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The spheres in symmetric spaces

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

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Introduction.

Our main purpose is to determine the totally geodesic spheres in every compact symmetric space. This includes finding of all the monomorphisms of the group $SU(2)\cong S^3$ into the compact Lie groups. The task is a part of the fundamental problem of determination of all the homomorphisms between symmetric spaces (1. 1); a smooth mapping $f: M\rightarrow N$ between symmetric spaces is a homomorphism if and only if f is totally geodesic, provided M is connected.

Historically, the one dimensional case of S^1 was done by E. Cartan himself [C]. The case of S^3 overlaps with Dynkin's monumental work [D] in the part where he determines all the three dimensional complex subalgebras of the complex simple Lie algebras. Wolf [W] studied the case of the spheres in the real, complex and quaternion Grassmann manifolds $G_n(\boldsymbol{R}^{2n})$, $G_n(\boldsymbol{C}^{2n})$ and $G_n(\boldsymbol{H}^{2n})$ under a certain condition to be explained later (4. 6), completing a work of Y. C. Wong. Helgason studied a sphere which corresponds to the highest root ([H], Chap. 7, § 11). Fomenko in [F-1], [F-2] and [F-3] discussed the homotopy and homology classes of totally geodesic spheres; Fomenko's book [F-4] (English translation) has just appeared. Finally, the case of the zero dimensional sphere or the pair of points was done in [CN] and [N-1]; in this case a homomorphism f: $\{o, p\}\rightarrow N$ is characterized by the property that $f(p)$ is fixed by the point symmetry $s_{f(o)}$ at $f(o)$.

Our method is more geometric in a way, based on the theory under development (See [CN], [N-1] and [N-2]); one can determine the spheres by using a huge induction mechanism coming from interrelationship between the symmetric spaces, at least all those spheres in certain classes (See the end of Section 1). The article [NS] might serve as another introduction.

In § 1 we will explain our geometric method along with basic concepts. Careful reading of this section and the next will help understand our results in the later sections. In § 2, we will prepare technical facts including the Dynkin numbers (" the characteristic numbers " in his term) and explain the finiteness of the congruence classes of the spheres in a symmetric space. In § 3, we will establish our main theorem 3. 2, precisely describe the spheres which are not contained in subspaces (called meridians 1. 3) and of the greatest curvature (or the smallest size) among such spheres; these spheres are intimately related to the Bott periodicity (or rather they make a geometric background for it) and were also studied by Wolf in certain cases. In § 4, we will apply the theorem 3. 2 to each individual space. In § 5, we will show how to deal with the other spheres by means of the representation theory.

It will be convenient to agree that the term " sphere " means a sphere of dimension $\geqq 2$ usually and a 1-dimensional sphere is called a circle. The symbols for the symmetric spaces are standard ones (as in [H] or E. Cartan's papers) except for the Grassmannians. The symbols and numberings related to root systems will follow Bourbaki [B].

§ 1. Basic concepts and the method.

A symmetric space M is a manifold which admits a transformation of M, called *the point symmetry* s_x, for every point x such that (1) s_x is involutive, (2) x is an isolated fixed point of s_x and (3) M admits a Riemannian metric of which every point symmetry s_x is an isometry.

1. 1 DEFINITION. *A homomorphism* $f: M \to N$ of a symmetric space M into another one N is a smooth map which commutes with every point symmetry; i. e. one has $f \circ s_x = s_{f(x)} \circ f$ for every point x of M. In particular *a subspace* means a space whose inclusion is a homomorphism.

1. 2 EXAMPLES. ① If M and N are compact Lie groups, a Lie group homomorphism $f: M \to N$ is a homomorphism of a symmetric space with the point symmetry s_x defined by $s_x(y) = xy^{-1}x$. ② The point symmetry s_x is an automorphism if the symmetric space M is connected, [N-2]. Thus the automorphism group $\mathrm{Aut}(M)$ is transitive on a connected M. Its identity component, denoted by G, G_M or $G(M)$, is also transitive on M then.

Since the symmetric space is defined in terms of its point symmetries, it is not surprising that the fixed point set of a point symmetry s_x, denoted by $F(s_x, M)$, plays a fundamental role; for instance, $F(s_x, M)$ is closely related to the topology of M (See [N-1]). Thus the next definitions would be reasonable.

1. 3 DEFINITIONS. *A polar* of a point x in M is an arbitrary connected component of the fixed point set $F(s_x, M)$. We call it *a pole* if it is a singleton $\{p\}$. For each polar $F(s_x, M)_{(p)}$ through the point p, *its meridian* at p is the connected component $F(s_x \circ s_p, M)_{(p)}$ of the fixed point set $F(s_x \circ s_p, M)$ of $s_x \circ s_p$ through p. More generally, an involution t of M which fixes a point p gives rise to two subspaces $F(t, M)_{(p)}$ and $F(t \circ s_p, M)_{(p)}$. We call these subspaces *the completely orthogonal space* or *c-orthogonal* space of one another; indeed their tangent spaces at p are the orthogonal complement of each other in the tangent space T_pM.

1. 3a For a later reference, we point out that the distance from x to every polar $F(s_x, M)_{(p)}$ is known (2. 14 in [N-2]). By the way, theoretical importance of the concepts of the polar and the meridian lies in the fact that a compact connected symmetric space M is completely determined by any pair $(M^+(p), M^-(p))$ of a polar and its meridian [N-2]. Its relevance to our problem will be seen shortly.

1. 3b We have a complete list of the poles and the meridians (as well as the centrosomes), the essential parts of which may be found in [CN], [N-1] and [N-2]. Our hope is that the reader finds later examples easy to see; but otherwise he could skip them. In Section 4, those individual results are important; the reader could reinterprete the last chapter of [M] in our context to obtain a clear picture.

Since a noncompact simple symmetric space contains no sphere, we will consider the compact connected symmetric spaces only. Also we will assume that they are simple; if a space M is not simple, M has a finite covering space which is a product $M_1 \times M_2$, into which a sphere S^m in M lifts and projects onto a sphere in each of M_1 and M_2 isomorphically or onto a single point in it, and one can reverse the process to make a sphere in M out of spheres in M_1 and/or M_2.

Given a point o of S^m, its pole $p \neq o$ in S^m lies in a polar $M^+(p)$ of o in M. Hence S^m is entirely contained in the meridian $M^-(p)$ by the above relationship. Since all the pairs $(M^+(p), M^-(p))$ were determined in [CN] and [N-1], our task of finding all the spheres in M is therefore reduced to easy induction arguments on the dimension of M, unless $M^+(p)$ is a pole $\{p\}$ of o in M so that the meridian $M^-(p)$ is the whole space M. Our problem is now the determination of the spheres in M which contains the pole p of o in M along with o. We will concentrate on this case.

In order to proceed further in this case, we consider the monomorphism of the equator S^{m-1} of S^m and we need the next concept.

1. 4 DEFINITION. If there is a pole p of a point o in M, the set of the midpoints of the geodesic segments joining o to p is called *the centrosome* and denoted by $C(o, p)$. $C(o, p)$ is disconnected in general.

1. 5 $C(o, p)$ is a subspace; in fact it can be characterized by the property that it projects onto the union of several polars under the unique double covering epimorphism $\pi: M \to M''$ which carries o and p into a single point (1. 9 of [N-1] and [N-2]). Thus $C(o, p)$ is known. The centrosome comes into the scene when $S^m \supset \{o, p\}$ in M is considered: S^m meets $C(o, p)$ in the subspace $S^{m-1} = S^m \cap C(o, p)$ of $C(o, p)$. Question arises whether or not a sphere S^{m-1} in $C(o, p)$ extends to such an S^m in M conversely. Uniqueness in the correspondence is another question. We denote by $\gamma: M \to M$ the corresponding covering automorphism (deck transformation) of M for a more precise describption of $p = \gamma(o)$ in case a point o has more than one poles. π is the projection of M onto the orbit space $M/\{1, \gamma\}$. γ centralizes G. $\gamma(x)$ is a pole of x in M for any point x. G is transitive on the set $\{(x, \gamma(x)) | x \in M\}$ of these pairs of the points and their specified poles. γ stabilizes $C(o, p)$ and the above S^{m-1}. One might remember that $C(o, p)$ was effectively used in a proof of the Bott periodicity [M].

1. 6 Our problem is to determine all the maximal (in terms of inclusions) spheres S^m, $m \geqq 2$, which are subspaces of the compact connected symmetric spaces $M = G/K$ up to G-congruence, that is, we ignore the difference between the spheres in M which some members of G carry one onto another; we will mention something about a weaker equivalence, the $\mathrm{Aut}(M)$-congruence in appropriate occasions. As a summary, we will explain our strategy. If p is the pole of o in S^m, then S^m is a subspace of the meridian $M^-(p) = G^-/K^+$ in which p is a pole of o. The G-congruence class $[S^m]$ restricts to the G^--congruence class $[S^m]^-$ injectively. In case p is not a pole of o in M, $M^-(p)$ is a well determined spece of a lower dimension; some kind of induction argumes will thus work to classify the spheres. Hence we will concentrate on the case of a pole $p = \gamma(o)$ in M. Then S^m meets the centrosome $C(o, p)$ in the equator S^{m-1}. Moreover, if x is a point of S^{m-1}, then its pole $y = \gamma(x)$ lies in S^{m-1} and y is a pole of x in $C(o, p)$, as mentioned in 1. 5. Thus we will have a similar situation in a lower dimensional space $C(o, p)$; that is, a sphere S^{m-1} in the space $C(o, p)$ contains a pair $(x, \gamma(x))$, $\dim C(o, p) < \dim M$.

