{x} \to \lambda^i_{\dot{x}}(f)$ is a ν-integrable function for each bounded Baire function f on X $(i=1, 2)$,
2. supp $(\lambda^i_{\dot{x}}) \subset \pi^{-1}(\dot{x})$ $(i=1, 2)$,
3. $\int_{X/G} \lambda^1_{\dot{x}}(f) d\nu(\dot{x}) = \int_{X/G} \lambda^2_{\dot{x}}(f) d\nu(\dot{x})$ for all bounded Baire functions f on X.

   Then we have $\lambda^1_{\dot{x}} = \lambda^2_{\dot{x}}$ ν-*a.a.* $\dot{x} \in X/G$.

Let $\mu \in \mathscr{M}(X)$ and $\eta \in \mathscr{M}^+(X/G)$. An <u>$\eta$-disintegration</u> of μ is a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $\mathscr{M}(X)$ satisfying (1)': $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X and (3)-(4) in (D. I). If in addition, $\eta = \pi(|\mu|)$ and $\|\lambda_{\dot{x}}\| = 1$ for all $\dot{x} \in X/G$, then following [4] we call $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ a canonical disintegration of μ.
a canonical disintegration of μ.
For $x \in X$, we put $G_x = \{g \in G : g \cdot x = x\}$, we define the map $B_x : G \to G \cdot x$ by $B_x(g) = g \cdot x$, we put $\dot{x} = \pi(x)$ and $m_{\dot{x}} = B_x(m_G)$. We also consider the map $\tilde{B}_x : G/G_x \to G \cdot x$ defined by: $\tilde{B}_x(g + G_x) = g \cdot x$.

LEMMA 3. *Let H be a closed subgroup of G, and Λ be a nicely placed subset of $\hat{G}$. Let (f_n) be a bounded sequence in $L^1_{\Lambda \cap H^\perp}(G/H)$ such that (f_n) converges to f in $m_{G/H}$- measure. Then spec f is contained in $\Lambda \cap H^\perp$. In particular, $\Lambda \cap H^\perp$ is a nicely placed subset of $H^\perp$.*

PROOF. Let $q : G \to G/H$ be the quotient map. We have that $(f_n \circ q)$ is a bounded sequence in $L^1(G)$. Moreover,

$$(f_n \circ q)^\wedge(\gamma) = \begin{cases} \hat{f}_n(\gamma) & \text{if } \gamma \in (G/H)^\wedge = H^\perp \\ 0 & \text{if } \gamma \in \hat{G} \setminus H^\perp. \end{cases}$$

Thus, $\operatorname{spec}(f_n \circ q) \subset \Lambda \cap H^{\perp}$.
Moreover $(f_n \circ q)$ converges to $f \circ q$ in m_G-measure. Then $\operatorname{spec}(f \circ q)$ is contained in Λ and spec f is contained in $\Lambda \cap H^{\perp}$.

■

LEMMA 4. *Let G be metrizable and σ be a measure in $\mathscr{M}^{+}(X)$ which is quasi-invariant. If (G, X) satisfies conditions* (D. I) *and* (D. II), *then the conclusion of theorem* 2 *holds.*

PROOF. Let μ be in $\mathscr{M}(X)$ with spec μ contained in Λ. By [4, Lemma 2.7] we may assume that $\eta = \pi(|\mu|) << \pi\ (\sigma)$. Let $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ be a canonical disintegration of $|\mu|$. Let h be an unimodular Baire function on X with $\mu = h|\mu|$.
We define $\mu_{\dot{x}} \in \mathscr{M}(X)$ by $\mu_{\dot{x}} = h\lambda_{\dot{x}}$. Then $\{\mu_{\dot{x}}\}_{\dot{x} \in X/G}$ is a canonical disintegration of μ. For each $\dot{x} \in X/G$, let $\lambda_{\dot{x}} = \lambda_{\dot{x}}^{a} + \lambda_{\dot{x}}^{s}$ and $\mu_{\dot{x}} = \mu_{\dot{x}}^{a} + \mu_{\dot{x}}^{s}$ be the Lebesgue decomposition of $\lambda_{\dot{x}}$ and $\mu_{\dot{x}}$ with respect to $m_{\dot{x}}$. Then $\mu_{\dot{x}}^{a} = h\ \lambda_{\dot{x}}^{a}$ and $\mu_{\dot{x}}^{s} = h\ \lambda_{\dot{x}}^{s}$. Since spec $\mu \subset \Lambda$, [4, Lemma 2.6] implies that

