Remarks on L²-well-posed mixed problems for hyperbolic equations of second order

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§ 1. Introduction and results

We shall start this paper with a general situation which clarifies our problem.

In the open half space $\boldsymbol{R}_+^{n+1}=\{x=(x', x_n);\ x'=(x_0, x_1, \cdots, x_{n-1})\in\boldsymbol{R}^n,\ x_n>0\}$ with boundary $x_n=0$, we consider a boundary value problem (P, B_j):

$$\begin{aligned} &P(x, D)u=f && && \text{in } \boldsymbol{R}_+^{n+1}, \\ &B_j(x', D)u=g_j && (j=1, \cdots, l) && \text{on } \boldsymbol{R}^n. \end{aligned}$$

Here $D=(D_0, \cdots, D_n)$, $D_j=-i\dfrac{\partial}{\partial x_j}$, $P=P(x, D)$ is strictly x_0-hyperbolic operator of order m, $B_j=B_j(x', D)$ is a boundary differential operator of order $m_j<m$ and $m_j\neq m_k$ if $j\neq k$. Furthermore the hyperplane $x_n=0$ is non-characteristic for P and B_j. The coefficients of P and B_j are C^∞-functions and constant outside a compact set of $\boldsymbol{R}^{n+1}$.

In this paper we use the functional spaces $H_{k,\gamma}(\boldsymbol{R}_+^{n+1})$ and $H_{s,\gamma}(\boldsymbol{R}^n)$ with non zero real parameter γ as follows:

$$H_{k,\gamma}(\boldsymbol{R}_+^{n+1})=\left\{u;\ e^{-\gamma x_0}u\in H^k(\boldsymbol{R}_+^{n+1})\right\} \qquad (k\geqq 0:\ \text{integer}),$$

$$H_{s,\gamma}(\boldsymbol{R}^n)=\left\{u;\ e^{-\gamma x_0}u\in H^s(\boldsymbol{R}^n)\right\} \qquad (s:\ \text{real}),$$

with norms defined by

$$\|u\|_{k,\gamma}^2=\sum_{j+|\alpha|=k}\int_{\boldsymbol{R}_+^{n+1}}|e^{-\gamma x_0}\gamma^j D^\alpha u(x)|^2 dx,$$

$$\langle\!\langle u\rangle\!\rangle_{s,\gamma}^2=\int_{\boldsymbol{R}^n}|e^{-\gamma x_0}\Lambda^s u(x')|^2 dx'$$

respectively, where

$$\Lambda^s u(x')=(2\pi)^{-n}\int_{\boldsymbol{R}^n} e^{i\tau x_0+i\sigma x''}\Lambda^s(\gamma, \xi, \sigma)\hat{u}(\tau, \sigma)d\xi d\sigma,$$

$$\hat{u}(\tau, \sigma)=\int_{\boldsymbol{R}^n} e^{-i\tau x_0-i\sigma x''}u(x')dx',$$

$$\Lambda(\gamma, \xi, \sigma)=(|\tau|^2+|\sigma|^2)^{\frac{1}{2}}=(|\tau|^2+\sigma_1^2+\cdots+\sigma_{n-1}^2)^{\frac{1}{2}},$$

$$\tau=\xi-i\gamma\,,\quad \sigma x''=\sigma_1x_1+\cdots+\sigma_{n-1}x_{n-1}\,,\quad x'=(x_0,\,x'')\,,$$
$$\sigma\in \boldsymbol{R}^{n-1}\,.$$

Remark that the norm $\langle\!\langle u\rangle\!\rangle^2_{k,\gamma}$ is equivalent to

$$\sum_{j+|\alpha'|=k}\int_{\boldsymbol{R}^n}|e^{-\gamma x_0}\gamma^j D'^{\alpha'}u(x')|^2dx'$$

where $D^\alpha=D'^{\alpha'}D_n^{\alpha_n}=D_0^{\alpha_0}\cdots D_{n-1}^{\alpha_{n-1}}D_n^{\alpha_n}$.

In this paper we use the following

DEFINITION. *A boundary value problem (P, B_j) is L^2-well-posed if and only if there exist positive constants C and γ_0 such that for every $\gamma\geqq\gamma_0$, $f\in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$ and $g_j\in H_{m-m_j+\frac{1}{2},\gamma}(\boldsymbol{R}^n)$ (P, B_j) has a unique solution $u\in H_{m,\gamma}(\boldsymbol{R}^{n+1}_+)$ which satisfies*

$$\gamma^2\|u\|^2_{m-1,\gamma}\leqq C\Big(\|f\|^2_{0,\gamma}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j-\frac{1}{2},\gamma}\Big). \tag{1.1}$$

This definition is equivalent to one in [7] and, in the case of constant coefficients, is also equivalent to one in [2]. In fact, it is proved in [7] that an L^2-well-posed problem (P, B_j) has a unique solution u with zero initial data on $x_0=0$ provided $f=0$ and $g_j=0$ in $x_0<0$.

Let P^0 and B_j^0 be the principal parts of P and B_j respectively. Let $(P^0, B_j^0)_{y'}$ be the constant coefficient problem resulting from freezing the coefficients at a boundary point $(y', 0)$. Then we obtain the following

THEOREM 1. *If a variable coefficient problem (P, B_j) is L^2-well-posed, then each constant coefficient problem $(P^0, B^0{}_j)_{y'}$ is also L^2-well-posed.*

We shall consider the following

PROBLEM A. *Is the converse of Theorem 1 true?*

When P is of second order and the coefficients of B^0 are real valued, Problem A is affirmatively solved by a slight modification of [1]. For a class of L^2-well-posed problems $(P^0, B_j^0)_{y'}$ with uniform Lopatinskii condition, the problem is solved affirmatively in [6], [9], [10].

The aim of this paper is to give an affirmative answer to the problem in a certain second order case where B^0 is non-real.

Let

$$P(x,D)=-D_0^2+2\sum_{j=1}^{n}a_j(x)D_0D_j+\sum_{j,k=1}^{n}a_{j,k}(x)D_jD_k \tag{1.2}$$
$$+(\textit{lower order term})\,,\ (a_{nn}=1,\ a_{jk}=a_{kj})\,,$$

$$B(x',D)=D_n-\sum_{j=1}^{n-1}b_j(x')D_j-c(x')D_0+(\textit{lower order term}). \tag{1.3}$$

Lopatinskii determinant $R(x', \tau, \sigma)$ for (P^0, B^0) is defined as follows:

$$R(x', \tau, \sigma) = B^0\left(x', \tau, \sigma, \lambda^+(x', 0, \tau, \sigma)\right),$$
$$= \lambda^+(x', 0, \tau, \sigma) - \sum_{j=1}^{n-1} b_j(x')\sigma_j - c(x')\tau,$$
$$\tau = \xi - i\gamma, \quad \gamma \geqq 0, \quad \sigma \in \boldsymbol{R}^{n-1},$$

where $\operatorname{Im} \lambda^+ > 0$ and $\operatorname{Im} \lambda^- < 0$ if $\gamma > 0$ and $P^0(x, \tau, \sigma, \lambda) = (\lambda - \lambda^+(x, \tau, \sigma))(\lambda - \lambda^-(x, \tau, \sigma))$. When $\lambda^+(x', 0, \xi, \sigma)$ is simple, $R(x', \tau, \sigma)$ is written by the following form:

$$R(x', \tau, \sigma) = R_0(x', \xi, \sigma) + \gamma R_1(x', \xi, \sigma) + \gamma^2 R_2(x', \xi, \sigma, \gamma)$$

Let assume the following conditions:

(I) *$R(x', \xi, \sigma) \neq 0$ if $\lambda^+(x', 0, \xi, \sigma) = \lambda^-(x', 0, \xi, \sigma)$*

(II) Re $R_0(x', \xi, \sigma)\overline{R_1(x', \xi, \sigma)} \geqq 0$ *in a neighbourhood of a point* (x'_0, ξ_0, σ_0) *where* $R(x'_0, \xi_0, \sigma_0) = 0$.