§ 2. Technical preliminaries.

Given a sphere S^m in M, take a maximal torus A in M which meets

S^m in a circle c. Let o be a point on c. Consider the symmetry decomposition $\boldsymbol{g}=\boldsymbol{k}+\boldsymbol{m}$ of the Lie algebra $\boldsymbol{g}=\mathscr{L}G$ at o (See [H] or [N-2]):

2.1 $\boldsymbol{g}=\boldsymbol{k}+\boldsymbol{m}$, where $\boldsymbol{k}$ is the eigenspace of $\mathrm{ad}(s_o)$ for the eigenvalue 1 and $\boldsymbol{m}$ is the one for -1. $\boldsymbol{k}$ is the Lie algebra of the isotropy subgroup K of G at o. When necessary, $\boldsymbol{k}$ and $\boldsymbol{m}$ will be denoted by $\boldsymbol{k}_o$ and $\boldsymbol{m}_o$ to indicate the point of reference o explicitly.

2.1a LEMMA. *The identity component $K_{(1)}$ of K is transitive on each polar $M^+(p)$ of o. In particular, one has $\boldsymbol{k}_p=\boldsymbol{k}_o$ if p is a pole of o.*

PROOF. Since s_o fixes p, s_o commutes with s_p. Hence one obtains the eigenspace decompositions of $\boldsymbol{k}$ and $\boldsymbol{m}$ with respect to s_p: $\boldsymbol{k}=\boldsymbol{k}^++\boldsymbol{k}^-$ and $\boldsymbol{m}=\boldsymbol{m}^++\boldsymbol{m}^-$. Thus the symmetry decomposition at p is $\boldsymbol{g}=(\boldsymbol{k}^++\boldsymbol{m}^+)+(\boldsymbol{k}^-+\boldsymbol{m}^-)$. It is immediate from the defintions that $\boldsymbol{k}^-$ gives the tangent space to $M^+(p)$ at p and $\boldsymbol{m}^-$ gives $T_p(M^-(p))$. If p is a pole of o, then one has $s_o=s_p$. □

2.1b COROLLARY. *$\boldsymbol{k}^-\subset\boldsymbol{m}_p$ and $\boldsymbol{k}^-$ gives the tangent space $T_pM^+(p)$; $T_pM^+(p)=\{v(p)|v\in\boldsymbol{k}^-\}$.*

One can easily verify or rather one knows

2.2 $[\boldsymbol{k},\boldsymbol{k}]\subset\boldsymbol{k}, [\boldsymbol{k},\boldsymbol{m}]\subset\boldsymbol{m}$, and $[\boldsymbol{m},\boldsymbol{m}]=\boldsymbol{k}$.

2.3 Let $\boldsymbol{a}$ denote a maximal abelian subalgebra contained in $\boldsymbol{m}$ which may be identified with the tangent spact T_oA to A at o; $\boldsymbol{a}$ contains the initial tangent $c'(0)$, $c(0)=o$. Then the adjoint action of $\boldsymbol{a}$ on $\boldsymbol{g}$ gives rise to the decomposition: $\boldsymbol{g}=\sum\boldsymbol{g}(\alpha)$, where α is a linear form on $\boldsymbol{a}$, such that $[\boldsymbol{a},\boldsymbol{g}(\alpha)]\subset\boldsymbol{g}(\alpha)$ and $\mathrm{ad}(H)^2$ acts on $\boldsymbol{g}(\alpha)$ as $-\alpha(H)^2$ times the identity for every member H of $\boldsymbol{a}$, with our convention $\boldsymbol{g}(-\alpha)=\boldsymbol{g}(\alpha)$. α is called *a root* of M if $\alpha\neq0$. Indeed the set of all the roots form a root system in the usual sense [B]. Since $\mathrm{ad}(H)$ skew-commutes with $\mathrm{ad}(s_o)$, $\boldsymbol{g}(\alpha)$ decomposes into $\boldsymbol{k}(\alpha)+\boldsymbol{m}(\alpha)$; $\boldsymbol{k}(\alpha)=\boldsymbol{k}\cap\boldsymbol{g}(\alpha)$ and $\boldsymbol{m}(\alpha)=\boldsymbol{m}\cap\boldsymbol{g}(\alpha)$. One has

2.3a $[\boldsymbol{a},\boldsymbol{k}(\alpha)]=\boldsymbol{m}(\alpha)$ and $[\boldsymbol{a},\boldsymbol{m}(\alpha)]=\boldsymbol{k}(\alpha)$ if α is a root.

2.4 REMARK. It is important to notice that the decompositions are closely related to the curvature and, more specifically, to the Jacobi equation for the geodesic with the initial tangent $H\in\boldsymbol{a}$; thus the Jacobi equation is reduced to $\mathrm{ad}(H)^2v+\alpha(H)^2v=0$ for $v\in\boldsymbol{g}(\alpha)$, where v is first thought of as a vector field on M then one restricts it to one along c. Based on this fact, one can determine conjugate points and the distance to them. For instance, there is a conjugate point of o on the open interval $(0, t_1)$ along the geodesic c: $t \to \exp(tH)(o)$, if and only if $\alpha(H)t_1 > \pi$ for some root α. Also, as mentined in (1. 3a), we have explicitly determined the distance between o and any of its polars, which we will talk about later.

2. 4a REMARK. Another geometric meaning of a root α is illustrated by the fact that $\boldsymbol{m}(\alpha)$ and α span the tangent space $T_o\Psi$ to a subspace $\Psi = \Psi(\alpha)$ of positive constant curvature unless 2α is another root. In fact, for any nonzero vector X in $\boldsymbol{m}(\alpha)$, one obtains $[\alpha, X] = \|\alpha\|^2 Y$, $[\alpha, Y] = -\|\alpha\|^2 X$ for some Y in $\boldsymbol{k}(\alpha)$ and hence a compact 3-dimensional subgroup $G^{(3)}$ of G, whose orbit through o is a 2-dimensional space of positive constant curvature; the desired space $\Psi(\alpha)$ is the union of all these spaces for X ranging over $\boldsymbol{m}(\alpha)$. In case 2α is a root, one obtains a complex, quaternin or Cayley projective space. For brevity, we call a subspace of positive constant curvature *a ψ-(pseudo-)sphere*; this is a sphere or a real projective space. And we call $\Psi(\alpha)$ *the ψ-sphere for the root α*. Helgason studied the ψ-spheres for the longest roots (Theorem 11. 1 on p. 334 and others in [H]).

2. 5 REMARK. We recall a formula relating the roots to the curvature. Let X be a unit vector in $\boldsymbol{m}$ and let H be a unit vector in $\boldsymbol{a}$. Then the sectional curvature at the 2-plane spanned by orthonormal X and H equals the sum $\sum \alpha(H)^2 \|X(\alpha)\|^2$, where $X(\alpha)$ is the component of $X = \sum X(\alpha)$ in $\boldsymbol{m}(\alpha)$ and the summation occurs over all the roots, taking one of $\pm\alpha$ only for each α. The greatest value of the sectional curvature of M is realized by the ψ-spheres for the roots of the longest roots clearly.

Now we try to label the G-congruence classes of the 2-dimensional ψ-spheres Ψ^2 which are subspaces of M, following Dynkn's idea for 3-dimensional subalgebras and hence aiming at some analogue of his "characterisitc numbers", which we will call *the Dynkin numbers*. We fix a maximal abelian subalgebra $\boldsymbol{a} = T_oA$. We may assume that a tangent vector H to a given Ψ^2 lies in $\boldsymbol{a}$ and further that the values $\alpha_i(H)$ are nonnegative for all simple roots α_i, $1 \leqq i \leqq \operatorname{rank}(M)$, corresponding to a fixed Weyl chamber. We so normalize H that the circle c: $t \to \exp(tH)(o)$ in Ψ^2 has the first conjugate point p of o at $t = \frac{1}{2}\pi$ along the geodesic c in Ψ^2; that is, $T_o\Psi^2$ has another tangent vector X satisfying $[H, X] = 2Y$, $[H, Y] = -2X$ and $[X, Y] = 2H$ for a member Y of $\boldsymbol{k}$. The Dynkin numbers of Ψ^2 are $d^i := \alpha_i(H)$, $1 \leqq i \leqq r(M)$, by definition.

