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Existence results for singular elliptic equations

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The purpose of this paper is to study the existence of positive solutions in $\boldsymbol{R}_n$ of the singular elliptic equation

$$(1) \qquad Lu=-\sum_{i,j=1}^{n}D_i(a_{ij}(x)D_ju)+c(x)u=g(x,u),$$

where the nonlinearity g is defined on $\boldsymbol{R}_n\times(0,\infty)$. Solutions of (1), which are defined on $\boldsymbol{R}_n$, are called entire solutions. The precise conditions on g, to be formulated later, show that equation (1) is a natural extension of the following equation

$$(1') \qquad -\Delta u=f(x)u^{-\gamma} \text{ in } \boldsymbol{R}_n,$$

where $\gamma>0$ is a constant. The equation (1′) is called in the existing literature the Lane-Emden-Fowler equation and arises in the boundary-layer theory of viscous fluids (see [4], [5], [6], [8] and the references given there). In papers [4] and [8] it is assumed that $f(x)$ depends "almost" radially on x in the sense that

$$c_1p(|x|)\leqq f(x)\leqq c_2p(|x|),$$

where $c_1>0$ and $c_2>0$ are constants and $p(|x|)$ is a positive function satisfying some integrability condition. The existence results are then obtained using the method of sub and supersolutions. In [5] the existence of positive solutions was obtained by replacing (1′) with an equivalent operator equation which can be solved using the Schauder-Tichonov fixed point theorem. In this paper we develop ideas from paper [1], where the existence of weak solutions, in the case $g(x,u)=f(x)u^{-\gamma}$, $0<\gamma<\infty$, has been considered. Here we consider more general nonlinearities g. Our method in based on approximation arguments. We first solve the Dirichlet problem in a bounded domain with zero boundary data. An entire solution is then obtained as a limit of solutions u_m of the Dirichlet problems on Ω_m, with $\{\Omega_m\}$ exhausting $\boldsymbol{R}_n$. The assumptions (g_1) and (g_2) ensure that solutions of the Dirichlet problem in a bounded domain Ω belong to $W_{\mathrm{loc}}^{1,2}(\Omega)\cap C(\bar{\Omega})$. We also point out that under some additional assumption a solution u is in $\overset{\circ}{W}^{1,2}(\Omega)$. Throughout this paper we assume that $n \geqq 3$ and we extensively use the Sobolev inequality

$$\|u\|_{\frac{2n}{n-2}} \leqq S\|Du\|_2,$$

which is true for any u in $\overset{\circ}{W}^{1,2}(\Omega)$ and for an arbitrary domain $\Omega \subset \boldsymbol{R}_n$ with the constant $S>0$ depending only on n. The case $n=2$ can be treated in a similar way, with suitable modifications, due to the fact that in this case the Sobolev inequality remains true with $\|\cdot\|_{\frac{2n}{n-2}}$ replaced with $\|\cdot\|_p$, $1<p<\infty$, and with $\|Du\|_2$ replaced by $\|u\|_{W^{1,2}}$. However, we do not consider this case here. Some results concerning the case $n=1$ can be found in [6].

Finally, I would like to express my gratitude to Professor A. M. Fink for his interest in this research and bringing my attention to the paper [6]

1. The Dirichlet problem in a bounded domain.

We commence by studying the Dirichlet problem

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju) + c(x)u = g(x, u) \text{ in } \Omega, \tag{2}$$

$$u(x) = 0 \text{ on } \partial\Omega, \tag{3}$$

where $\Omega \subset \boldsymbol{R}_n$ is a bounded domain with the boundary $\partial\Omega$ satisfying the exterior cone condition.

Throughout this section we make the following assumptions

(A) There exists a constant $\lambda>0$ such that

$$\lambda|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

for all $x \in \boldsymbol{R}^n$ and $\xi \in \boldsymbol{R}_n$. Moreover, we assume that a_{ij} $(i, j=1, \cdots, n)$ and c are in $L^\infty(\Omega)$ with $c(x) \geqq 0$ on Ω.