Then we obtain the following

THEOREM 2. *Let the conditions (I) and (II) be fulfilled*[1)]. *Then Problem A is affirmatively solved.*

To prove Theorem 2 we use the following

THEOREM 3. *Under the same assumption as in Theorem 2, if each constant coefficient problem $(P, B)_{y'}$ is L^2-well-posed then there exist positive constants C and γ_0 such that it holds for every $\gamma \geqq \gamma_0$ and $u \in H_{2,\gamma}(\boldsymbol{R}^{n+1}_+)$*

$$\gamma^2 \|u\|^2_{1,\gamma} \leqq C(\|Pu\|^2_{0,\gamma} + \langle\!\langle Bu \rangle\!\rangle^2_{\frac{1}{2},\gamma}). \tag{1.4}$$

We shall finally remark on semigroup estimates of L^2-well-posed problems. For two cases mentioned above where Problem A was affirmatively solved, the semigroup estimate, i.e., the energy inequality with non-zero initial date, holds ([1], [10]). However, it is in general an open problem whether the semigroup estimate holds for an L^2-well-posed problem, even for the following simple example satisfying the assumption of Theorem 2 (§ 5):

$$P = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2},$$
$$B = \frac{\partial}{\partial x} - ib\frac{\partial}{\partial y}, \quad 0 < |b| < 1 \qquad (b\text{: real}).$$

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1) see Added in proof.

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§ 2. Proof of Theorem 1

Let P^* be the formal adjoint of P. Then by the assumptions on P and B_j there exist boundary differential operators C_j, B_k^* and C_k^* $(j=1,\cdots,l,k=l+1,\cdots,m)$ such that

$$(Pu, v)-(u, P^*v) = \sum_{j=1}^{l} \langle B_j u, C_j v\rangle + \sum_{k=l+1}^{m} \langle C_k^* u, B_k^* v\rangle,,$$
$$u,\ v\in C_0^\infty(\overline{\boldsymbol{R}_+^{n+1}}),$$

where both $\{B_j, C_k^*\}$ and $\{C_j, B_k^*\}$ are Dirichlet sets and $(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle$ are the inner products in $L^2(\boldsymbol{R}_+^{n+1})$ and $L^2(\boldsymbol{R}^n)$ respectively. We denote the order of B_k^* by m_k^*.

The following lemma is proved in [7], Theorem 1 and 2.

LEMMA 2. 1. *Suppose that (P, B_j) is L^2-well-posed. Then there exist constants C^*, C_1^* and γ_0^* such that for every $\gamma\geqq\gamma_0^*$, $f^*\in H_{1,-\gamma}(\boldsymbol{R}_+^{n+1})$* and *$g_k^*\in H_{m-m_k^*+\frac{1}{2},-\gamma}(\boldsymbol{R}^n)$ the dual boundary value problem (P^*, B_k^*) has a unique solution $v\in H_{m,-\gamma}(\boldsymbol{R}_+^{n+1})$ which satisfies*

$$\gamma^2\|v\|^2_{m-1,-\gamma}\leqq C^*\Big(\|f^*\|^2_{0,-\gamma}+\sum_{k=l+1}^{m}\langle\!\langle g_k^*\rangle\!\rangle^2_{m-m_k-\frac{1}{2},-\gamma}\Big),$$

$$\gamma^2\|v\|^2_{m,-\gamma}\leqq C_1^*\Big(\|f^*\|^2_{1,-\gamma}+\sum_{k=l+1}^{m}\langle\!\langle g_k^*\rangle\!\rangle^2_{m-m_k+\frac{1}{2},-\gamma}\Big).$$

Furthermore the solution u of (P, B_j) satisfies, in addition to (1. 1),

$$\gamma\,\|u\|^2_{m,\gamma}\leqq C_1\Big(\|f\|^2_{1,\gamma}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j+\frac{1}{2},\gamma}\Big)\qquad(\gamma\geqq\gamma_1),$$

where C_1 and $\gamma_1\geqq\gamma_0$ are constants.

i) The existence of a solution of $(P^0, B_j^0)_{y'}$.

LEMMA 2. 2. *Suppose that (P, B_j) is L^2-well-posed. Then for every $\mu>0$, $f\in H_{1,\mu}(\boldsymbol{R}_+^{n+1})$ and $g_j\in H_{m-m_j+\frac{1}{2},\mu}(\boldsymbol{R}^n)$ each constant coefficient problem $(P^\circ, B_j^\circ)_{y'}$ has a solution $u\in H_{m,\mu}(\boldsymbol{R}_+^{n+1})$ which satisfies*

$$\mu^2\|u\|^2_{m-1,\mu}\leqq C\Big(\|f\|^2_{0,\mu}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j-\frac{1}{2},\mu}\Big)$$

where the constant C is the same as one in (1. 1).

PROOF. We use a similar method in [3]. Let $\mu>0$ be arbitrary but fixed. Let

$$f_\varepsilon(x)=\varepsilon^{-m}f\left(\varepsilon^{-1}(x'-y'),\ \varepsilon^{-1}x_n\right),$$
$$g_{j,\varepsilon}(x)=\varepsilon^{-m_j}g_j\left(\varepsilon^{-1}(x'-y')\right)$$

where $\varepsilon>0$, $f\in H_{1,\mu}(\boldsymbol{R}^{n+1}_+)$ and $g_j\in H_{m-m_j+\frac{1}{2},\mu}(\boldsymbol{R}^n)$.

Then we see that $f_\varepsilon\in H_{1,\frac{\mu}{\varepsilon}}(\boldsymbol{R}^{n+1}_+)$ and $g_{j,\varepsilon}\in H_{m-m_j+\frac{1}{2},\frac{\mu}{\varepsilon}}(\boldsymbol{R}^n)$. In fact,

$$\|f_\varepsilon\|^2_{1,\frac{\mu}{\varepsilon}}=e^{-2\frac{\mu}{\varepsilon}y_0}\ \varepsilon^{-2m+n-1}\|f\|^2_{1,\mu}$$

and since

$$\hat{g}_{j,\varepsilon}\left(\xi-i\frac{\mu}{\varepsilon},\ \sigma\right)=e^{-i(\xi-i\frac{\mu}{\varepsilon})y_0-i\sigma y''}\ \varepsilon^{-m_j+n}\hat{g}_j(\varepsilon\xi-i\mu,\ \varepsilon\sigma),$$

we have

$$\begin{aligned}\langle\!\langle g_{j,\varepsilon}\rangle\!\rangle_{m-m_j+\frac{1}{2},\frac{\mu}{\varepsilon}}&=e^{-2\frac{\mu}{\varepsilon}y_0}\ \varepsilon^{-2m_j+2n}\int_{\boldsymbol{R}^n}\left\{\left(\frac{\mu}{\varepsilon}\right)^2+\xi^2+|\sigma|^2\right\}^{m-m_j+\frac{1}{2}}\\&\quad\times|g_j(\varepsilon\xi-i\mu,\ \varepsilon\sigma|^2d\xi d\sigma\\&=e^{-2\frac{\mu}{\varepsilon}y_0}\ \varepsilon^{-2m+n-1}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j+\frac{1}{2},\mu},\end{aligned}$$