$$\operatorname{spec} \mu_{\dot{x}} \subset \Lambda \qquad \eta\text{-a.a.}\ \dot{x} \in X/G. \tag{2}$$

Let $x \in \pi^{-1}(\dot{x})$ and $\xi_{\dot{x}} \in \mathscr{M}(G/G_x)$ such that $\widetilde{B}_x(\xi_{\dot{x}}) = \mu_{\dot{x}}$. Then by (2) and [4, Proposition 1.2] $\xi_{\dot{x}} \in \mathscr{M}_{\Lambda \cap G_{\dot{x}}^{\perp}}(G/G_x)\ \eta-\text{a.a.}\ \dot{x} \in X/G$.
Let $\xi_{\dot{x}} = \xi_{\dot{x}}^{a} + \xi_{\dot{x}}^{s}$ be the Lebesgue decomposition of $\xi_{\dot{x}}$ with respect to m_{G/G_x}. As G/G_x is a metrizable compact abelian group, there exists an identity approximation (f_n) in the unit ball of $L^1(G/G_x)$ such that

$$\begin{cases} f_n \star \xi_{\dot{x}}^{a} \to \xi_{\dot{x}}^{a} \text{ in } L^1\text{-norm} \\ f_n \star \xi_{\dot{x}}^{s} \to 0 \text{ in } m_{G/G_x}\text{-measure (see [2]).} \end{cases}$$

Then,
$f_n \star \xi_{\dot{x}} \to \xi_{\dot{x}}^{a}$ in m_{G/G_x}-measure and spec $f_n \star \xi_{\dot{x}}$ is contained in $\Lambda \cap G_{\dot{x}}^{\perp}$ η-a.a. $\dot{x} \in X/G$. From Lemma 3 it follows that spec $\xi_{\dot{x}}^{a}$ is also contained in $\Lambda \cap G_{\dot{x}}^{\perp}$ η-a.a. $\dot{x} \in X/G$.
By [4, Propositions 1.5 and 1.2] it follows that

$$\operatorname{spec} \mu_{\dot{x}}^{a} \subset \Lambda \quad \eta\text{-a.a.}\ \dot{x} \in X/G. \tag{3}$$

By [4, Lemma 2.8], the functions:

$$\dot{x} \to \lambda_{\dot{x}}^{a}(f), \quad \dot{x} \to \lambda_{\dot{x}}^{s}(f)$$

are η-measurable for each bounded Baire function f on X. Hence, the functions

$$\dot{x} \to \mu_{\dot{x}}^{a}(f), \quad \dot{x} \to \mu_{\dot{x}}^{s}(f)$$

are η-measurable for each bounded Baire function f on X.
We can then define $\omega_1,\omega_2\in\mathscr{M}^+(X)$ and $\mu_1,\mu_2\in\mathscr{M}(X)$ as follows:

$$\omega_1(f)=\int_{X/G}\lambda_{\dot{x}}^a(f)d\eta(\dot{x}), \qquad \omega_2(f)=\int_{X/G}\lambda_{\dot{x}}^s(f)d\eta(\dot{x})$$
$$\mu_1(f)=\int_{X/G}\mu_{\dot{x}}^a(f)d\eta(\dot{x}), \qquad \mu_2(f)=\int_{X/G}\mu_{\dot{x}}^s(f)d\eta(\dot{x})$$