The function g: $\Omega \times (0, \infty) \to (0, \infty)$ is a Carathéodory function, that is, $g(\cdot, u)$ is a measurable function for each $u \in (0, \infty)$ and $g(x, \cdot)$ is continuous on $(0, \infty)$ for a.e. $x \in \Omega$.

Further, we impose the following two conditions on g:

(g_1) for each $a>0$ there exists $f_a \in L^p(\Omega)$, with $p>n$, such that

$$g(x, u) \leqq f_a(x) \text{ on } \Omega \text{ for all } a \leqq u < \infty,$$

(g_2) the function $g(x, \cdot)$: $(0, \infty) \to (0, \infty)$ is nonincreasing for a. e. $x \in \Omega$.

THEOREM 1. *The Dirichlet problem* (2), (3) *admits a positive solution* $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$.

PROOF: Let $\varepsilon>0$ and consider the Dirichlet problem for the equation

$$(2\varepsilon) \qquad Lu=g(x, |u|+\varepsilon) \text{ in } \Omega,$$

with the boundary condition (3). By the Schauder fixed point theorem and the Sobolev-Rellich embedding theorem, the problem (2ε), (3) has a unique solution u_ε in $\overset{\circ}{W}{}^{1,2}(\bar{\Omega})$, which by (g_1) belongs to $C(\bar{\Omega})$ (see Theorem 8.30 in [6]). The uniqueness follows from the assumption (g_2). It follows from the mximum principle that $u_\varepsilon>0$ on Ω. We now show that $\{u_\varepsilon\}$ is an increasing sequence as $\varepsilon\searrow 0$. Let $0<\varepsilon_1<\varepsilon_2$. Taking $(u_{\varepsilon_1}-u_{\varepsilon_2})_+$ as a test function, we obtain on substitution

$$\int_\Omega\left[\sum_{i,j=1}^n a_{ij}(x)D_i(u_{\varepsilon_2}-u_{\varepsilon_1})_+D_j(u_{\varepsilon_2}-u_{\varepsilon_1})_+ + c(x)(u_{\varepsilon_2}-u_{\varepsilon_1})_+^2\right]dx$$
$$=\int_\Omega(g(x, u_{\varepsilon_2}+\varepsilon_2)-g(x, u_{\varepsilon_1}+\varepsilon_1))(u_{\varepsilon_2}-u_{\varepsilon_1})_+dx.$$

Since by the condition (g_2), $g(x, u_{\varepsilon_2}+\varepsilon_2)-g(x, u_{\varepsilon_1}+\varepsilon_1)\leqq 0$ a.e. on the set $\{u_{\varepsilon_2}>u_{\varepsilon_1}\}$, we deduce using the ellipticity condition that

$$\int_\Omega|D(u_{\varepsilon_2}-u_{\varepsilon_1})_+|^2\,dx\leqq 0$$

and consequently $u_{\varepsilon_2}\leqq u_{\varepsilon_1}$ a.e. on Ω. In the next step of the proof we show that the sequence $\{u_\varepsilon+\varepsilon\}$ is decreasing as $\varepsilon\searrow 0$. Let $\varepsilon_1>\varepsilon_2$ and since $u_{\varepsilon_1}-u_{\varepsilon_2}=0$ on $\partial\Omega$, $(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-\in\overset{\circ}{W}{}^{1,2}(\Omega)$ and on substitution we obtain

$$-\int_\Omega\left[\sum_{i,j=1}^n a_{ij}(x)D_i(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-D_j(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-\right.$$
$$\left.+c(x)(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-^2\right]dx$$
$$=\int_\Omega(g(x, u_{\varepsilon_1}+\varepsilon_1)-g\ (x, u_{\varepsilon_2}+\varepsilon_2))(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-dx$$
$$+\int_\Omega c(x)(\varepsilon_1-\varepsilon_2)(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-dx.$$