Let $\varepsilon\leq\mu\gamma_1^{-1}$. Then by Lemma 2.1 there exists a unique solution $v_\varepsilon\in H_{m,\frac{\mu}{\varepsilon}}(\boldsymbol{R}^{n+1}_+)$ of the problem:

$$\text{(2.1)}\qquad\begin{aligned}&P(x,D)v_\varepsilon=f_\varepsilon &&\text{in } \boldsymbol{R}^{n+1}_+,\\&B_j(x,D)v_\varepsilon=g_{j,\varepsilon}\qquad(j=1,\cdots,l)&&\text{on } \boldsymbol{R}^n\end{aligned}$$

such that

$$\text{(2.2)}\qquad\left(\frac{\mu}{\varepsilon}\right)^2\|v_\varepsilon\|^2_{m-1,\frac{\mu}{\varepsilon}}\leqq C\left(\|f_\varepsilon\|_{0,\frac{\mu}{\varepsilon}}+\sum_{j=1}^{l}\langle\!\langle g_{j,\varepsilon}\rangle\!\rangle_{m-m_j-\frac{1}{2},\frac{\mu}{\varepsilon}}\right),$$

$$\text{(2.3)}\qquad\left(\frac{\mu}{\varepsilon}\right)^2\|v_\varepsilon\|^2_{m,\frac{\mu}{\varepsilon}}\leqq C_1\left(\|f_\varepsilon\|_{1,\frac{\mu}{\varepsilon}}+\sum_{j=1}^{l}\langle\!\langle g_{j,\varepsilon}\rangle\!\rangle^2_{m-m_j+\frac{1}{2},\frac{\mu}{\varepsilon}}\right).$$

Put $u_\varepsilon(x)=v_\varepsilon(y'+\varepsilon x',\varepsilon x_n)$. Using relations

$$\text{(2.4)}\qquad D^\alpha v_\varepsilon(y'+\varepsilon x',\ \varepsilon x_n)=\varepsilon^{-|\alpha|}D^\alpha u_\varepsilon(x),$$

it follows from (2.3) and changes of variables that

$$\mu^2\|u_\varepsilon\|^2_{m,\mu}\leqq C_1\left(\|f\|^2_{1,\mu}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j+\frac{1}{2},\mu}\right).$$

Hence there exists a weak limit $u_{\varepsilon_j}\rightarrow u(\varepsilon_j\rightarrow 0)$ in $H_{m,\mu}(\boldsymbol{R}^{n+1}_+)$ *so that*

$$\mu^2\|u\|^2_{m,\mu}\leqq C_1\left(\|f\|^2_{1,\mu}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j+\frac{1}{2},\mu}\right).$$

The same argument gives from (2.2)

$$\mu^2\|u\|^2_{m-1,\mu}\leqq C\left(\|f\|^2_{0,\mu}+\sum_{j=1}^{l}\langle\!\langle g_j\rangle\!\rangle^2_{m-m_j-\frac{1}{2},\mu}\right).$$

From (2.1) and (2.4) we obtain

$$P(y'+\varepsilon x', \varepsilon x_n, \varepsilon^{-1}D)u_\varepsilon(x)=\varepsilon^{-m}f(x) \qquad \text{in } \boldsymbol{R}^{n+1}_+,$$

$$B_j(y'+\varepsilon x', \varepsilon^{-1}D)u_\varepsilon(x', 0)=\varepsilon^{-m_j}g_j(x', 0) \qquad (j=1,\cdots,l) \quad \text{on } \boldsymbol{R}^n$$

Hence, if $\varepsilon\to 0$ in the equations, then the u satisfies

$$P^0(y', 0, D)u=f \qquad \text{in } \boldsymbol{R}^{n+1}_+,$$

$$B^0_j(y', D)u=g_j \qquad (j=1,\cdots,l) \qquad \text{on } \boldsymbol{R}^n.$$

ii) The uniqueness of solutions of $(P^0, B^0_j)_{y'}$.

In virture of Lemma 2.1 and the proof of Lemma 2.2 we see that each dual constant coefficient problem $(P^{*0}, B^{*0}_k)_{y'}$ has a solution $v\in H_{m,\mu}(R^{n+1}_+)$ $(\mu<0)$. Therefore the uniqueness of solutions of $(P^0, B^0_j)_{y'}$ follows immediately from Green's formula

$$(P^0u, v)-(u, P^{*0}v)=\sum_{j=1}^{l}\langle B^0_ju, C^0_jv\rangle+\sum_{k=l+1}^{m}\langle C^{*0}_ku, B^{*0}_kv\rangle.$$

§ 3. Lemmas

In this section we shall state the properties of pseudo-differential operators with positive parameter γ ([4], [5], [8]) and the facts derived from a characterization of L^2-well-posed problems with constant coefficients ([2]).

Let $a(x'\xi, \sigma, \gamma)$ be a C^∞-function in $x'\in\boldsymbol{R}^n$, $\xi\in\boldsymbol{R}$, $\sigma\in\boldsymbol{R}^{n-1}$. That $a(x', \xi, \sigma, \gamma)$ belongs to a symbol class S^k_+ (k: real) means that for every α', α'', j it holds with a positive constant $C_{\alpha',\alpha'',j}$

$$(3.1) \qquad |D^{\alpha'}_{x'}D^j_\xi D^{\alpha''}_\sigma a(x', \xi, \sigma, \gamma)|\leqq C_{\alpha',\alpha'',j}(\gamma^2+\xi^2+|\sigma|^2)^{(k-j-|\alpha''|)/2},$$

for any $\gamma>0$, $(x', \xi, \alpha)\in\boldsymbol{R}^{2n}$. When $a(x', x_n, \xi, \sigma, \gamma)$ has a compact support in x_n and (3.1) holds uniformly in x_n, we say that a belongs to $\mathring{S}^k_+$. For $a\in S^k_+$ and $u\in H_{k,\gamma}(\boldsymbol{R}^n)$ we define a pseudo-differential operator $a(x', D', \gamma)$ by

$$a(x', D', \gamma)u(x')=a(x', D'_0, D'', \gamma)u(x')$$

$$=(2\pi)^{-n}e^{\gamma x_0}\int_{\boldsymbol{R}^n}e^{i\xi x_0+i\sigma x''}a(x', \xi, \sigma, \gamma)\hat{u}(\tau, \sigma)d\xi d\sigma,$$

$$D'_0=(D_0+i\gamma), \qquad \tau=\xi-i\gamma.$$

The well-known basic properties for ordinary pseudo-differential operators hold analogously for pseudo-differential operators with positive parameter γ. In particular, a sharp form of Gårding inequality plays an important role.