2. 6 PROPOSITION. (i) *The Dynkin numbers defined above of a 2-dimensional ψ-sphere Ψ^2 equal* $0, 1$ *or* 2. (ii) *These are all even numbers if Ψ^2 is a sphere and contains a pole $\gamma(o)$ of o in M for a point o in Ψ^2.* (iii) *These are all even numbers also if Ψ^2 is not a sphere.* (iv) *At least one of the Dynkin numbers is odd if Ψ^2 is a sphere such that $p=c\left(\frac{1}{2}\pi\right)$ lies on a polar which is not a pole in M.*

PROOF. The point $c(\pi)=\exp(\pi H)(o)$ is o by the normalization. Hence $\mathrm{ad}\circ\exp(\pi H)$ stabilizes $\boldsymbol{k}$ and every root space $\boldsymbol{k}(\alpha)$. Thus $\alpha(H)$ is an integer. The adjoint representation of $\boldsymbol{g}$ restricts to that of the subalgebra $\boldsymbol{g}(\Psi^2)$ spanned by $\{H, X, Y\}$ with the Cartan subalgebra $\boldsymbol{R}H$. The positive weights of $\boldsymbol{g}(\Psi^2)$-modules which meet the root space $\boldsymbol{g}(\alpha_k)$ nontrivially are integral multiples of $\alpha_k(H)$ for every simple root α_k. Therefere $\alpha_k(H)$ cannot exceed 2; (i) obtains. Ψ^2 is a sphere if and only if $p\neq o$. Since one has d^k even $\Longleftrightarrow$ a nonzero vector field $Y_k\in\boldsymbol{k}(\alpha_k)$ vanishes at p, one sees that all d^k are even $\Longleftrightarrow$ $\mathrm{ad}\circ\exp\left(\frac{1}{2}\pi H\right)$ stabilizes $\boldsymbol{k}$ $\Longleftrightarrow$ p is either o or a pole $\gamma(o)$ by 2. 1a. Hence one has (ii), (iii) and (iv), which exhaust all the cases. □

2. 6a REMARK. The part (i) of the proposition is basically Dynkin's theorem about three-dimensional subalgebra of simple Lie algebras (Thm 8. 3 [D], also Prop. 5 on p. 164, Chap. 8 of [B]).

2. 6b REMARK. As opposed to the case of the Lie subalgebras (Same Thm 8. 3 [D] and Prop. 6 on p. 165, Chap. 8 of [B]), the Dynkin numbers do not uniquely determine the congruence class of Ψ^2, as this example shows. The real projective space $\boldsymbol{R}P^3$ is a polar in the adjoint group $\mathrm{ad}\,SO(4)$, its meridian $\mathrm{ad}\,SO(3)$ share a subspace $\boldsymbol{R}P^2$ with $\boldsymbol{R}P^3$, which is a polar in both; $\boldsymbol{R}P^3$ is certainly not congruent with the isomorphic space $\mathrm{ad}\,SO(3)$. The space $\mathrm{ad}\,SO(4)$ is not simple, but it is easy to construct a simple one out of this; see (3. 4). For a later reference we observe: by lifting these two $\boldsymbol{R}P^3$'s to the universal covering space $Spin(4)=:SO(4)^{\sim}$, one obtains two 3-spheres which meet in S^2 in the common centrosome of different pairs $(o, \gamma(o))$ and $(p, \gamma(p))$.

2. 7 THEOREM. *The congruence classes of the ψ-spheres of dimensions $\geqq 2$ in a compact symmetric space M are finite in number.*

We have no direct, simple proof; what 2. 6 immediately gives is the next 2. 8. As it will turn out alter, it suffices to show that the finiteness of the conjugate classes of the orthogonal groups in a Lie group on the Lie algebra level. Also the fact will be revealed by later developments of this paper.

2. 8 COROLLARY. *The set of the diameters of the spheres S^m, $m\geqq 2$, in M is finite. Similarly for the ψ-spheres, (but in sharp contrast with the circles).*

2. 9 PROPOSITION. *There is a unique congruence class of the 2-dimensional ψ-spheres with all the Dynkin numbers $d^i=2$, $1\leqq i\leqq r(M)$, called principal.*

PROOF. First we will prove the existence. Let $H\in \boldsymbol{a}$ denote $\sum_i 2H_i$, where $H_i=2^2\|\alpha_i\|^{-2}\varpi_i$ and ϖ_i is the i-th fundamental weight. H is a linear combination $\sum_i c_i\alpha_i$ of the simple roots. Choose $X_i\in \boldsymbol{m}(\alpha_i)$ such that $[X_i, Y_i]=c_i\alpha_i$, where $Y_i:=[H, X_i]$; this is a matter of normalization. We have $-\mathrm{ad}(H)^2X_i=2^2X_i$. Then H and $X=\sum_i X_i$ span the tangent space $T_o\Psi^2$ to a 2-dimensional ψ-sphere Ψ^2; in fact $[X, Y]:=[\sum X_i, \sum Y_j]=\sum[X_i, Y_i]=2H$, since $[H,[X_i, Y_j]+[X_j, Y_i]]=0$ which, together with $(\alpha_i+\alpha_j)(H)=4\neq 0$, implies that $[X_i, Y_j]+[X_j, Y_i]\in \boldsymbol{k}(\alpha_i+\alpha_j)$ vanishes. Now we will show the uniqueness. The component $X(\alpha)$ of X in the earlier notation does not vanish if and only if α is a simple root α_i, $1\leqq i\leqq r(M)$, since every other positive root is the sum of two or more simple roots, while one has $\alpha(H)^2X(\alpha)=-\mathrm{ad}(H)^2X(\alpha)=2^2X(\alpha)$ by $-\mathrm{ad}(H)^2X=2^2X$, which gives $\alpha(H)^2=2^2$ for $X(\alpha)\neq 0$. From $[X, Y]=2H$, one obtains $\sum\|X(\alpha)\|^2\alpha=2H$. Since the simple roots make a basis for $\boldsymbol{a}$, this shows that the length $\|X(\alpha)\|$ is uniquely determined by H for each root α. Suppose there is another ψ-sphere to which H and X' are tangent at o with the similar relationship to the pair (H, X); so one has $\|X'(\alpha)\|=\|X(\alpha)\|$. If the multiplicity $m(\alpha):=\dim \boldsymbol{m}(\alpha)$ equals 1 for a simple root $\alpha=\alpha_i$, then one has $X'(\alpha)=\pm X(\alpha)$ and one can carry $X'(\alpha)$ into $X(\alpha)$ by means of the subgroup generated by ϖ_i without affecting H or the other components. If $m(\alpha)>1$, then one can do the same, by following the proof of Proposition 6 in Chap. VIII, n° 3 or p. 165 of [B] basically or by a geometric study of individual cases using 2.25, 2.26a and the proof of 2.27 of [N-2] (among which the first two imply that the isotropy subgroup contains a subgroup which leaves $\boldsymbol{m}(\alpha)$ invariant and acts transitively on the unit sphere in $\boldsymbol{m}(\alpha)$). □

2. 10 REMARK. The principal ψ-spheres have the smallest curvature among all the ψ-spheres in M.

2. 11 QUESTION. A maximal ψ-sphere which contains a principal

one has dimension$\leqq 1+\min$, where min denotes the minimum of the multiplicities $m(\alpha)$ of the roots α of M. Does it attain $1+\min$? And are the maximal ones unique up to congruence? These are known to be the case for the groups M, as one can easily deduce from 2. 9 also; $\min=2$.

§ 3. The spheres of maximal curvature.

In this section, we will explain how to determine the spheres of the maximal curvature (See 2. 8) for a pair $(o, \gamma(o))$ in a compact simple symmetric space M which contain a point o together with its pole $\gamma(o)$ in M. A sphere of dimension>1 in a given M has a greater curvature than another if and only if it has a smaller diameter.

3. 1 DEFINITION. When a sphere has the smallest diameter among the spheres which contain a fixed point, say o, as well as its pole $\gamma(o)$ in M, we say it has *size* S, for brevity.

If S^m contains $\{o, \gamma(o)\}$, then S^m meets the centrosome $C(o, \gamma(o))$ in the equator S^{m-1}. The next theorem asserts the converse under certain conditions. If S^m has size S, then S^{m-1} has size S too and the component of $C(o, \gamma(o))$ which contains S^{m-1} is one of the closest to o among the components of $C(o, \gamma(o))$. The distance between a point o and its polar or a component of the centrosome is known for each space (1. 3a). To a polar, there corresponds a simple root α_j such that, if the highest root of the root system (2. 3) of M is expressed as a linear combination of the simple roots, the coefficient of α_j is 1 or 2. Roughly speaking, the desired distance is the length of the vector $2\|\alpha_j\|^{-2}\varpi_j$, where ϖ_j is the j-th fundamental weight.

3. 2 THEOREM. *Let γ be the covering automorphism of a compact connected symmetric space M for a double covering epimorphism: $M\rightarrow M''$. Let S^{m-1} be a sphere in a centrosome $C(o, \gamma(o))$, $m\geqq 1$. Assume*: (1) $\gamma(S^{m-1})=S^{m-1}$; (2) S^{m-1} *has size* S: *and* (3) *the distance between the points of S^{m-1} and o equals $\frac{1}{2}\,dis(o, \gamma(o))$, that is, the components of $C(o, \gamma(o))$ which contain S^{m-1} are the closest to o. Then there exists a unique sphere S^m which meets $C(o, \gamma(o))$ in S^{m-1}.* (The assumptions (2) and (3) are necessary for the conclusion only if M is $SO(4n)^{\sim}$, $SO(4n)^{\#}$, E_7 or EV.)