It is easy to see that the right hand side is nonnegative and, as before, we conclude that $|D(u_{\varepsilon_1}+\varepsilon_1-u_{\varepsilon_2}-\varepsilon_2)_-|=0$ a.e. on Ω, that is $u_{\varepsilon_1}+\varepsilon_1\geqq u_{\varepsilon_2}+\varepsilon_2$ a.e. on Ω. From these two claims we see that if $0<\varepsilon<\delta$, Then

$$0<u_\varepsilon-u_\delta<\delta-\varepsilon \text{ on } \bar{\Omega}.$$

This means that there exists $u\in C(\bar{\Omega})$ such that $\lim_{\delta\to 0}u_\delta=u$ uniformly on $\bar{\Omega}$. We now show that $u\in W^{1,2}_{loc}(\Omega)$ and satisfies (2) in the distributional sense. Let $B(x_0, r)$ be a ball with a center at x_0 of radius r and assume that $\overline{B(x_0, 2r)} \subset \Omega$. Let Φ be a function in $C^1(\boldsymbol{R}_n)$ such that $\Phi(x)=1$ on $B(x_0, r)$, $\Phi(x)=0$ on $\boldsymbol{R}_n - B(x_0, 2r)$ and $0<\Phi(x)\leqq 1$ on $\boldsymbol{R}_n$. Taking $u_\varepsilon\Phi^2$ as a test function we obtain on substitution

$$(4) \qquad \int_\Omega \Big[\sum_{ij=1}^n a_{ij} D_i u_\varepsilon D_i u_\varepsilon D_j u_\varepsilon \Phi^2 + 2\sum_{ij=1}^n a_{ij} D_j u_\varepsilon u_\varepsilon \Phi D_i \Phi + c u_\varepsilon^2 \Phi^2 \Big] dx = \int_\Omega g(x, u_\varepsilon + \varepsilon) u_\varepsilon \Phi^2 dx.$$

Since u_ε is positive and increases to u as $\varepsilon \searrow 0$, we may assume that there exist constants $\varepsilon_0 > 0$, $a>0$ and $A>0$ such that

$$a \leqq u_\varepsilon \leqq A \text{ on supp } \Phi,$$

for all $0<\varepsilon\leqq\varepsilon_0$. Using the Young inequality and the assumptions (A) and (g_1) we easily derive from (4) that

$$(5) \qquad \int_{B(x_0,r)} |Du_\varepsilon|^2 dx \leqq C\Big[\int_{B(x_0,2r)} f_a u_\varepsilon dx + \int_{B(x_0,2r)} u_\varepsilon^2 dx\Big] \leqq C\int_{B(x_0,2r)} (f_a A + A^2)\, dx,$$

where $C>0$ is a constant independent of ε. The inequality (5) shows that $\{u_\varepsilon\}$ is bounded in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Finally, the Sobolev-Rellich embedding theorem, applied on each compact subset K of Ω, shows that u satisfies (2) in the distributional sense.

REMARK 1. The assumptions (g_1) and (g_2) are satisfied in each following example:

(a) $g(x, u) = f(x)u^{-r}$, with $0<r<\infty$,

(b) $g(x, u) = \dfrac{f(x)}{\ln(1+u)}$,

(c) $g(x, u) = \dfrac{f(x)}{\sin\left(\dfrac{u}{1+u}\dfrac{\pi}{2}\right)}$,

(d) $g(x, u) = f(x) \exp \dfrac{1}{u}$,

(e) $g(x, u) = \dfrac{f(x)}{(|x|^2+u)^{\gamma}}$, with $0<\gamma<\infty$,

where $f>0$ on Ω and $f\in L^p(\Omega)$, *with* $p>n$.

The functions g from the examples (a)–(d) have the property that $\lim_{u\to+0} g(x, u) = \infty$ for all $x\in\Omega$. Theorem 1 is related to the results of papers [2-3], where the existence of classical solutions have been investigated under a different set of assumptions including $\lim_{u\to 0} g(x, u)=\infty$

uniformly on Ω.

We now impose an additional assumption on g guaranteeing that $u\in W^{1,2}(\Omega)$.