LEMMA 3.1. *Let* $a(x', \xi' \sigma, \gamma)\in S_{+}^{0}$ *and* $\operatorname{Re} a(x', \xi, \sigma, \gamma)\geqq 0$. *Then there exist a positive constant* C *such that for any* $u\in H_{0,\gamma}(\boldsymbol{R}^n)$ *and* $\gamma\geqq\gamma_0>0$

$$\operatorname{Re}\langle a(x', D', \gamma)u, u\rangle_{0,\gamma}\geqq -C\langle\!\langle u\rangle\!\rangle^2_{-\frac{1}{2},\gamma}.$$

COROLLARY 3.1. *Let* $a(x', \xi, \sigma, \gamma)\in S_{+}^{0}$ *and* $\operatorname{Re} a(x', \xi, \sigma, \gamma)\geqq c>0$. *Then there exist positive constants* C *and* γ_0 *such that for any* $\gamma\geqq\gamma_0$ *and* $u\in H_{0,\gamma}(\boldsymbol{R}^n)$

$$\operatorname{Re}\langle a(x', D', \gamma)u, u\rangle_{0,\gamma}\geqq C\langle\!\langle u\rangle\!\rangle^2_{0,\gamma}.$$

COROLLARY 3.2. *Let* $a(x', \xi, \sigma, \gamma)\in S_{+}^{1}$ *and* $\operatorname{Re} a(x', \xi, \sigma, \gamma)\geqq c\gamma,\ c>0$. *Then there exist positive constants* C *and* γ_0 *such that for any* $\gamma\geqq\gamma_0$ *and* $u\in H_{1,\gamma}(\boldsymbol{R}^n)$

$$\operatorname{Re}\langle a(x', D', \gamma)u, u\rangle_{0,\gamma}\geqq C\gamma\langle\!\langle u\rangle\!\rangle^2_{0,\gamma}.$$

Furthermore we enumerate the facts obtained by applying results in [2] and [11] to our second order problem.

Rewrite simply the characteristic polynomials corresponding to (1.2) and (1.3) as follows:

$$\text{(3.1)}\qquad \begin{aligned} P^0(x, \tau, \sigma, \lambda) &= \lambda^2-\alpha_1(x, \tau', \sigma')\Lambda\lambda+\alpha_2(x, \tau', \sigma')\Lambda^2 \\ &= \left(\lambda-\lambda^+(x, \tau', \sigma')\Lambda\right)\left(\lambda-\lambda^-(x, \tau', \sigma')\Lambda\right), \end{aligned}$$

$$\text{(3.2)}\qquad B^0(x', \tau, \sigma, \lambda) = \lambda-\beta(x', \tau', \sigma')\Lambda,$$

where $\Lambda=(|\tau|^2+|\sigma|^2)^{\frac{1}{2}}$, $\tau'=\xi'-i\gamma'=\tau\Lambda^{-1}$, $\sigma'=\sigma\Lambda^{-1}$, α_1 and α_2 are real valued for $\gamma'=0$. Then Lopatinskii determinant R and the reflection coefficient Q/R are written as

$$R(x', \tau' \sigma') = \lambda^+(x', 0, \tau', \sigma')-\beta(x', \tau', \sigma'),$$

$$\frac{Q(x', \tau', \sigma')}{R(x', \tau', \sigma')} = \frac{\lambda^-(x', 0, \tau', \sigma')-\beta(x', \tau', \sigma')}{R(x', \tau', \sigma')}$$

where $|\tau'|^2+|\sigma'|^2=1$, $\tau'=\xi'-i\gamma'$, $\gamma'\geqq 0$. Hereafter we denote the normalized variables by (τ', σ').

From [11], Theorem 3 we have the following

LEMMA 3.2. *Suppose that a constant coefficient problem* $(P^0, B^0)_{y'}$ *is* L^2*-well-posed. Then* $R(y', \tau, \sigma)\neq 0$ *if either* $\gamma>0$ *or* $\gamma=0$ *and* $\lambda^+(y', 0, \xi, \sigma)$ *is a real simple root.*

LEMMA 3.3. *Suppose that a constant coefficient problem* $(P^0, B^0)_{y_0'}$ *is*

L^2-well-posed. Then for every point (ξ_0', σ_0') satisfying $\operatorname{Im}\lambda^+(y_0', 0, \xi_0', \sigma_0')\neq 0$ *and* $R(y_0', \xi_0', \sigma_0')=0$ *there exist a constant* $C(\xi_0', \sigma_0')$ *and a neighbourhood* $U(\xi_0', \sigma_0')$ *in* $\gamma'>0$ *such that*

$$\left|\frac{Q(y', \tau', \sigma')}{R(y', \tau', \sigma')}\right| \leqq C(\xi_0', \sigma_0')\gamma'^{-1} \tag{3.3}$$

for any $(\tau', \sigma')\in U(\xi_0', \sigma_0')$.

PROOF. Applying Fourier transform to a constant coefficient problem $(P^0, B^0)_{y_0'}$, we obtain a boundary value problem for ordinary differential equations with parameters (τ, σ):

$$P^0(y_0', 0, \tau, \sigma, D_n)\hat{u}(\tau, \sigma, x_n) = \hat{f}(\tau, \sigma, x_n) \qquad \text{in } x_n>0,$$
$$B^0(y_0', \tau, \sigma, D_n)\hat{u}(\tau, \sigma, 0) = 0 \qquad \text{on } x_n=0.$$

If $R(y_0', \tau, \sigma)\neq 0$, then the compensating function $G(y_0', \tau, \sigma, x_n, z_n)$ is defined by the equation

$$\hat{u}(y_0', \tau, \sigma, x_n) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{e^{ix_n\lambda}\hat{f}(\tau, \sigma, \lambda)}{P^0(y_0', \tau, \sigma, \lambda)}\,d\lambda + \int_0^{\infty} G(y_0', \tau, \sigma, x_n, z_n)\hat{f}(\tau, \sigma, z_n)\,dz_n\right),$$

where

$$G(y_0', \tau, \sigma, x_n, z_n) = -\frac{e^{i\lambda^+(y_0', 0, \tau, \sigma)x_n}}{R(y_0', \tau, \sigma)}\int_\Gamma \frac{B^0(y_0', \tau, \sigma, \lambda)}{P^0(y_0', 0, \tau, \sigma, \lambda)}e^{-i\lambda z_n}\,d\lambda$$

and Γ denotes a closed Jordan curve in the lower half λ-plane enclosing $\lambda^-(y_0', 0, \tau, \sigma)$.

Form Lemma 3.2. we see that $R(y_0', \tau, \sigma)\neq 0$ if $\operatorname{Im}\tau=-\gamma<0$. Then Theorem 4.1, [2] shows that, $(P^0, B^0)_{y_0'}$ is L^2-well-posed if and only if for every (ξ_0', σ_0') with $R(y_0', \xi_0', \sigma_0')=0$ there exist a constant $C(\xi_0', \sigma_0')$ and a neighbourhood $U(\xi_0', \sigma_0')$ in $\gamma'>0$ such that

$$\|D_{x_n}^k G(y_0', \tau', \sigma', x_n, z_n)\|_{\mathscr{L}(L^2(z_n>0), L^2(x_n>0))} \leqq \frac{C(\xi_0', \sigma_0')}{\gamma'} \qquad (k=0, 1) \tag{3.4}$$

for any $(\tau', \sigma')\in U(\xi_0', \sigma_0')$, where $\|\cdot\|_{\mathscr{L}(L^2(z_n>0), L^2(x_n>0))}$ denotes the operator norm from $L^2(z_n>0)$ to $L^2(x_n>0)$.