for $f\in K(X)$.
One has $\mu_1<<\omega_1$ and $\mu_2<<\omega_2$.
By [4, Lemma 2.5] it follows that

$$\omega_1<<\sigma, \quad \omega_2\perp\sigma.$$

And also,

$$\mu_1<<\sigma, \quad \mu_2\perp\sigma.$$

Since $\mu=\mu_1+\mu_2$, one has: $\mu_1=\mu_a$. Let $\gamma\notin\Lambda$ and $f\in K(X)$, then,

$$\gamma\star\mu_a(f)=\gamma\star\mu_1(f)=\int_{X/G}\gamma\star\mu_{\dot{x}}^a(f)d\eta(\dot{x})$$
$$=0. \quad \text{(by (3))}$$

Hence spec $\mu_a\subset\Lambda$.

■

In [4], Yamaguchi introduced conditions (C. I) and (C. II).

(C. I) For any closed subgroup H of G with $H^\perp$ countable and any $\mu\in\mathscr{M}^+(X/H)$, put $\eta=\pi(\mu)$, where $\pi: X/H\rightarrow Y=X/H/G/H(\cong X/G)$ is the canonical map. Then there exists a family $\{\lambda_y\}_{y\in Y}$ of measures in $\mathscr{M}^+(X/H)$ with the following properties:

1. $y\rightarrow\lambda_y(f)$ is η-measurable for each bounded Baire function f on X/H,
2. $\|\lambda_y\|=1$,
3. $\text{supp}(\lambda_y)\subset\pi^{-1}(y)$,
4. $\mu(f)=\int_Y\lambda_y(f)d\eta(y)$ for each bounded Baire function f on X/H.

(C. II) Let H be any closed subgroup of G with $H^\perp$ countable. Let Y and π be as in (C. I). Let $\eta\in\mathscr{M}^+(Y)$, and let $\{\lambda_y^1\}_{y\in Y}$ and $\{\lambda_y^2\}_{y\in Y}$ be families of measures in $\mathscr{M}(X/H)$ satisfying the following properties:

1. $y\rightarrow\lambda_y^i(f)$ is η-integrable for each bounded Baire function f on $X/H(i=1,2)$,
2. $\text{supp}(\lambda_y^i)\subset\pi^{-1}(y)$ $(i=1,2)$,

3. $\int_Y \lambda_y^1(f) d\eta(y) = \int_Y \lambda_y^2(f) d\eta(y)$ for all bounded Baire functions f on X/H.

Then $\lambda_y^1 = \lambda_y^2$ η-a.a. $y \in Y$.

REMARK. [4] When X is a metric space then (G, X) satisfies (C. I) and (C. II).

LEMMA 5. *Assume that* (G, X) *satisfies conditions* (C. I) *and* (C. II) *then the conclusion of theorem* 2 *holds.*

PROOF. In fact, we may suppose that σ is a measure in $\mathscr{M}^+(X)$ that is quasi-invariant [4].
We will prove that spec $\mu_s \subset \Lambda$. We may assume that $\mu_s \neq 0$. Suppose there exists $\gamma_0 \in (\text{spec } \mu_s) \setminus \Lambda$. Then $\gamma_0 \star \mu_s \neq 0$. By [4, Lemmas 2. 11 and 2. 13] there exists a countable subgroup Γ of $\widehat{G}$ with $\gamma_0 \in \Gamma$ such that

$$\begin{cases} \pi(\gamma_0 \star \mu_s) \neq 0, \\ \pi(|\mu_s|) \perp \pi(\sigma), \end{cases}$$

where $\pi : X \to X/\Gamma^{\perp}$ is the quotient map.
Then $\pi(\mu_s)$ is the singular part of $\pi(\mu)$ with respect to $\pi(\sigma)$. The measure $\pi(\sigma)$ is also quasi-invariant. The group Γ is countable and $(G/\Gamma^{\perp}, X/\Gamma^{\perp})$ satisfies conditions (D. I) and (D. II). Since spec $\pi(\mu) \subset \Gamma \cap \Lambda$(cf. [4, Lemma 2. 10]), Lemmas 3 and 4 imply spec $\pi(\mu_s) \subset \Gamma \cap \Lambda$. By [4, Lemma 2. 9],