THEOREM 2. *Suppose that there exist constants $b>0$ and $0<\alpha\leqq 1$ and a function $f\in L^1(\Omega)$ such that*

(6) $$g(x, u)u^\alpha\leqq f(x)$$

for all $u\in(0, b]$ and a. e. on Ω. Then the solution u of the problem (2), (3) *belongs to $\mathring{W}^{1,2}(\Omega)\cap C(\overline{\Omega})$.*

PROOF: The proof is straightforward. Let u_ε be a solution of the problem (2ε), (3). Taking u_ε as a test function, we get on substitution

$$\begin{aligned}\int_\Omega\Big[\sum_{i,j=1}^n a_{ij}D_iu_\varepsilon D_ju_\varepsilon+cu_\varepsilon^2\Big]dx&=\int_\Omega g(x, u_\varepsilon+\varepsilon)u_\varepsilon dx\\ &\leqq\int_{0<u_\varepsilon<b} fu_\varepsilon^{1-\alpha}dx+\int_{b<u_\varepsilon} f_bu_\varepsilon dx\\ &\leqq\int_\Omega fA^{1-\alpha}dx+\int_\Omega f_bA\,dx,\end{aligned}$$

where $A=\sup_{\varepsilon>0, x\in\Omega}u_\varepsilon(x)$. This inequality, together with the ellipticity condition (A), yields that $\{u_\varepsilon\}$ is bounded in $W^{1,2}(\Omega)$ and the result follows.

REMARK 2. The condition (6) is obviously satisfied in examples (b),(c) and (a), (e) with $0<r\leqq 1$. If (6) holds with $\alpha>1$, one can easily show that

$$\int_\Omega|Du(x)|^2u(x)^{\alpha-1}dx<\infty \text{ and } u^{\frac{1+\alpha}{2}}\in\mathring{W}^{1,2}(\Omega)$$

This result has been obtained in the paper [1].

In the next result we briefly examine the behaviour of Du near the boundary. Let $r(x)=\mathrm{dist}(x, \partial\Omega)$ for $x\in\Omega$.

THEOREM 3. *Suppose that $\partial\Omega$ is of class C^2 and that there exist constants $s>0$, $0<\alpha\leqq 1$, $b>0$ and a function $f>0$, with $r^{1+s}f\in L^1(\Omega)$ such that*

(7) $$g(x, u)u^\alpha\leqq f(x) \text{ for } 0<u\leqq b \text{ and a. e. } x\in\Omega.$$

Then the solution u of (2), (3) *has the property*

$$\int_\Omega|Du(x)|^2r(x)^{1+s}dx<\infty.$$

PROOF: It follows from the regularity of $\partial\Omega$ that there exists δ_0 such that $\partial\Omega_\delta$ is of class C^2 for $\delta\in(0, \delta_0]$, where $\Omega_\delta=\{x\in\Omega\,;\ \operatorname{dist}(x, \partial\Omega)>\delta\}$ (see [6] Lemma 14.16, p. 355). We now define a function $\rho\in C^2(\Omega)$ such that $\rho(x)=r(x)$ on $\Omega-\Omega_{\delta_0}$ and $c_1 r(x)\leqq\rho(x)\leqq c_2 r(x)$ on $\overline{\Omega}$ for some constants $c_1>0$ and $c_2>0$. Let $0<\delta<\delta_0$ and set

$$v(x)=\begin{cases} u(x)(\rho(x)-\delta)^{1+s} & \text{for } x\in\Omega_\delta, \\ 0 & \text{for } x\in\Omega-\Omega_\delta. \end{cases}$$

It is clear that $v\in \overset{\circ}{W}{}^{1,2}(\Omega)$ and taking v as a test function we obtain

(8)

$$\int_{\Omega_\delta}\sum_{i,j=1}^n a_{ij}(x)D_iuD_ju(\rho-\delta)^{1+s}dx+(1+s)\int_{\Omega_\delta}\sum_{i,j=1}^n a_{ij}(x)D_iuuD_j\rho(\rho-\delta)^s dx$$

$$+\int_{\Omega_\delta}c(x)u^2(\rho-\delta)^{1+s}dx\leqq A^{1-\alpha}\int_{u<b}f(x)(\rho-\delta)^{1+s}dx$$