Since $\operatorname{Im}\lambda^+(y_0', 0, \xi_0', \sigma_0')\neq 0$, the coefficients of P^0 are real and P^0 is of second order, we see that $\operatorname{Im}\lambda^-(y_0', 0, \xi_0', \sigma')\neq 0$. Hence we have by the Residue formula

$$G(y_0', \tau', \sigma', x_n, z_n) = -\frac{e^{i(\lambda^+ x_n - \lambda^- z_n)} Q(y_0', \tau', \sigma')}{(\partial_\lambda P)(y_0', 0, \tau', \sigma') R(y_0', \tau', \sigma')}.$$

in a neighbourhood of (ξ_0', σ_0') in $\gamma' > 0$, where $\lambda^\pm = \lambda^\pm(y_0', 0, \xi' \sigma')$. It follows from the simplicity of $\lambda^\pm$ and the definition of the operator norm that

$$\begin{aligned} \left| \frac{Q(y_0', \tau', \sigma')}{R(y_0', \tau', \sigma')} \right| & \\ &\leqq C_1(\xi_0', \sigma_0') \| e^{\overline{(i\lambda^+ x_n - \lambda^- z_n)}} G(y_0', \tau', \sigma', x_n, z_n) \|_{L^2(x_n>0, z_n>0)} \\ &\leqq C_1(\xi_0', \sigma_0') \| G(y_0', \tau', \sigma', x_n, z_n) \|_{\mathscr{L}(L^2(z_n>0), L^2(x_n>0))} \end{aligned} \tag{3.5}$$

in a neighbourhood of (ξ_0', σ_0').Therefore (3. 3) follows immediately from (3. 5).

§4. Proofs of Theorem 2 and 3

PROOF OF THEOREM 3. We reduce formally a problem (P, B) to a boundary value problem for a first order system.

Let us put for $u \in H_{2,\gamma}(\boldsymbol{R}_+^{n+1})$

$$V = \begin{pmatrix} \Lambda u \\ D_n u \end{pmatrix}, \quad N = \begin{pmatrix} 1, & 1 \\ \lambda^+, & \lambda^- \end{pmatrix}, \quad U = N^{-1} V = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then it follows from (3. 1) and (3. 2) that (P, B) becomes to

$$\begin{aligned} LU &= D_n U - K\Lambda U + (l.\ o.\ t.) && \text{in } \boldsymbol{R}_+^{n+1}, \\ Bu &= (R, Q) U + (l.\ o.\ t.) && \text{on } \boldsymbol{R}^n, \end{aligned} \tag{4.1}$$

where

$$K = \begin{pmatrix} \lambda^+, & 0 \\ 0, & \lambda^- \end{pmatrix}, \quad LU = N^{-1} \begin{pmatrix} 0 \\ Pu \end{pmatrix}.$$

Put

$$M = \begin{pmatrix} -m_1, & 0 \\ 0, & m_2 \end{pmatrix},$$

where $m_1 = c_1$ or $d_1 \gamma \Lambda^{-1}$, $m_2 = c_2$ or $d_2 \gamma^{-1} \Lambda$ respectively

and c_j, d_j $(j=1, 2)$ are positive constants determined later on. Then the integration by parts gives formally that

$$\begin{aligned} &2 \operatorname{Im} (LU, MU)_{0,\gamma} \\ &\quad = \langle U, MU \rangle_{0,\gamma} + 2 \operatorname{Im} (U, MK\Lambda U)_{0,\gamma} + R(U, U), \end{aligned} \tag{4.2}$$

where $R(U, U)$ is the sesquilinear form which satisfies

$$|R(U, U)| \leqq C \|U\|_{0,\gamma}^2 \quad \text{or} \quad C(\|u_1\|_{0,\gamma}^2 + \gamma^{-2} \|\Lambda u_2\|_{0,\gamma}^2).$$

Remark that

$$\text{(4. 3)} \qquad \operatorname{Im} \overline{MK} = \begin{pmatrix} m_1 \operatorname{Im} \lambda^+, & 0 \\ 0 \quad , & -m_2 \operatorname{Im} \lambda^- \end{pmatrix}$$

Since the coefficients of P and B is constant outside a compact set of $\boldsymbol{R}^{n+1}$ and the normalized space $\{(\tau', \sigma'); |\tau'|^2+|\sigma'|^2=1, \gamma'\geqq 0\}$ is compact, there exists a finite partition of the unity such that

$$\hat{u}(\tau, \sigma, x_n) = \sum_{j,k\geqq 1} \psi_j(\tau, \sigma)\varphi_k(x)\varphi_0(x_n)\hat{u}(\tau, \sigma, x_n) + \big(1-\varphi_0(x_n)\big)\hat{u}(\tau, \sigma, x_u),$$

where the φ_0, φ_k have compact supports, $\varphi_0=1$ in $0\leq x_n\leq \delta$, $\varphi_1=1$ for $|x|\to\infty$, $\psi'_j(\tau', \sigma')$ has a compact support in the normalized space and $\psi_j(\tau, \sigma)=\psi'_j(\tau\Lambda^{-1}, \sigma\Lambda^{-1})\in S^0_+$. From a priori estimate for a strictly hyperbolic operator P, we have for any $\gamma>0$

$$\text{(4. 4)} \qquad \gamma^2 \|(1-\varphi_0)u\|^2_{1,\gamma} \leqq C\|P(1-\varphi_0)u\|^2_{0,\gamma}\,.$$

Therefore considering $\hat{u}$ as $\psi_j\varphi_k\varphi_0\hat{u}$, we may assume that the support of $\hat{u}$ is contained in a neighbourhood of a point $(x'_0, 0, \tau_0, \sigma_0)$. Hereafter $(x'_0, \tau'_0, \sigma'_0)$ is arbitrary but fixed and by C denote positive constants depending on $(x'_0, \tau'_0, \sigma'_0)$.

We derive a priori estimate in each of the following four cases:

i) Case where $\gamma'_0>0$. Since P is of second order and strictly hyperbolic, $\lambda^\pm(x'_0, 0, \tau'_0, \sigma'_0)$ are simple and

$$\text{(4. 5)} \qquad |\operatorname{Im} \lambda^\pm(x, \tau', \sigma')| \geqq C \qquad (\gamma'>0)$$

in a neighbourhood of $(x'_0, 0, \tau'_0, \sigma'_0)$. We may assume from the simplicity of $\lambda^\pm$ that $\lambda^\pm(x, \tau', \sigma')$, $R(x', \tau', \sigma')$, $Q(x', \tau', \sigma'),\cdots$ belong to S^0_+ or $\mathring{S}^0_+$. By Lemma 3. 2 we have

$$\text{(4. 6)} \qquad |R(x', \tau', \sigma')| \geqq C \qquad (\gamma'>0)$$

in a neighbourhood of $(x'_0, \tau'_0, \sigma'_0)$. Using Corollary 3. 1 it follows from (4. 3), (4. 5) and (4. 6) that for $\gamma\geqq\gamma_1>0$

$$\text{(4. 7)} \qquad \langle\!\langle Ru_1\rangle\!\rangle^2_{0,\gamma} \geqq C\langle\!\langle u_1\rangle\!\rangle^2_{0,\gamma}\,,$$

$$\text{(4. 8)} \qquad \operatorname{Im}(U, MK\Lambda U)_{0,\gamma} \geqq C\|\Lambda^{\frac{1}{2}}U\|^2_{0,\gamma} \geqq C\gamma\|U\|^2_{0,\gamma}\,,$$