PROOF. Let p be a point on S^{m-1}. Then there is a unique shortest circle $c=c_H$ which passes through p and o by (3). Here H is a member of $\mathbf{m}=\boldsymbol{m}_o$ for the symmetry decomposition at o and tangent to c along it; $c(t)=\exp(tH)(o)$ for every $t\in\boldsymbol{R}$. If $m=1$, c is the desired S^m; one has

$\gamma(c)=c$, in particular $c \supset S^{m-1}$, since c contains $\gamma(o)$ and the epimorphism $\pi: M \to M''=M/\{1, \gamma\}$ must project c into a circle. Assume $m>1$. Let c_Y be a circle on S^{m-1} which passes through p: Y is a member of $\boldsymbol{m}_p$ for the symmetry decomposition at p and tangent to c_Y along it. Y is also a member of $\boldsymbol{k}_o$, since every component of $C(o, \gamma(o))$ is an orbit of the identity component $K_{(1)}$ of the isotropy subgroup K of G at o as well as every polar of o (See 2. 1a and 2. 1b). We will construct a 2-sphere which contains c_H and c_Y. For that we will show that H is a member of $\boldsymbol{k}_q$ for a point q on c_Y. Let q be the midpoint of an arc connecting p and $\gamma(p)$ on c_Y. Then the point symmetry s_q exchanges p and $\gamma(p)$ as well as o and $\gamma(o)$. Hence the double covering map π projects c_H into a polar of $\pi(q)$ in M''. Therefore H is a member of $\boldsymbol{k}_q$ by the proof of 2. 1a. Since both c_H and c_Y have the property that every arc on the curve is one of the shortest connecting its end points by (2) and (3), we can use conjugate point arguments. The vector field Y restricted to c is a Jacobi field on c and vanishes at o and $\gamma(o)$ only. If we normalize H so $\exp(tH)(o)$ reaches $\gamma(o)$ at $t=\pi$ for the first time, then one has $\mathrm{ad}(H)^2 Y=-Y$. Similarly one has $\mathrm{ad}(Y)^2 H=-H$. Therefore one obtains a 2-sphere which contains c_H and c_Y; the Lie algebra of its automorphism group is spanned by H, Y and $[H, Y]$. (The assumptions (2) and (3) are thus crucial for the existence of this 2-sphere.) The desired S^m should contain it. To prove it, may be convenient to recall a theorem of Cartan's: a subspace $\boldsymbol{m}'$ of $\boldsymbol{m}_p$ gives the tangent space T_pN to a subspace N of M if and only if $[\boldsymbol{m}', [\boldsymbol{m}', \boldsymbol{m}']] \subset \boldsymbol{m}'$. Take a subspace $\boldsymbol{m}''$ of $\boldsymbol{m}_p$ which gives the tangent space T_pS^{m-1} through the evaluation: $Z \in \boldsymbol{m}'' \to Z(p)$. Let $\boldsymbol{m}'$ be the subspace spanned by $\boldsymbol{m}''$ and H. We want to see that $\boldsymbol{m}'$ satisfies Cartan's condition. $\boldsymbol{m}''$ already satisfies it. We have proven $\mathrm{ad}(H)^2\boldsymbol{m}''=\boldsymbol{m}''$. Next we assert $[H,[\boldsymbol{m}'', \boldsymbol{m}'']]=\{0\}$. Let K'' be the connected subgroup of G whose Lie algebra is generated by $[\boldsymbol{m}'', \boldsymbol{m}'']$. K'' leaves $C(o, \gamma(o))$ invariant (2. 1a). K'' fixes not only p but o by the fact $[\boldsymbol{k}^-, \boldsymbol{k}^-] \subset \boldsymbol{k}^+$ from 2. 2 in the notation of the proof 2. 1a, recalling that $\pi(p)$ is a point of a polar $\subset \pi(C(o, \gamma(o)))$ of $\pi(p)$ (1. 5). Therefore the members of $[\boldsymbol{m}'', \boldsymbol{m}'']$ vanish identically on c_H (by the non-existence of conjugate points on an open half circle in c_H); so one has $[H,[\boldsymbol{m}'', \boldsymbol{m}'']]=\{0\}$. This implies $[Y_1,[Y_2, H]]=[[Y_1, Y_2],H]+[Y_2,[Y_1, H]]=[Y_2,[Y_1, H]]$ is symmetric in Y_1 and Y_2 for Y_1 and $Y_2 \in \boldsymbol{m}''$. Thus we conclude from $\mathrm{ad}(Y)^2H=-H$ that $[\boldsymbol{m}'',[H, \boldsymbol{m}'']]$ is spanned by H. Finally the obtained S^m is unique, simply because c_H is unique for the point p. □

3. 2a REMARK. The above uniqueness implies that the completely

orthogonal subspace to $C(o, \gamma(o))$ is a local product $c_H \cdot N$ with another subspace N. This helps pick up the components of $C(o, \gamma(o))$ which meet the spheres of size S from the tables (in [CN]; see [N-1] also) of the meridians in M. For example, $C(o, \gamma(o))$ of the group E_7 has two components EV and $EVII$, whose c-orthogonal spaces are $SU(8)/\boldsymbol{Z}_2$ and $c \cdot E_6$, c a circle, respectively. Hence the spheres of size S meet $EVII$ and never EV.

3. 2b REMARK. The shortest geodesics joining points p and their poles $\gamma(p)$ within the centrosome are shortest in the whole space M, although we can only verify it case by case.

The next (3. 3) supplements (3. 2); so we have a complete process of determining the S-sized spheres$\supset\{o, \gamma(o)\}$.

3. 3 PROPOSITION. *The bijection (given by* 3. 2) *of the spheres of size S in* G/K *onto those of size S in the centrosome* G_C/K_C *which carries* S^m *into* $S^{m-1}=S^m \cap G_C/K_C$ *subject to the conditions in Theorem* 3. 2, *induces a bijection of the set of the corresponding G-congruence classes onto that of the* G_C*-concgruence classes. Here, strictly speaking,* G_C/K_C *is the component of the centrosome C that contains* $S^{m-1}=S^m \cap C$, $G_C=G(G_C/K_C)$, *and* G_C/K_C *might have to be replaced with two isomorphic components* $G_C/K_C \amalg G_C/K_C$ *if* $m=1$.

PROOF. This is another way of summarizing 3. 2. If a member b of G carries a sphere into another and fixes the point o, then b stabilizes G_C/K_C and hence b may be thought of as a member of G_C. Conversely, every member of G_C extends to that of G, since G_C is the 1-component of K.

□

3. 4 EXAMPLE. Let $M=SO(6)^{\sim}\cong SU(4)$ and S^m be a maximal S-sized sphere $\supset\{o, \gamma(o)\}$. Then S^m meets the centrosome $G^o{}_2(\boldsymbol{R}^6)\cong G_2(\boldsymbol{C}^4)$ in S^{m-1}, which meets the centrosome $U(2)\cong c \cdot SU(2)$ in S^{m-2}. One sees that S^{m-2} is either $c\cong S^1$ or $SU(2)\cong S^3$ by maximality; $m=3$ or 5. Hence there are exactly two congruence classes, $[S^5]$ and $[S^3]$, of the maximal S-sized sphere$\supset\{o, \gamma(o)\}$ by 3. 3. Notice that a sphere of $[S^5]$ meets some of $[S^3]$ in S^2; indeed the subspaces c and $SU(2)$ of $U(2)$ meet at a point p, one of the two and hence the uniqueness of c_H mentioned in 3. 2a, applied twice, gives that those spheres in $[S^5]$ and $[S^3]$ share an S^2 if they share p. This fact does not contradict 3. 2, of course; cf. the final remark in 2. 6b. The second example in 2. 6b is contained in this one: embed $SO(4)^{\sim}$ into $SO(6)^{\sim}$in the usual way and the two 3-spheres (which share S^2) are contained in S^5 and S^3 which share

the same S^2.

§ 4. Applications of Theorem 3.2; case studies.

We now apply Theorem 3. 2 and the theory briefly explained in the previous sections to the individual spaces, by using the information about the centrosomes in [N-2] (See 1. 3b); see 4. 12 for the go al of this section. As mentioned in 1. 3b, our method is to use geometric relationship between spaces systematically, which is hinted in the last chapter of [M] in the cases of the first two series of classical spaces (4. 2 and 4. 4). We begin with the unitary group $U(n)$, which is the most important case because of the next lemma.