$$+A\int_{u\geqq b}f_b(x)(\rho-\delta)^{1+s}dx,$$

where $A=\max_{\bar{\Omega}}u(x)$. We now observe that by the Young inequality we have

$$(1+s)\int_{\Omega_\delta}\sum_{i,j=1}^n a_{ij}(x)D_iuuD_j\rho(\rho-\delta)^s dx\leqq\frac{\lambda}{2}\int_{\Omega_\delta}|Du(x)|^2(\rho-\delta)^{s+1}dx$$

$$+C\int_{\Omega_\delta}u(x)^2(\rho-\delta)^{s-1}dx,$$

where $C>0$ is a constant depending on $\|a_{ij}\|_{L_\infty}$, s and λ. Hence

$$\frac{\lambda}{2}\int_{\Omega_\delta}|Du(x)|^2(\rho-\delta)^{1+s}dx\leqq A^{1-\alpha}\int_\Omega f(x)\rho^{1+s}dx+A\int_\Omega f_b(x)\rho^{1+s}dx$$

$$+CA^2\int_\Omega \rho^{s-1}dx.$$

Since $\int_\Omega \rho^{s-1}\,dx<\infty$, the result follows from the Lebesgue Monotone Convergence Theorem.

REMAMK 3. The assertion of Theorem 3 can be slightly improved if $a_{ij}\in C^1(\overline{\Omega}-\Omega_{\delta_0})$ and the condition (7) holds with $s=0$. Then the solution u of the problem (2), (3) has the property

$$\int_\Omega |Du(x)|^2 r(x)\,dx<\infty.$$

Indeed, using a truncation we may assume that $a_{ij}\in C^1(\overline{\Omega})$. Then we proceed as in the proof of Theorem 3. Taking

$$v(x)=\begin{cases} u(x)(\rho(x)-\delta) & \text{for } x\in\Omega_\delta \\ 0 & \text{for } x\in\Omega-\Omega_\delta, \end{cases}$$

with $0<\delta<\delta_0$, as a test function we arrive at the relation (8) with s=0. The second integral can be handled by integration by parts

$$\int_{\Omega_\delta}\sum_{ij=1}^n a_{ij}D_j u u D_i\rho dx=\frac{1}{2}\int_{\Omega_\delta}\sum_{ij=1}^n a_{ij}D_j(u^2)D_i\rho dx=$$

$$+\frac{1}{2}\int_{\partial\Omega_\delta}\sum_{ij=1}^n a_{ij}u^2 D_i\rho D_j\rho|D\rho|^{-1}dS_x-\frac{1}{2}\int_{\Omega_\delta}u^2\sum_{ij=1}^n D_j(a_{ij}D_i\rho)dx,$$

which shows that this integral is bounded independently of δ.

2. Entire solutions of (1).

We now use the results of Section 1 to obtain the existence of positive solutions of (1).

We assume that the hypothesis (A) holds on $\boldsymbol{R}_n$ and that the nonlinearity g satisfies the Carathéodory condition and moreover

(g_1') For each $a>0$ there exists a positive function $f_a\in L^p_{loc}(\boldsymbol{R}_n)\cap L^{\frac{2n}{n+2}}(\boldsymbol{R}_n)$, with $p>n$, such that

$$g(x,u)\leqq f_a(x)$$

for all $u\geqq a$ and a.e. on $\boldsymbol{R}_n$.

(g_2') The function $g(x\cdot)$ is nonincreasing on $(0,\infty)$ for a.e. $x\in\boldsymbol{R}_n$.

We need the follwing lemma.

LEMMA 1. *For each number $\delta>0$ the equation*

(9) $$Lu=g(x,u+\delta) \text{ in } \boldsymbol{R}_n$$

admits a positive solution $v^\delta\in W^{1,2}_{loc}(\boldsymbol{R}_n)$ such that $Dv^\delta\in L^2(\boldsymbol{R}_n)$ and $v^\delta\in L^{\frac{2n}{n-2}}(\boldsymbol{R}_n)$.