where m_j $(j=1, 2)$ are taken as constants c_j. The relations (4. 1) and (4. 7) gives

$$\langle\!\langle u_2\rangle\!\rangle^2_{0,\gamma} + \langle\!\langle Bu\rangle\!\rangle^2_{0,\gamma} + \langle\!\langle U\rangle\!\rangle^2_{-1,\gamma} \geqq C\langle\!\langle u_1\rangle\!\rangle^2_{0,\gamma}\,,$$

which implies for $\gamma\geqq\gamma_1$

$$\langle U, MU\rangle_{0,\gamma}=-c_1\langle\!\langle u_1\rangle\!\rangle^2_{0,\gamma}+c_2\langle\!\langle u_2\rangle\!\rangle^2_{0,\gamma}$$
$$\geqq\left(c_2-\frac{c_1+1}{C}\right)\langle\!\langle u_2\rangle\!\rangle^2_{0,\gamma}+\langle\!\langle u_1\rangle\!\rangle^2_{0,\gamma}-\frac{c_1}{C}\langle\!\langle U\rangle\!\rangle^2_{-1,\gamma}-\frac{c_1}{C}\langle\!\langle Bu\rangle\!\rangle^2_{0,\gamma}.$$

Let $c_2C>c_1+1$. Then it follows from this, (4. 2) and (4. 8) that for $\gamma\geqq\gamma_2\geqq\gamma_1$ and $\varepsilon>0$

$$\frac{1}{\varepsilon\gamma}\|LU\|^2_{0,\gamma}+\varepsilon\gamma\|U\|^2_{0,\gamma}+\langle\!\langle Bu\rangle\!\rangle^2_{0,\gamma}\geqq C(\langle\!\langle U\rangle\!\rangle^2_{0,\gamma}+\gamma\|U\|^2_{0,,}).$$

Hence this shows that for $\gamma\geqq\gamma_2$

$$\gamma^2\|u\|^2_{1,\gamma}+\gamma(\langle\!\langle u\rangle\!\rangle^2_{1,\gamma}+\langle\!\langle D_nu\rangle\!\rangle^2_{0,\gamma})\leqq C(\|Pu\|^2_{0,\gamma}+\gamma\langle\!\langle Bu\rangle\!\rangle^2_{0,\gamma}). \tag{4.9}$$

ii) Case where $\gamma_0'=0$ and $\lambda^+(x_0', 0, \xi_0', \sigma_0')$ is a real simple root. Since P is strictly hyperbolic we have

$$|\mathrm{Im}\,\lambda^\pm(x, \tau', \sigma'|\geqq C\gamma' \qquad (\gamma'>0)$$

in a neighbourhood of $(x_0', 0, \tau_0', \sigma_0')$. Moreover we have from Lemma 3. 2

$$|R(x', \tau', \sigma'|\geqq C \qquad (\gamma'>0)$$

in a neighbourhood of $(x_0', \tau_0', \sigma_0')$. Therefore the same argument as in (i) gives a priori estimate (4. 9), because (4. 7) and (4. 8) also hold by Corollary 3. 1 and 3. 2.

iii) Case where $\gamma_0'=0$ and $\lambda^+(x_0', 0, \xi_0', \sigma_0')=\lambda^-(x_0', 0, \xi_0', \sigma_0')$. In this case $\lambda^\pm, R,\cdots$ does not belong to our symbol classes. However, a priori estimate (4. 9) has been proved in [6], [9], because of the condition (I)

iv) Case where $\gamma_0'=0$ and $\mathrm{Im}\,\lambda^+(x_0', 0, \xi_0',\sigma_0')\neq0$. By the simplicity of λ^+, we see that $\lambda^\pm, R, Q,\cdots$ belong to S^0_+ or S^0_+ and

$$|\mathrm{Im}\,\lambda^\pm(x, \tau', \sigma')|\geqq C \qquad (\gamma'>0) \tag{4.10}$$

in a neighbourhood of $(x_0', 0, \xi_0', \sigma_0')$. If $R(x_0', \xi_0', \sigma_0')\neq0$ then a priori estimate (4. 9) is proved as in (i). Hence we may assume $R(x_0', \xi_0', \sigma_0')=0$. By the definitions of R and Q, the simplicity of $\lambda^\pm$ implies that $Q(x_0', \xi_0', \sigma_0')\neq0$ if $R(x_0', \xi_0', \sigma_0')=0$. Therefore it follows from Lemma 3. 3 that

$$|R(x_0', \tau', \sigma')|\geqq C\gamma' \qquad (\gamma'>0)$$

in a neighbourhood of (ξ_0', σ_0'). Hence $R_1(x_0', \xi_0', \sigma_0')\neq0$, where $R(x', \tau', \sigma')=R_0(x', \xi', \sigma')+\gamma' R_1(x', \xi', \sigma')+\gamma'^2R_2(x', \xi', \sigma', \gamma')$. In fact, if $R_1(x_0', \sigma_0', \xi_0')=0$ then $R(x_0', \xi_0'-i\gamma',\sigma_0')=\gamma'^2R_2(x_0', \xi_0', \sigma_0', \gamma')$. Since $\mathrm{Re}\,R_0\bar{R}_1\geqq0$ (Condition (II)) and $0<\gamma'<\delta$, we have

$$\mathrm{Re}\frac{\Lambda R}{R_1}=\frac{\mathrm{Re}\, R_0\bar{R}_1}{|R_1|^2}+\gamma+\gamma\gamma'\,\mathrm{Re}\frac{R_2}{R_1}\geqq\gamma-\gamma C_1\delta\geqq C\gamma\,.$$

Using Corollary 3.2 and Schwarz inequality we see that for $v\in H_{1,\gamma}(\boldsymbol{R}^n)$ and $\gamma\geqq\gamma_1>0$

$$\left\langle\!\!\left\langle\frac{\Lambda R}{R_1}v\right\rangle\!\!\right\rangle_{0,\gamma}\geqq C\gamma\langle\!\langle v\rangle\!\rangle_{0,\gamma}\,. \tag{4.11}$$

Put $v=\Lambda^{-\frac{1}{2}}u$. Then it follows form (4.1) and (4.11) that for $\gamma\geqq\gamma_2\geqq\gamma_1$

$$\gamma^2\langle\!\langle\Lambda^{-\frac{1}{2}}u_1\rangle\!\rangle^2_{0,\gamma}\leqq C(\langle\!\langle\Lambda^{\frac{1}{2}}Bu\rangle\!\rangle^2_{0,\gamma}+\langle\!\langle\Lambda^{\frac{1}{2}}u_2\rangle\!\rangle^2_{0,\gamma}+\langle\!\langle U\rangle\!\rangle_{-\frac{1}{2},\gamma})\,. \tag{4.12}$$