4. 1 LEMMA. *If S^m contains a point o together with its pole p in M, then M is a subspace of $SU(2k)$ for some k in which p is still a pole of o.*

PROOF. First assume that M is a simple group. Then M is a subgroup of $SU(n)$ for some n. The point p lies in a polar of o in $SU(n)$. The corresponding meridian contains M and its semisimple part has the form $SU(2k)\times SU(n-2k)$. If o is located at $(1,1)$ in the meridian, then p is $(-1, 1)$ in it. And M is a subgroup of $SU(2k)$. If M is not a group but a simple space G/K, then M is a subspace of some finite covering group $G^\wedge$ of the simple group G (See Thm 1. 9 in [N-2]; the local result was known to E. Cartan). The point p is a pole of o in $G^\wedge$, since Cartan's homomorphism: $M\to G: x\to s_o\circ s_x$ carries o and p into a single point. And we are in the previous case. □

4. 2 PROPOSITION (*Case of $U(n)$ and $SU(n)$*). *Let $\Sigma^{(m)}$ be the set of the G-congruent classes of the maximal spheres S^m, $m\geqq 2$, in the unitary group $U(n)$ which contain $\{1, -1\}$ and have size S. Then the dimension m is an odd number $2i+1\geqq 3$ such that 2^i divides n. And there is a bijection of $\Sigma^{(m)}$ onto the set of the integers h satisfying $0\leqq h< 2^{-i}n-h$; the correspondence is explicitly given in the proof. Similarly for S^m in $SU(n)$; indeed every sphere S^m, $m\geqq 2$, in $U(n)$ is contained in $SU(n)$.*

PROOF. The centrosome $C(1, -1)$in $U(n)$ is the disjoint union of the subspaces which are isomorphic with Grassmannians $G_h(\boldsymbol{C}^n)$ of the h-dimensional linear subspaces of $\boldsymbol{C}^n$, which we identify with those subspaces here; these make the connected components of $C(1, -1)$. $S^{m-1}= S^m\cap C(1, -1)$ is a subspace of $G_{\frac{n}{2}}(\boldsymbol{C}^n)$, since the pole $\gamma(p)$ of a point p in $G_h(\boldsymbol{C}^n)\cap C(1, -1)$ lies in $G_{n-h}(\boldsymbol{C}^n)$. Now the centrosome of $G_k(\boldsymbol{C}^{2k})$ is $U(k)$, $k\geqq 1$; thus one has the " period 2"-sequence of centrosomes

$U(1)\subset\cdots\subset U(k)\subset G_k(\boldsymbol{C}^{2k})\subset U(2k)\subset\cdots$. If one repeatedly takes the intersections of the spheres with the centrosomes, one ends up with a circle S^1 in the unitary group $U(2^{-i}n)$, $2i=m-1$, and not in a Grassmannian, simply because the centrosome of $G_k(\boldsymbol{C}^{2k})$ is the connected space $U(k)$ and hence $S^1\cap U(k)$ would be contained in a circle, contramy to the maximality assumption of S^m (which implies the maximality of S^1 by Thm 3. 2). The intersection of our S^1 with the centrosome are two points in some subspace $G_h(\boldsymbol{C}^j)\cup G_{j-h}(\boldsymbol{C}^j)$, $j=2^{-i}n$. Two pairs of a point p and its pole $\gamma(p)$ in this centrosome are congruent to each other if and only if they lie in the same $G_h(\boldsymbol{C}^n)\cup G_{n-h}(\boldsymbol{C}^n)$. Hence S^m is completely classified by (h, i), $2i=m-1$ and $0\leqq h<2^{-i}n-h$, the correspondence being bijective. S^m, $m\geqq 2$, in $U(n)$ is necessarily a subspace of $SU(n)$. □

The Grassmannian $G_h(\boldsymbol{C}^n)$ contains poles if and only if $n=2h$.

4. 3 Corollary (*Case of* $G_n(\boldsymbol{C}^{2n})$). *Let* $\Sigma^{(m)}$ *be the set of the* G-*congruence classes of the maximal spheres* S^m, $m\geqq 2$, *in* $G_n(\boldsymbol{C}^{2n})$ *which contain a fixed point together with its pole and have size S. Then the dimension m is an even number* $2i+2\geqq 2$ *such that* 2^i *divides n. And there is a bijection of* $\Sigma^{(m)}$ *onto the set of the integers h satisfying* $0\leqq h<2^{-i}n-h$.

For other spaces, one has a sequence of "period 8", which is schematically described by the diagram below, the arrows indicating embeddings as centrosomes.

$$\begin{tikzcd}
& G_n(\boldsymbol{R}^{2n}) \arrow[r] & UI \arrow[dr] & \\
O \arrow[ur] & & & CI \arrow[d] \\
OIII \arrow[u] & & & Sp \arrow[dl] \\
& UII \arrow[ul] & G_m(\boldsymbol{H}^{2m}) \arrow[l] &
\end{tikzcd}$$

More precisely, the poles and hence the centrosomes are unique for given points in these spaces and the real Grassmannian $G_n(\boldsymbol{R}^{2n})$ is one of the components, $G_k(\boldsymbol{R}^{2n})$, of the centrosome in $UI(2n)=U(2n)/O(2n)$, $UI(n)$ the centrosome of $CI(n)=Sp(n)/U(n)$, $CI(n)$ that of $Sp(n)$, $Sp(n)$ that of the quaternion Grassmann manifold $G_n(\boldsymbol{H}^{2n})$, $G_n(\boldsymbol{H}^{2n})$ one of the components, $G_k(\boldsymbol{H}^{2n})$, of the centrosome in $UII(2n)=U(4n)/$

$Sp(2n)$, $UII(n)$ that of $DIII(2n)=SO(4n)/U(2n)$, $OIII(2n)=O(4n)/U(2n)$ that of $SO(2n)$, and $O(n)$ that of $G_n(\boldsymbol{R}^{2n})$. The selected components are those which satisfy the conditions in Theorem 3. 2. One has the next proposition.

4. 4 PROPOSITION (*Case of those* 8 *spaces*). *Let* $M(n)$ *be one of the* 8 *spaces in the first line of the table below. We write* $n=2^q w$, *w odd, and* $q=4a+b$, $b=0, 1, 2$ *or* 3. *Then* (i) *the table gives the highest dimension,* $8a+k$, *of the spheres* $S^m \subset M(n)$ *of size S which contain the unique poles* $\gamma(o)$ (*in* $M(n)$) *of points* $o \in S^m$. (ii) *If* $0 \leqq a' \leqq a$, *then* $M(n)$ *also contains the spheres* $S^m \subset M(n)$ *of that property,* $m=8a'+b'$, *where* b' *is any number in the column of b such that the entry of the line of* b' *and of the column of* $M(n)$ *is not marked* (2) *and, in case* $a'=a$, *it has to satisfy an additional condition* $b' \leqq b$. *Finally,* (iii) *two spheres of equal dimensions among these spheres belong to the same congruence class if and only if they end up with the same* (*or congruent, strictly speaking*) *pair of connected components of a centrosome after one successively takes the intersections with the centrosomes.*

b	$UI(n)$	$CI(n)$	$Sp(n)$	$G_n(\boldsymbol{H}^{2n})$	$UII(n)$	$DIII(2n)$	$SO(n)$	$G_n(\boldsymbol{R}^{2n})$
0	$8a+1$	$8a+2$	$8a+3$	$8a+4$	$8a+1$	$8a$	$8a$ (2) F	$8a+1$ (2) F
1	$8a+2$ (2) F	$8a+3$ (2) F	$8a+4$ (2) F	$8a+5$ (2) F	$8a+5$	$8a+2$	$8a+1$ (2)	$8a+2$ (2)
2	$8a+3$ (2)	$8a+4$ (2)	$8a+5$ (2)	$8a+6$ (2)	$8a+6$ (2) F	$8a+6$	$8a+3$	$8a+4$
3	$8a+5$	$8a+6$	$8a+7$	$8a+8$	$8a+7$ (2)	$8a+7$ (2) F	$8a+7$	$8a+8$

4. 5 REMARK. If the entry $m=8a+k$ is marked (2) in the column of $M(n)$, then the corresponding sphere S^m gives the generator of the homotopy group $\pi_m(M(n))$ which is of order 2. If not, $\pi_m(M(n))$ is infinite cyclic and S^m generates it. Our table is naturally compatible with the well known periodicity of the homotopy groups of those 8 spaces in the stable range. Cf. [F-1], to which one could add the spheres corresponding to the mark (F). In order to obtain the spheres, subspaces of $M(n)$, in [F-1], one has only to look at every meridian M^- in $M(n)$ and find the sphere of the above property in M^-.

4. 6 REMARK. Wolf [W] classified the spheres of the above property (characterized differently as the isoclinic ones) for $M(n)=G_n(\boldsymbol{R}^{2n})$ and

$G_n(\boldsymbol{H}^{2n})$ as well as $G_n(\boldsymbol{C}^{2n})$. His results agrees perfectly with ours. If $M(n)$ is $G_n(\boldsymbol{H}^{2n})$, for instance, one has $b'=0$ or 3 and the number of the congruence classes of the $(8a'+b')$-spheres equals $\left[2^{-4a'-b'-1}n+\frac{1}{2}\right]$. His index thus has the explained geometric meaning. Also one sees that the isoclinic spheres generate $\pi_m(M(n))\cong\boldsymbol{Z}$.