PROOF: Let $\{\Omega_m\}$, $m\geqq 1$, be an increasing sequence of bounded domains with smooth boundaries $\{\partial\Omega_m\}$, such that $\boldsymbol{R}_n=\cup_{m\geqq1}\Omega_m$. For each $m\geqq 1$, the Dirichlet problem

(10) $$Lu=g(x,|u|+\delta) \text{ in } \Omega_m,$$

(11) $$u(x)=0 \text{ on } \partial\Omega_m$$

admits a positive solution $v_m^\delta \in \overset{\circ}{W}{}^{1,2}(\Omega_m) \cap C(\overline{\Omega}_m)$. This follows by applying the Schauder fixed point theorem. We now extend each function v_m^δ by 0 outside Ω_m. Since $v_m^\delta \leqq v_{m+1}^\delta$ on $\partial\Omega_m$ and $g(x, \cdot)$ is deacreasing it is easy to show that sequence $\{v_m^\delta\}$ is increasing as $m \nearrow \infty$. Let $\lim_{m\to\infty} v_m^\delta(x) = v^\delta(x)$ on $\boldsymbol{R}_n$. The following estimates show that the function v^δ has all desired properties. Indeed, for each m we have

$$\int_{\Omega_m}\left[\sum_{i,j=1}^n a_{ij}D_i v_m^\delta D_j v_m^\delta + c(v_m^\delta)^2\right]dx = \int_{\Omega_m} g(x, v_m^\delta + \delta)v_m^\delta dx$$

$$\leqq \left[\int_{\Omega_m} f_\delta^{\frac{2n}{n+2}} dx\right]^{\frac{n+2}{2n}} \left[\int_{\Omega_m} |v_m^\delta|^{\frac{2n}{n-2}} dx\right]^{\frac{n-2}{2n}}$$

On the other hand by the Sobolev inequality we have

$$\lambda\left[\int_{\Omega_m} |v_m^\delta|^{\frac{2n}{n-2}} dx\right]^{\frac{n-2}{n}} \leqq S\lambda\int_{\Omega_m} |Dv_m^\delta|^2 dx \leqq S \int_{\Omega_m}\sum_{i,j=1}^n a_{ij}D_i v_m^\delta D_j v_m^\delta dx,$$

where $S>0$ is a constant independent of m. These two estimates yield that the integrals

$$\int_{\Omega_m} |Dv_m^\delta|^2 dx \text{ and } \int_{\Omega_m} |v_m^\delta|^{\frac{2n}{n-2}} dx$$

are bounded independently of m and the result easily follows.

THEOREM 4. *The equation* (1) *admits an entire positive solution* $u \in W_{loc}^{1,2}(\boldsymbol{R}_n)$ *such that*

(12) $\quad v^\delta(x) \leqq u(x) \leqq v^\delta(x) + \delta$ *on* $\boldsymbol{R}_n$

for each $\delta>0$, *where* v^δ *is a solution of the equation* (9), *constructed in the proof of Lemma* 1.

PROOF: As in the proof of Lemma 1, let $\{\Omega_m\}$ be sequence of bounded domains with smooth boundaries such that $\boldsymbol{R}_n = \bigcup_{m\geq 1}\Omega_m$. According to Theorem 1 for each m the Dirichlet problem

(13) $\quad Lu = g(x, u)$ in Ω_m,

(14) $\quad u(x) = 0$ on $\partial\Omega_m$

admits a positive solution u_m in $W_{loc}^{1,2}(\Omega_m) \cap C(\overline{\Omega}_m)$. It follows from the proof of Theorem 1 that for each m, $u_m(x) = \lim_{\delta\to 0} v_m^\delta(x)$ uniformly on $\overline{\Omega}_m$, where v_m^δ is a solution of the problem (10), (11). Moreover, we have for $0<\varepsilon<\delta$

$$0 \leqq v_m^\varepsilon - v_m^\delta \leqq \delta - \varepsilon \text{ in } \Omega_m.$$