$$\begin{aligned} \pi(\gamma_0 \star \mu_s) &= q(\gamma_0) \star \pi(\mu_s) \\ &= \gamma_0 \star \pi(\mu_s), \end{aligned}$$

where $q : G \to G/\Gamma^{\perp}$ is the canonical map and $\gamma_0 \in$ spec $\pi(\mu_s)$; hence $\gamma_0 \in \Gamma \cap \Lambda$.
This gives the contradiction.

■

PROOF OF THEOREM 2. We may suppose that X is a σ-compact locally compact Hausdorff space and σ is a quasi -invariant measure in $\mathscr{M}^+(X)$.
We will prove that spec $\mu_s \subset \Lambda$. We may suppose that $\mu_s \neq 0$. Suppose there exists $\gamma_0 \in (\text{spec } \mu_s) \setminus \Lambda$. Then $\gamma_0 \star \mu_s \neq 0$.
By [5, Lemma 3. 1], there exists an equivalence relation "$\sim$" on X such that:

(1) $X/\sim$ is a (σ-compact) metrizable, locally compact Hausdorff space with respect to the quotient topology;

(2) $(G, X/\sim)$ becomes a transformation group by the action $g\cdot\tau(x)=\tau(g\cdot x)$ for $g\in G$ and $x\in X$, where $\tau: X\rightarrow X/\sim$ is the canonical map;

(3) $\tau(\gamma_0\star\mu_s)\neq 0$;

(4) $\tau(|\mu_s|)\perp\tau(\sigma)$.

By (4) $\tau(\mu_s)$ is the singular part of $\tau(\mu)$.
By [5, Lemma 2.2], spec $\tau(\mu)\subset$spec $\mu\subset\Lambda$.
$\tau(\sigma)$ is a quasi-invariant measure in $\mathscr{M}^+(X/\sim)$ and since $X/\sim$ is metrizable, $(G, X/\sim)$ satisfies conditions (C. I) and (C. II).
Lemma 5 implies that spec $\tau(\mu_s)\subset\Lambda$.
But

$$\begin{aligned}\tau(\gamma_0\star\mu_s)&=\gamma_0\star\tau(\mu_s)\\&\neq 0.\quad\text{(by (3))}\end{aligned}$$

And $\gamma_0\in$spec $\tau(\mu_s)\subset\Lambda$. This gives the contradiction.

■

References

[1] C. FINET, V. TARDIVEL, Lacunary sets on transformation groups. Preprint.

[2] G. GODEFROY, On Riesz subsets of abelian discrete groups. Isr. J. Math. 61 (1988), 301-333.

[3] S. L. GULICK, P. S. LIN, A. C. VAN ROOIJ, Group algebra modules II. Canad. J. Maths. 19(1967), 151-173.

[4] H. YAMAGUCHI, The F. and M. Riesz theorem on certain transformation groups. Hokkaido Math. J. 17 (1988), 289-32.

[5] H. YAMAGUCHI, The F. and M. Riesz theorem on certain transformation groups II. Hokkaido Math. J. 19 (1990), 345-359.

[6] H. YAMAGUCHI, A property of spectrums of measures on certain transformation groups. Hokkaido Math. J. 20 (1991), 109-121.

Université de Mons-Hainaut
Institut de Mathématique et d'Informatique
15, Avenue Maistriau, 7000 MONS-BELGIQUE
Université Paris 7

Université Paris 6, Equipe d'Analyse
U. A. 754 au C.N.R.S.
Tour 46-00, 4ème étage, 4, place Jussieu
75252 PARIS CEDEX 05-FRANCE
Université Catholique de l'Ouest