4. 7 REMARK. We like to point out certain details. The centrosome $C(o, \gamma(o))$ is disconnected if and only if $M(n)$ is $UI(n)$, $UII(n)$, $SO(n)$ (with n even) or $G_n(\boldsymbol{R}^{2n})$. There are more than one components of $C(o, \gamma(o))$ that are the closest to o if and only if $M(n)$ is $SO(n)$ (with n even) or $G_n(\boldsymbol{R}^{2n})$; they are $OIII(n/2)=DIII(n/2) \amalg DIII(n/2)$ and $O(n) =SO(n) \amalg SO(n)$ respectively. However, the two components of $OIII(n/2)$ are congruent in $G_n(\boldsymbol{R}^{2n})\supset SO(n)$, and similarly for those of $O(n)$ in $UI(n)$.

Among the above 8 spaces, $SO(n)$ and $G_n(\boldsymbol{R}^{2n})$ have the fundamental groups of order 2. We now examine the spheres in their universal covering spaces $SO(n)^{\sim}=Spin(n)$ and $G^o{}_n(\boldsymbol{R}^{2n})$. Let $\{1, \varepsilon\}$ be the kernel of the projection $\pi: SO(n)^{\sim}\to SO(n)$. ε is the only pole of 1 in $SO(n)^{\sim}$ unless $n=4k$ is a multiple of 4. Let $\delta=\delta_n$ denote a member of $SO(n)^{\sim}$ which projects onto -1, if n is $4k$. Also we will study $SO^{\#}(n):= SO(n)^{\sim}/\{1, \delta_n\}$ and $G^{\#}{}_n(\boldsymbol{R}^{2n}):=G^o{}_n(\boldsymbol{R}^{2n})/\{1, \delta_{2n}\}$ later, which are isomorphic with the quotient spaces by $\{1, \delta\varepsilon\}$.

4. 8 PROPOSITION (*Case of* $Spin(n)$, $G^o{}_p(\boldsymbol{R}^n)$, $SO^{\#}(n)$ *and* $G^{\#}{}_n(\boldsymbol{R}^{2n})$). (i) *A maximal sphere of size S passing* 1 *and* ε *in* $SO(n)^{\sim}$ *is* S^3 *or* S^{n-1}; *there are exactly two congruence classes.* (ii) *The spheres (of size S or not) passing* 1 *and* δ (*or* $\delta\varepsilon$) *in* $SO(n)^{\sim}$ *project onto spheres in* $SO(n)$ *isomorphically*; *they are in a one-to-one correspondence with the spheres passing* ±1 *in* $SO(n)$. (iii) *For* $G^o{}_p(\boldsymbol{R}^n)\subset SO(n)^{\sim}$, *it is* S^p *or* S^{n-p} *if the pole is* ε *and it projects into a sphere in* $G_p(\boldsymbol{R}^n)$ *similarly.* (iv) *These spheres through* 1 *and* ε *in* $SO(n)^{\sim}$ *project into spheres of that property in* $SO^{\#}(n)$ *and* $G^{\#}{}_p(\boldsymbol{R}^n)$, $2p=n$, *respecively, and vice versa.*

PROOF. The right (that is, the nearest to 1) component of the centrosome $C(1, \varepsilon)$ in $SO(n)^{\sim}$ is $G^o{}_2(\boldsymbol{R}^n)$ (See 1. 3a and 3. 2a), of which the centrosome is $G^o{}_1(\boldsymbol{R}^2)\cdot G^o{}_1(\boldsymbol{R}^{n-2})\cong S^1\cdot S^{n-3}$. The centrosome of $G^o{}_p(\boldsymbol{R}^n)$ is the disjoint union of $G^o{}_a(\boldsymbol{R}^p)\cdot G^o{}_b(\boldsymbol{R}^{n-p})$, $a+b=p$ and b odd. The right component corresponds to $b=1$, which is $S^{p-1}\cdot S^{n-p-1}$. The sphere through 1 and δ or $\delta\varepsilon$ project onto spheres through 1 and -1 in

$SO(n)$, since the kernel of the projection is $\{1, \varepsilon\}$; conversely those spheres in $SO(n)$ lift to spheres through 1 and δ or $\delta\varepsilon$ in $SO(n)^{\sim}$. (iii) and (iv) are similarly proved. □

We now consider the quotient groups $SU(n)/\boldsymbol{Z}_p$, p a divisor of n. $SU(n)/\boldsymbol{Z}_p$ has a pole p_p if and only if n/p is even.

4. 9 PROPOSITION (*Case of $SU(n)/\boldsymbol{Z}_p$ and the other spaces of root system A_{n-1}*). *Suppose $SU(n)/\boldsymbol{Z}_p$ has a pole p_p of 1. If p is odd, the congruence classes of the maximal spheres S^m, $m \geqq 2$, of size S and containing $\{1, p_p\}$ in $SU(n)/\boldsymbol{Z}_p$ correspond bijectively to those containing ± 1 in $SU(n)$ by the projection. If p is even, $SU(n)/\boldsymbol{Z}_p$ has no such sphere S^m, $m \geqq 2$. Similarly for the other spaces of type A.*

PROOF. Notice that pole -1 of 1 in $SU(n)$ projects to that of 1 in $SU(n)/\boldsymbol{Z}_p$ in case p is odd and it projects to 1 in the other case. Let S^m, $m \geqq 2$, be a sphere in $SU(n)/\boldsymbol{Z}_p$ which passes through 1 and its pole. S^m lifts to a sphere $\ni 1$ in $SU(n)$. The pole of 1 in the lift is either a pole in $SU(n)$ or a point of another polar. The second case does not occur, since a polar of positive dimension projects to a polar of positive dimension. Conversely, a sphere containing ± 1 in $SU(n)$ projects to a sphere in $SU(n)/\boldsymbol{Z}_p$ in case p is odd and it projects to a real projective space in the even case. The size is well preserved under the projection; and so is congruence. □

4. 10 PROPOSITION (*Case of E_7 and EV*). *There are two congruence classes of the maximal spheres S^m, $m \geqq 2$, of size S and passing ± 1 in E_7. The dimension $m=3$ for both. In $EV \subset E_7$, there is only one congruence class of the maximal spheres S^m, $m \geqq 2$, of size S and passing ± 1; $m=2$.*

PROOF. The centrosome of the (1-connected) E_7 is $EV \Vert EVII$, of which $EVII$ is the right one (3. 2a). The centrosome of $EVII$ is $T \cdot EIV$ and that of $T \cdot EIV$ is the union of two points and $2 \times FII$, the disjoint union of 2 copies of FII. Those two points are the poles of each other, while the pole of a point in a component of $2 \times FII$ lies in the other. The minimality condition in Theorem 3. 2 can be verified for both of these cases directly or by observing $U(3) \subset U\,II(3) \subset T \cdot EIV$ in view of 4. 2. $EV \subset E_7$ has the centrosome $AI(8)/\boldsymbol{Z}_2 \Vert AII(4)/\boldsymbol{Z}_2$, of which $AII(4)/\boldsymbol{Z}_2$ is the right one, because this is contained in $EVII$ and because EV has the same root system as E_7. The centrosome of $AII(4)/\boldsymbol{Z}_2$ is $G_1(\boldsymbol{H}^4) \Vert$

$G_3(\boldsymbol{H}^4)$, both of which have no pole. Hence EV has only one class as stated. □

4. 11 PROPOSITION (*Case of EVII*). *There are two congruence classes of the maximal spheres* S^m, $m \geqq 2$, *of size S and passing poles in EVII each containing* 2-*spheres.*

PROOF. This is immediate from the preceding proof. □

4. 11a REMARK. One may note the subspace $DIII(6) \subset EVII$ has the same property.

4. 12 A concluding remark. The spheres of size S have been completely classified. Their congruence classes correspond bijectively with the pairs of their dimensions and the Wolf indices in our generalized sense.

§ 5. The other spheres.

It remains to determine the maximal spheres S^m in M which are not of size S and yet pass through a point o along with its pole $\gamma(o)$ in M. However, our classifications in this section will be somewhat less than complete. In the next proposition of elementary nature, we use an embedding of $M = G/K$ into a finite covering $G^\wedge$ of G (as in the proof of 4. 1); in case M is a group, we choose $G^\wedge = M \times M$ and identify M with its subspace $\{(b, b^{-1}) | b \in M\}$.