Letting $\varepsilon \to 0$ we get

$$0 \leqq u_m - v_m^\delta \leqq \delta \text{ on } \Omega_m$$

and consequently

(15) $$v_m^\delta \leqq u_m \leqq v_m^\delta + \delta \text{ on } \Omega_m.$$

Let $B(x_0, r)$ be a ball with center at x_0 of radius r. Let Φ be a C^1-function defined as in the proof of Theorem 1. We now choose an integer $q \geqq 1$ such that $\overline{B(x_0, 2r)} \subset \Omega_q$. Since $\{v_m^\delta\}$ is as increasing sequence as $m \nearrow \infty$, we have $v_q^\delta(x) \leqq u_m(x)$ for all $m > q$. Let $0 < a = \inf_{\overline{B(x_0, 2r)}} v_q^\delta(x)$, then taking $u_m \Phi^2$ as a test function we get

$$\int_{\Omega_m} \sum_{i,j=1}^n a_{ij} D_i u_m D_j u_m \Phi^2 dx + 2\int_{\Omega_m} \sum_{i,j=1}^n a_{ij} D_i u_m u_m D_j \Phi \Phi dx + \int_{\Omega_m} c u_m^2 \Phi^2 dx$$
$$= \int_{\Omega_m} g(x, u_m) u_m \Phi^2 dx \leqq \int_{\Omega_m} f_a u_m \Phi^2 dx.$$

From this inequality we easily derive the following estimate

(16) $$\int_{\Omega_m} |Du_m|^2 \Phi^2 dx \leqq K\left(\int_{\Omega_m} u_m^2 |D\Phi|^2 dx + \int_{\Omega_m} f_a u_m \Phi^2 dx\right),$$

for some constant $K > 0$ independent of m. On the other hand we have for v_m^δ

$$\int_{\Omega_m} \left[\sum_{i,j=1}^n a_{ij} D_i v_m^\delta D_j v_m^\delta + c(v_m^\delta)^2\right] dx = \int_{\Omega_m} g(x, v_m^\delta + \delta) v_m^\delta dx$$
$$\leqq \int_{\Omega_m} f_\delta v_m^\delta dx \leqq \left[\int_{\Omega_m} f_\delta^{\frac{2n}{n+2}} dx\right]^{\frac{n+2}{2n}} \left[\int_{\Omega_m} |v_m^\delta|^{\frac{2n}{n-2}} dx\right]^{\frac{n-2}{2n}}$$

Consequently using the ellipticity and the Sobolev inequality we get

(17) $$\left[\int_{\Omega_m} |v_m^\delta|^{\frac{2n}{n-2}} dx\right]^{\frac{n-2}{2n}} \leqq C\left[\int_{\Omega_m} f_\delta^{\frac{2n}{n+2}} dx\right]^{\frac{n+2}{2n}}$$

for some constant $C > 0$ independent of m. Since $\dfrac{2n}{n-2} = 2 + \dfrac{4}{n-2}$, we deduce from (15), (16) and (17) that the sequence $\{u_m\}$ is bounded in $W_{\mathrm{loc}}^{1,2}(\boldsymbol{R}_n)$ and we may assure that $\lim_{m \to \infty} u_m = u$ strongly in $L^2(K)$, and weakly in $W^{1,2}(K)$ for each bounded domain $K \subset \boldsymbol{R}_n$. The inequality (12) is a consequence of Lemma 1 and the inequality (15). It is obvious that u

is a solution of (1) in the distributional sense.

The following result is an analogue of Theorem 2.

THEOREM 5. *Suppose that*

(18) $g(x, u)u^{\alpha}\leqq f(x)$ *on* $\boldsymbol{R}_n\times(0, a]$

for some constants $a>0$ *and* $0<\alpha\leqq 1$, *where* $f(x)>0$ *on* $\boldsymbol{R}_n$ *and* $f\in L^{\frac{2n}{n+2+\alpha(n-2)}}(\boldsymbol{R}_n)$. *Then there exsits an entire positive solution* u *of* (1) *such that* $Du\in L^2(\boldsymbol{R}_n)$ *and* $u\in L^{\frac{2n}{n-2}}(\boldsymbol{R}_n)$.