5. 1 PROPOSITION. *Let* S^m, $m \geqq 2$, *be a maximal sphere in a compact space* $M = G/K$ *which contains the pole* $\gamma(p)$ *of a point* $p \in S^m$. *Then* (i) *the group* $G^\wedge \supset M$ *contains a subgroup* $G^\wedge(S^m) = SO(m+1)^\sim$ *or* $SO(m+1)^\#$ *which contains* S^m; $\varpi_1 = \varpi_1(O(m+1))$ *is a tangent vector to* S^m *up to conjugacy in the notation of* [*B*]. (ii) $\gamma(p)$ *is a pole of* p *in* $G^\wedge$. (iii) *If* M *is a group, them* $m \geqq 3$ *and, if* $m > 3$, M *contains* $G^\wedge(S^m)$. (iv) *In case* $m = 3$ *and* M *is a group,* S^3 *is congruent with a subgroup if and only if the restriction of the projection*: $M \times M \to M$ (*onto either one of the* 2 *copies of* M) *to* $G^\wedge(S^m)$ *has the image of* 3 *dimensions.*

PROOF. M is a subspace of a covering group $G^\wedge$. For the same reason, S^m is a subspace of $G^\wedge(S^m)$, the counterpart of $G^\wedge$ for S^m. $G^\wedge(S^m)$ is a subgroup of $G^\wedge$ if one agrees that S^m contains the unit element 1 of $G^\wedge$. We show that the pole $\gamma(1)$ of 1 in S^m is one in $G^\wedge$. By the assumption $\gamma(1)$ is a pole in M. Hence M is contained in the meridian $G^{\wedge -}(\gamma(1))$, a subgroup of $G^\wedge$. Since the Lie algebra of $G^\wedge$ is generated by the tangent space $\boldsymbol{m} = T_1M \subset T_1G^\wedge$, one sees $G^{\wedge -}(\gamma(1)) = G^\wedge$; $\gamma(1)$ is a pole in $G^{\wedge}$ consequently. In particular $\gamma(1)$ is a pole in $G^{\wedge}(S^m)$. Since $SO(m+1)$ does not contain $S^m = SO(m+1)/SO(m)$, the group $G^{\wedge}(S^m)$ is either $SO(m+1)^{\sim}$ or $SO(m+1)^{\#}$; $G^{\wedge}(S^m)$ can act on S^m as the connected automorphism group. Clearly ϖ_1, the first fundamental weight [B], is a tangent vector to S^m; we always identify a linear form on a metric vector space V with a member of V. Now assume M is a group. If $m+1 \neq 4$, $G^{\wedge}(S^m)$ is simple, and hence the first projection: $M \times M \to M$ carries $G^{\wedge}(S^m)$ onto a subgroup which is isomorphic with $SO(m+1)^{\sim}$ or $SO(m+1)^{\#}$; in fact $(\gamma(1), \gamma(1))$ projects to $\gamma(1) = \varepsilon$. If $m=2$ in particular, M thus contains $SO(3)^{\sim} \cong S^3$ which in turn contains a congruent S^2 with the given S^m, contrary to its maximality. Finally assume $m=3$. Then it can happen that $G^{\wedge}(S^m)$ projects onto $SO(3)^{\sim}$, in which case S^3 is congruent with this group. Otherwise the image is $SO(4)^{\sim} = Sp(1) \times Sp(1)$ or $SO(4)^{\#} = Sp(1) \times SO(3)$. Let p denote an epimorphism of the first factor $Sp(1)$ onto the second. Then the sphere S^3 in question is congruent with the subspace $\{(b, pb^{-1}) | b \in Sp(1)\}$. $\square$

5. 2 REMARK. There is something subtle about this. Let $M = Sp(1) \times Sp(1)$. If S^3 is a subgroup, then it is congruent with one of these factors or the diagonal subgroup $\{b, pb) | b \in Sp(1))\}$, where p may be taken as the identity. But M admits the group automorphism: $(b, c) \to (c, b)$, by which those factors are congruent each other. Also the space M admits the automorphisms defined by the point symmetry s_1: $b \to b^{-1}$, by which the subspace $\{b, pb^{-1}) | b \in Sp(1)\}$ is congruent with the diagonal subgroup. Thus there are only two different congruence classes of the 3-spheres; those two are different because of the difference in curvature. But if M is a subspace of some space N, then the above automorphism $1 \times s_1$ does not necessarily extend to that of N. In the proposition and other places, we mean by congruence the one in terms of the connected automorphism group G unless otherwise mentioned.

5. 3 REMARK. If M is a simple connected group and contains $S^3 \supset \{1, \gamma(1)\}$, then M is one of $SU(2n)/\boldsymbol{Z}_k$, $Sp(n)$, $SO(2n)$, $SO(n)^{\sim}$, $SO(4w)^{\#}$ and E_7. The divisor k of $2n$ here is odd; in fact S^3 lifts to a sphere and every circle$\supset \{1, \gamma(1)\}$ on it lifts to a circle c, which shows that the point -1 coresponding to $\gamma(1)$ on c is involutive. Besides the congruence classes of the spheres $S^3 \supset \{1, \gamma(1)\}$ in $SU(2n)/\boldsymbol{Z}_k$ correspond bijectively with those containing $\{\pm 1\}$ in $SU(2n)$; thus one has only to investigate $SU(2n)$ among the groups $SU(2n)/\boldsymbol{Z}_k$, k odd divisor of $2n$. One can effectively use the representation theory to determine the congruent classes of the spheres. In case M is not a group, the connected simple

space M has a pole if and only if M has the same root system and the same center [N-2] as one of the above groups.

5. 4 PROPOSITION. *Let* $M=SU(2n)$. *Then* (i) *every subgroup* $S^3\supset\{\pm 1\}$ *is conjugate with the diagonal in the product of the principal subgroup* $SU(2)$ *of subgroups* $SU(2n_k)$, $\sum_k n_k=n$ *and* $\prod_k SU(2n_k)\subset SU(2n)$. (ii) *If a subspace* $S^3\supset\{\pm 1\}$ *is not congruent with a subgroup, then a minimal subgroup* $G^\wedge(S^3)$ *is determined similarly (See* 5. 2 *and the proof below).* (iii) *The other spheres* $S^m\supset\{\pm 1\}$, $m>3$, *are determined by faithful representations* ρ_k: $SO(m+1)^\sim$ *or* $SO(m+1)^\#\to SU(2n_k)$, $\sum_k n_k=n$: *thus, if* $m+1=2r+1$ *is odd for instance, the highest weight of* ρ_k *is of the form* $\sum_j h^j\varpi_j$, h^r *odd, in the notation of* [*B*]. (iv) *In all these cases, the congruence classes are completely determined thereby.*

PROOF. (i) The subgroup $S^3=SU(2)$ defines the tautological representation on $\boldsymbol{C}^{2n}$ through the inclusion map into $SU(2n)$. The linear space $\boldsymbol{C}^{2n}$ is the direct sum $\oplus_k V_k$ of simple $SU(2)$-modules V_k. Since $-1\in SU(2)$ acts on V_k as such, V_k has an even dimension $2n_k$, $\sum_k n_k=n$. A subgroup $SU(2)$ of $SU(2n_k)$ is irreducible on V_k if and only if it is principal; in particular the representation is unique up to conjugacy. Collecting all these representations, one obtains the description of $SU(2)$ in the theorem. (ii) Similarly, $\boldsymbol{C}^{2n}$ is the direct sum of simple $G^\wedge(S^3)$-modules V_k. Since $G^\wedge(S^3)$ is the direct product $G_1\times G_2=Sp(1)\times Sp(1)$ or $Sp(1)\times SO(3)$, each V_k is the tensor product $U_k\otimes W_k$ of a simple G_1-module U_k and a simple G_2-module W_k; here a possible trivial action on $\boldsymbol{C}=U_k$ or W_k is allowed. Since $-1\in G^\wedge(S^3)$ acts on V_k as such, G_1 or G_2 must be faithfully represented on U_k or W_k; that is, either G_1 or G_2 is a principal subgroup of $SU(U_k)$ or $SU(W_k)$ respectively. Since $G^\wedge(S^3)$ is a subgroup of $SU(2n)$, $G_1=Sp(1)$ must be a principal for some k and G_2 must be faithfully represented on W_k for another k. (iii) and (iv) must be obvious by now. □

5. 5 REMARK. In (iii), S^m has size S if and only if $(h^j)=(0,0,\cdots,0,1)$ with $n_k=n$, provided $m+1$ is odd.

5. 6 REMARK. For the other classical groups $SO(2n)$, $SO(n)^\sim$, $SO(4k)^\#$, and $Sp(n)$, one can determine the conjugate classes of the spheres$\supset\{1,\gamma(1)\}$ similarly to 5. 4 with slightly more representation theory. $SO(2n)$ and $Sp(n)$ are subgroups of $SU(2n)$. It is well known (See [D_0], e.g.) how to determine whether a representation: $G\to SU(2n)$ has the image in $SO(2n)$ or $Sp(n)$. The congruence classes of $S^m\supset\{1,-1\}$ in $SO(2n)$ correspond bijectively with those of $S^m\supset\{1,\delta\}$ in $SO(2n)^\sim$

and with those of $S^m \supset \{1, \delta\varepsilon\}$ in it by the projection, while those of $S^m \supset \{1, \varepsilon\}$ in $SO(4k)^{\#}$ correspond with those of $S^m \supset \{1, \varepsilon\}$ in $SO(2n)^{\sim}$ similarly.

5. 7 PROPOSITION. *Let $M = E_7$. Then* (i) *every maximal sphere $S^m \supset \{1, \gamma(1)\}$ in M has dimension $m \leqq 3$.* (ii) *As to the Dynkin numbers (d^j) of $S^3 \supset \{1, \gamma(1)\}$, d^j are all even and one of d^2, d^5 and d^7 equals* 2. *Assume in* (iii) *and* (iv) *that the given $S^3 \supset \{1, \gamma(1)\}$ is not congruent with a group.* (iii) *If $G^\wedge(S^3) = SO(4)^{\sim}$, then one has $G^\wedge(S^3) \subset Sp(1) \cdot SO(12)^{\sim}$, a meridian in E_7; hence* 5. 4 *and* 5. 6 *apply. Thus there are two cases: either* $1°$ *$G^\wedge(S^3) \subset SO(12)^{\sim}$ or* $2°$ *$G^\wedge(S^3)$ meets $SO(12)^{\sim}$ in $Sp(1)$. In case of* $1°$, *the class of $G^\wedge(S^3)$ is unique; under the projection of $SO(12)^{\sim}$ onto $SO(12)$, $G^\wedge(S^3)$ projects to the group $SO(4)^{\#}$ described by $\boldsymbol{R}^3 \otimes \boldsymbol{R}^4 = \boldsymbol{R}^{12}$. In the case* $2°$, *$G^\wedge(S^3)$ projects to $Sp(1)$ in $SO(12)$ described by $\boldsymbol{R}^4 \oplus \boldsymbol{R}^4 \oplus \boldsymbol{R}^4$ or $\boldsymbol{R}^4 \oplus \boldsymbol{R}^3$.* (iv) *If $G^\wedge(S^3) = SO(4)^{\#}$, then $G^\wedge(S^3)$ is one of the subgroups listed below in* 5. 8.

PROOF. (i) As mentioned in the proof of 4. 10, the centrosome of the E_7 is $EV \perp\!\!\!\perp EVII$. That of EV is $AI(8)/\boldsymbol{Z}_2 \perp\!\!\!\perp AII(4)/\boldsymbol{Z}_2$; hence the given S^m meets EV in a sphere of dimension $\leqq 2$ by 5. 3. The centrosome of $EVII$ is $T \cdot EIV$, of which EIV has the center of order 3. And (i) is proven. (ii) is immediate from the fact that $\gamma(1)$ lies in the direction of ϖ_7. (iii) Since $SO(4)^{\sim}$ has the center C of order 4, it meets a polar of 1 (other than the pole $\gamma(1)$) at a member δ of C. Hence it is contained in the meridian of the polar $M^+(\delta)$; $M^+(\delta)$ is EVI whose meridian is $Sp(1) \cdot SO(12)^{\sim}$. The rest of (iii) is easy to see. (iv) We have no simple argument to deal with the case of $G^\wedge(S^3) = SO(4)^{\#}$. The 3-dimensional subgroups of E_7 are listed in table 19 of [D]; by the way, there is a misprint in that table: the characteristic for the subgroup of index 60 has Dynkin number $d^i = 2$ at the extreme right (i. e. $2\varpi_i$ should be added to H). On the other hand, one has $E_7 \subset Sp(28) \subset SU(56)$ by the representation ϖ_7. Its restriction to $SO(4)^{\#}$ gives the decomposition of $\boldsymbol{C}^{56}$ into the sum of the simple $SO(4)^{\#}$-modules as in 5. 8, which one could determine by picking up $Sp(1)$ and $SO(3)$ from Dynkin's table, reading off the restrictions ρ_1, ρ_2 of the representation ϖ_7 to these groups and checking whether or not the pairs of these groups are compatible. We have not yet verified these actually give subgroups of E_7, however. □

5. 8 Supplement. For instance, pick up $Sp(1)$ and $SO(3)$ with indices (3″) and (8) from Dynkin's table 19. Then ρ_1 is $\boldsymbol{C}^4 \otimes \boldsymbol{C} \oplus 26\boldsymbol{C}^2 \otimes \boldsymbol{C} = \boldsymbol{C}^{56}$, while ρ_2 is $2\boldsymbol{C} \otimes \boldsymbol{C}^5 \oplus 14\boldsymbol{C} \otimes \boldsymbol{C}^3 \oplus 4\boldsymbol{C} \otimes \boldsymbol{C} = \boldsymbol{C}^{56}$. Out of these, one

could get the simple $SO(4)^{\#}$-modules $\boldsymbol{C}^4\otimes\boldsymbol{C}\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^5\oplus 7\boldsymbol{C}^2\otimes\boldsymbol{C}^3=\boldsymbol{C}^{56}$.
From (3″), (36′) : $\boldsymbol{C}^4\otimes\boldsymbol{C}\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^9\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^7\oplus 2\boldsymbol{C}^2\otimes\boldsymbol{C}^5$.
(3″), (156) : $\boldsymbol{C}^4\otimes\boldsymbol{C}\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^9\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^{17}$.
(7), (24) : $\boldsymbol{C}^4\otimes\boldsymbol{C}^7\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^5\oplus 3\boldsymbol{C}^2\otimes\boldsymbol{C}^3$.
(7), (28) : $\boldsymbol{C}^4\otimes\boldsymbol{C}^7\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^7\oplus 7\boldsymbol{C}^2\otimes\boldsymbol{C}$.
(7), (56) : $\boldsymbol{C}^4\otimes\boldsymbol{C}^7\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^{11}\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^3$.
(15), (24) : $\boldsymbol{C}^6\otimes\boldsymbol{C}^3\oplus\boldsymbol{C}^4\otimes\boldsymbol{C}^7\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^5$.
(35), (28) : $\boldsymbol{C}^{10}\otimes\boldsymbol{C}\oplus\boldsymbol{C}^6\otimes\boldsymbol{C}^7\oplus\boldsymbol{C}^4\otimes\boldsymbol{C}$.
(31), (8) : $\boldsymbol{C}^8\otimes\boldsymbol{C}^3\oplus\boldsymbol{C}^6\otimes\boldsymbol{C}^3\oplus\boldsymbol{C}^4\otimes\boldsymbol{C}\oplus\boldsymbol{C}^2\otimes\boldsymbol{C}^5$.

We turn to the non-group spaces for short discussions. In continuation of 5. 3, if a simple space M is not a group but contain $S^m\supset\{o, \gamma(o)\}$, then we have to study (1) $AI(2n)$, or $AII(2n)$ if $R(M)=A_{2n-1}$; (2) $CI(n)$, $G_n(\boldsymbol{C}^{2n})$, $G_n(\boldsymbol{H}^{2n})$, $DIII(2n)$, or $EVII$ (for $n=3$) if $R(M)=C_n$; (3) $G_n(\boldsymbol{R}^{2n})$, $G^\circ_n(\boldsymbol{R}^{2n})=G_n(\boldsymbol{R}^{2n})^\sim$, or $G_n(\boldsymbol{R}^{2n})^{\#}$ (for $n=2w$) if $R(M)=D_n$; and (4)EV if $R(M)=E_7$.

In order to use the results in the case of groups, one could use the following known inclusions [N-2], in which the symbol $\to$ means inclusion as a component of a centrosome. (1) $SO(n)\to G_n(\boldsymbol{R}^{2n})$; (2) $U(n)\to G_n(\boldsymbol{C}^{2n})$; (3) $Sp(n)\to G_n(\boldsymbol{H}^{2n})$; (4) $G_n(\boldsymbol{R}^{2n})\subset G_n(\boldsymbol{C}^{2n})\subset G_n(\boldsymbol{H}^{2n})$; and (5) $SO(n)^\sim\to G_n(\boldsymbol{R}^{2n})^\sim$. If $S^m\supset\{o, \gamma(o)\}$ is a sphere in M, one of these Grassmannians, then S^{m-1} is a subspace in the group in the centrosome. Conversely, if S^{m-1} is a sphere in that group, then S^{m-1} is a subspace in M. And this way the spheres$\supset\{o, \gamma(o)\}$ in M would be determined for the following reason. Let S^m be a sphere in $SU(2n)$ which is obtained by a faithful representation: $SO(m+1)^\sim\to SU(2n)$ with the highest weight $\sum h^k\varpi_k$. If S^m is the equator of S^{m+1} in $G_{2n}(\boldsymbol{C}^{4n})$, then $SO(m+2)^\sim\supset S^{m+1}$ must be a subgroup of $SU(4n)$ by a representation such that a highest weight $\sum c^k\varpi_k$ restricts to $\sum h^k\varpi_k$ for $SO(m+1)^\sim$; if $m+1$ is, say, even $m+1=2r$, then one has $c^k=h^k$ for $1\leqq k<r$ and $h^r=c^{r-1}+c^r$. From Weyl's formula one concludes that the degree of the representation $\sum c^k\varpi_k$ is far greater than $4n$ in general.

For the rest of spaces, one could use further inclusions. (6) $G_n(\boldsymbol{R}^{2n})\to AI(2n)\subset SU(2n)$; (7) $SU(2n)\subset AII(2n)\subset G_n(\boldsymbol{H}^{2n})$; (8) $AI(n)\subset UI(n)\to CI(n)\to Sp(n)$ in which one notes that $S^m\supset\{o, \gamma(o)\}$ in $CI(n)$ meets $UI(n)$ in S^{m-1} contained in $AI(n)$; (9) $AII(n)\to DIII(2n)\subset OIII(2n)\to SO(4n)$; (10) $DIII(6)\subset EVII\to E_7$; and (11) $EV\to E_7$.

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