PROOF: It is sufficient to show that the sequence of solutions $\{u_m\}$ of the Dirichlet problems (13), (14) has the properties: (i) $\{u_m\}$ is bounded in $L^{\frac{2n}{n-2}}(\boldsymbol{R}_n)$, (ii) $\{Du_m\}$ is bounded in $L^2(\boldsymbol{R}_n)$. By Theorem 2 $u_m\in \mathring{W}^{1,2}(\Omega_m)$ and we extend functions u_m by 0 outside Ω_m. Taking u_m as a test function we obtain on substitution

$$\int_{\Omega_m}\Big[\sum_{i,j=1}^{n} a_{ij}D_iu_mD_ju_m+cu_m^2dx\Big]\,dx\leqq\int_{\Omega_m}g(x, u_m)u_mdx$$

$$\leqq\int_{u_m\leqq a}fu_m^{1-\alpha}dx+\int_{u_m\geqq a}f_au_mdx$$

$$\leqq\Big[\int_{\Omega_m}u_m^{\frac{2n}{n-2}}dx\Big]^{\frac{(n-2)(1-\alpha)}{2n}}\Big[\int_{\Omega_m}f^{\frac{2n}{n+2+\alpha(n-2)}}dx\Big]^{\frac{n+2+\alpha(n-2)}{2n}}$$

$$+\Big[\int_{\Omega_m}f_a^{\frac{2n}{n+2}}dx\Big]^{\frac{n+2}{2n}}\Big[\int_{\Omega_m}u_m^{\frac{2n}{n-2}}dx\Big]^{\frac{n-2}{2n}}$$

On the other hand by the Sobolev inequality we have

$$S^{-1}\lambda\Big[\int_{\Omega_m}u_m^{\frac{2n}{n-2}}dx\Big]^{\frac{n-2}{n}}\leqq\int_{\Omega_m}\sum_{ij=1}^{n}a_{ij}D_iu_mD_ju_mdx$$

and the combination of these two estimates gives first the boundedness of $\{u_m\}$ in $L^{\frac{2n}{n-2}}(\boldsymbol{R}_n)$ and then the boundedness of $\{Du_m\}$ and the result follows.

REMARK 4. If (18) holds with $\alpha>1$, we assume that $f\in L^1(\boldsymbol{R}_n)$ and (g_1') holds with $f_a\in L^p_{loc}(\boldsymbol{R}_n)\cap L^{\frac{2n}{n+2}}(\boldsymbol{R}_n)\cap L^{\frac{n(\alpha+1)}{n+2\alpha}}(\boldsymbol{R}_n)$. Under these assumptions $\int_{\boldsymbol{R}_n}u^{\frac{n(\alpha+1)}{n-2}}dx<\infty$ and $\int_{\boldsymbol{R}_n}|Du|^2u^{\alpha-1}dx<\infty$. To prove this we use as a test function u_m^{α}.

References

[1] J. CHABROWSKI & M. KÖNIG, *On entire solutions of elliptic equations with singular nonlinearity,* Commet. Math. Univ. Carolinae 31, 4 (1990), 645-654.

[2] M. M. COCLITE & G. PALMIERI, *On a singular nonlinear Dirichlet problem,* Comm. in Partial Diff. Equations 14(10) (1989), 1315-1327.

[3] M. CRANDALL, P. H. RABINOWITZ & L. TARTAR, *On a Dirichlet problem with a singular nonlinearity,* Comm. in Partial Diff. Equations 2(2) (1977), 1983-222.

[4] R.DALMASSO, *Solutions d'ēquations elliptiques semi-linēaires singuliēres,* Ann, Mat. Pura Appl. 4(153) (1989), 191-201.

[5] A. L. EDELSON, *Entire solutions of singular elliptic equations,* J. Math. Anal. Appl. 139 (2) (1989), 523-532.

[6] A. M. FINK, J. A. GATICA, G. E. HERNANDEZ & P. WALTMAN, *Approximation of solutions of singular second order boundary value problems,* proprint.

[7] D. GILBARG & N. S. TRUDINGER, "Elliptic partial differential equations of second order," Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (second edition), 1983.

[8] T. KUSANO & C. A. SWANSON, *Entire positive solutions of singular elliptic equations,* Japan J. Math. 11(1) (1985), 145-155.

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