Let us put $m_1=d_1\gamma'$ and $m_2=d_2\gamma'^{-1}$, where $\gamma'=\gamma\Lambda^{-1}$ and d_j $(j=1,2)$ are positive constants. Then, using (4.12), the boundary integral of (4.2) becomes to

$$\begin{aligned}\langle U,\, MU\rangle_{0,\gamma}&=-d_1\langle\gamma' u_1,\, u_1\rangle_{0,\gamma}+d_2\langle\gamma'^{-1}u_2,\, u_2\rangle_{0,\gamma}\\&\geqq\gamma\langle\!\langle\Lambda^{-\frac{1}{2}}u_1\rangle\!\rangle^2_{0,\gamma}+\frac{1}{\gamma}\Big(d_2-(d_1+1)C\Big)\langle\!\langle\Lambda^{\frac{1}{2}}u_2\rangle\!\rangle^2_{0,\gamma}\\&\quad-\frac{C}{\gamma}\langle\!\langle\Lambda^{\frac{1}{2}}Bu\rangle\!\rangle^2_{0,\gamma}-\frac{C}{\gamma}\langle\!\langle U\rangle\!\rangle^2_{-\frac{1}{2},\gamma}\,.\end{aligned}$$

Let $d_2-(d_1+1)C>0$. Then we have for $\gamma\geqq\gamma_3\geqq\gamma_2$.

$$\langle U.\ MU\rangle_{0,\gamma}\geqq-\frac{C}{\gamma}\langle\!\langle Bu\rangle\!\rangle^2_{\frac{1}{2},\gamma} \tag{4.13}$$

Consider the volume integral of (4.2). It follows from (4.2) and (4.13) that for $\gamma\geqq\gamma_3$ and $\varepsilon>0$

$$\begin{aligned}&\frac{1}{\varepsilon\gamma}\|LU\|^2_{0,\gamma}+\varepsilon\gamma\|MU\|^2_{0,\gamma}+\frac{C}{\gamma}\langle\!\langle Bu\rangle\!\rangle^2_{\frac{1}{2},\gamma}\\&\geqq2\,\mathrm{Im}\,(u_1,\,-m_1\lambda^+\Lambda u_1)_{0,\gamma}+2\,\mathrm{Im}\,(u_2,\,m_2\lambda^-\Lambda u_2)_{0,\gamma}\\&\quad-C\left(\|u_1\|^2_{0,\gamma}+\frac{1}{\gamma^2}\|\Lambda u_2\|^2_{0,\gamma}\right).\end{aligned} \tag{4.14}$$

On the other hand, we have by the choice of M

$$\|MU\|^2_{0,\gamma}\leqq C\left(\|u_1\|^2_{0,\gamma}+\frac{1}{\gamma^2}\|\Lambda u_2\|^2_{0,\gamma}\right). \tag{4.15}$$

Using Corollary 3.1 and 3.2, it follows from (4.10) and the choice of M that

$$\mathrm{Im}\,(u_1,\,-m_1\lambda^+\Lambda u_1)_{0,\gamma}\geqq C\gamma\|u_1\|^2_{0,\gamma}\,, \tag{4.16}$$

$$\mathrm{Im}\,(u_2,\, m_2\lambda^- \Lambda u_2)_{0,\gamma} \geqq C\frac{1}{\gamma}\|\Lambda u_2\|_{0,\gamma}^2$$

for $\gamma \geqq \gamma_4 \geqq \gamma_3$. Hence it follows from (4. 14), (4. 15) and (4. 16) that for a small fixed $\varepsilon > 0$ and any $\gamma \geqq \gamma_5 \geqq \gamma_4$

$$\gamma^2\|u\|_{1,\gamma}^2 \leqq C(\|Pu\|_{0,\gamma}^2 + \langle\!\langle Bu \rangle\!\rangle_{\frac{1}{2},\gamma}^2). \tag{4. 17}$$

Using the finite partition of the unity a priori estimate (1. 4) follows directly from (4. 4), (4. 9) and (4. 17).

PROOF OF THEOREM 2. Let

$$P'^0(x, D) = P^{*0}(-x_0, x'', x_n, -D_0, D'', D_n),$$
$$B'^0(x', D) = B^{*0}(-x_0, x'', -D_0, D'', D_n).$$

From Lemma 2. 1 we see that if $(P^0, B^0)_{y'}$ is L^2-well-posed then $(P'^0, B'^0)_{y'}$ is so. Hence the statements of Lemma 3. 2 and 3. 3 are valid for (P'^0, B'^0). From Green formula

$$\int_0^\infty P^0(y_0', 0, \tau, \sigma, D_n)\, u(x_n)\overline{v(x_n)}dx_n$$
$$-\int_0^\infty u(x_n)\overline{P^0(y_0', 0, \tau, \sigma, D_n)v(x_n)}dx_n$$
$$= B^0(y_0', \tau, \sigma)u(0)\overline{v(0)} = u(0)\overline{B^{*0}(y_0', \tau, \sigma, D_n)v(0)}.$$

Let $u(x_n) = e^{ix_n\lambda^+(y_0', 0, \tau, \sigma)}$ and $v(x_n) = e^{ix_n\lambda^-(y_0', 0, \bar{\tau}, \sigma)}$. Then

$$B^0\left(y_0', \tau, \sigma. \lambda^+(y_0', 0, \tau, \sigma)\right) = B^{*0}\left(\overline{(y_0', \tau, \sigma, \lambda^-(y_0', 0, \tau, \sigma)}\right).$$

Hence

$$B^{*0}\left(y_0', -\tau, \sigma, \lambda^-(y_0', 0, -\tau, \sigma)\right) = \overline{R(-y_0, y_0'', -\tau, \sigma)}.$$

Since the left hand side of the above equality is Lopatinski determinant for (P'^0, B'^0), the assumption in Theorem 2 is also valid for the problem (P', B').

By the above considerations we can apply Theorem 3 to (P', B'). Therefore there exist positive constants C^* and γ_0^* such that it holds for any $\gamma \geqq \gamma_0^*$ and $v \in H_{2,-\gamma}(\boldsymbol{R}_+^{n+1})$

$$\gamma^2\|v\|_{1,-\gamma}^2 \leqq C^*(\|P^*v\|_{0,-\gamma}^2 + \langle\!\langle B^*v \rangle\!\rangle_{\frac{1}{2}-\gamma}^2). \tag{4. 18}$$

A priori estimates (1. 4) and (4. 18) assure the existence of solutions of (P, B) ([7]).

§ 5. Examples

In this section we present some examples of L^2-well-posed mixed problems which satisfy the condition (I) and (II).

Let $P(D)=-D_t^2+D_y^2+D_x^2$ and let $B(D)=D_x-bD_y-cD_t$, where $(t, y, x) =(x_0, x_1, x_2)$ and b, c are complex constants. In this case, $\lambda^{\pm}(\xi, \sigma)=\mp sgn\ \xi\sqrt{\xi^2-\sigma^2}$ $(\xi^2>\sigma^2)$, $\lambda^{\pm}(\xi, \sigma)=0$ $(\xi^2=\sigma^2)$ and $\lambda^{\pm}(\xi, \sigma)=\pm i\sqrt{\sigma^2-\xi^2}$ $(\sigma^2>\xi^2)$. Apply the results in [2] and [11] to our case. Then it can be proved by a similar argument as in the proof of Lemma 3. 3 that (P, B) is L^2-well-posed if and only if the following conditions are fulfilled:

(i) $R(\tau, \sigma)\neq 0$ for $\mathrm{Im}\,\tau=-\gamma<0$,

(ii) $R(\xi, \sigma)\neq 0$ for $\xi^2>\sigma^2$,

(iii) if $R(\xi_0', \sigma_0')=0$ for some (ξ_0', σ_0') $(\xi_0'^2=\sigma_0'^2)$, then there exist a positive constant $C(\xi_0', \sigma_0')$ and a neighbourhood $U(\xi_0', \sigma_0')$ such that

$$\left|\frac{Q(\tau', \sigma')}{R(\tau', \sigma')}\right|^2 \leqq C(\xi_0', \sigma_0')\frac{|\mathrm{Im}\,\lambda^+(\tau', \sigma')||\mathrm{Im}\,\lambda^-(\tau', \sigma')||D_\lambda P(\tau', \sigma', \lambda^-)(\tau', \sigma')|^2}{\gamma'^2} \tag{5. 1}$$

for any $(\tau', \sigma')\in U(\xi_0', \sigma_0')\cap\{\gamma'>0\}$,

(iv) if $R(\xi_0', \sigma_0')=0$ for some (ξ_0', σ_0') $(\sigma_0'^2>\xi_0'^2)$, then there exist a positive constant $C(\xi_0', \sigma_0)$ and a neighbourhood $U(\xi_0', \sigma_0')$ such that $|R(\tau', \sigma')|\geqq C(\xi_0', \sigma_0')\gamma'^{-1}$ for any $(\tau', \sigma')\in U(\xi_0', \sigma_0')\cap\{\gamma'>0\}$. Here (τ', σ') denotes the normalized variable.

Example 1. Let $B(t, y, D)=D_x-ib(t, y)D_y$ (b: real). Then we see from (5. 1) that $(P, B)_{(t_0, y_0)}$ is L^2-well-posed if and only if $|b(t_0, y_0)|<1$. The condition (I) is not satisfied if $b(t_0, y_0)=0$. *If* $0<|b(t, y|<1$ *then* (P, B) *satisfies the conditions* (*I*) *and* (*II*). In fact, the condition (I) follows from the fact that $ib(t, y)\sigma'$ is pure imaginary, because $R(t, y, \xi', \sigma')=ib(t, y)\sigma'$. To verify the condition (II) we remark that for every (t, y) there exists a point (σ_0, ξ_0) $(\sigma_0^2>\xi_0^2)$ where $R(t, y, \xi_0, \sigma_0)=0$. In a neighbourhood of (σ_0, ξ_0),

$$\begin{aligned} R(t, y, \tau, \sigma) &= i\left(\sqrt{\sigma^2-\xi^2}-b(t, y)\sigma\right)-\gamma\left(\frac{\xi}{\sqrt{\sigma^2-\xi^2}}\right)+0(\gamma^2) \\ &= R_0+\gamma R_1+\gamma^2 R_2 . \end{aligned}$$

Hence we obtain $\mathrm{Re}\, R_0\bar{R}_1=0$.

Example 2. Let $B(t, y, D)=D_x-b(t, y)D_y-D_t$, where $b(t, y)=b_1(t, y)$

$+ib_2(t, y)$. Then it follows from (5. 1) that $(P, B)_{(t_0,y_0)}$ is L^2-well-posed if and only if $|b(t_0, y_0)|\leqq 1$. The condition (I) is not satisfied if $b_2(t_0, y_0)=0$ and $b_1(t_0, y_0)=\pm 1$. If $|b(t, y)|<1$ then (P, B) satisfies the uniform Lopatinski condition. *If* $|b(t, y)|=1$ *and* $|b_2(t, y)|>0$ *then* (P, B) *satisfies the conditions* (*I*) *and* (*II*). In fact, the condition (I) follows from $b_2(t, y)\neq 0$. To verify the condition (II) we note that for every (t, y) there exists a point (ξ_0, σ_0) $(\sigma_0^2>\xi_0^2)$ where $R(t, y, \xi_0, \sigma_0)=0$. More precisely, $R(t, y, \xi_0, \sigma_0)=0$ is equivalent that

$$\sigma_0^2>\xi_0^2,\ b_2(t, y)\sigma_0>0 \quad \text{and} \quad b_1(t, y)\sigma_0+\xi_0=0\,. \tag{5. 2}$$

In a neighbourhood of (σ_0, ξ_0) with (5. 2), we expaned $R(t, y, \tau, \sigma)$ in $\gamma>0$:

$$R(t, y, \tau, \sigma)=i\left(\sqrt{\sigma^2-\xi^2}-b_2(t, y)\sigma\right)-\left(b_1(t, y)\sigma+\xi\right)$$
$$+\gamma\left(-\frac{\xi}{\sqrt{\sigma^2-\xi^3}}+i\right)+0(\gamma^2)\,.$$

Hence we have

$$\operatorname{Re} R_0\bar{R}_1=\frac{\xi}{\sqrt{\sigma^2-\xi^2}}\left(b_1(t, y)\sigma+\xi\right)+\left(\sqrt{\sigma^2-\xi^2}-b_2(t, y)\right).$$

Let $f(\xi, \sigma)=\sqrt{\sigma^2-\xi^2}\operatorname{Re} R_0\bar{R}_1$ for a fixed (t, y). Then, at a point (ξ_0, σ_0) with (5. 2), the following relations hold:

$$f=0\,,\ \frac{\partial f}{\partial\sigma}=0\,,\ \frac{\partial f}{\partial\xi}=0\,,\ \frac{\partial^2 f}{\partial\sigma^2}=\frac{\sigma_0^2}{b_2^2\sigma_0^2}\,, \tag{5. 3}$$
$$\frac{\partial^2 f}{\partial\sigma\partial\xi}=\frac{-\xi_0\sigma_0}{b_2^2\sigma_0^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial\xi^2}=\frac{\xi_0^2}{b_2^2\sigma_0^2}\,.$$

Under the condition $\xi^2+\sigma^2=1$ we may show that $f\geqq 0$ in a neighbourhood of (ξ_0, σ_0). Let $g(\xi)=f(\xi, \sigma(\xi))$. Then we have

$$\frac{dg}{d\xi}=\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\sigma}\frac{\partial\sigma}{\partial\xi}\,, \tag{5. 4}$$
$$\frac{d^2g}{d\xi^2}=\frac{\partial^2 f}{\partial\xi^2}+2\frac{\partial^2 f}{\partial\xi\partial\sigma}\frac{\partial\sigma}{\partial\xi}+\frac{\partial^2 f}{\partial\sigma^2}\left(\frac{\partial\sigma}{\partial\xi}\right)^2+\frac{\partial f}{\partial\sigma}\frac{\partial^2\sigma}{\partial\xi^2}\,.$$

Note that $\dfrac{\partial\sigma}{\partial\xi}=-\dfrac{\xi}{\sigma}$. Then it follows from (5. 3) and (5. 4) that, at a point (ξ_0, σ_0) with (5. 2),

$$g=0\,,\quad \frac{dg}{d\xi}=0 \quad \text{and} \quad \frac{d^2g}{d\xi^2}=\frac{1}{b_2^2\sigma_2^4}>0\,.$$

This means the condition (II).

EXAMPLE 3. We consider the problem for symmetric system of first order:

$$\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}-1&0\\0&1\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u_1\\u_2\end{pmatrix}\qquad(x>0),$$

$$(1,\ -b)\begin{pmatrix}u_1\\u_2\end{pmatrix}=0\qquad(x=0)$$

where $b=b_1+ib_2$ is constant. This problem is dissipative if and only if $|b|\leqq 1$. In particular, if $|b|=1$ then it is conservative. Furthermore Lopatinskii determinant and the reflection coefficient for this problem are the same ones as in example 2. However, it is unknown whether semigroup estimates hold for problems in example 2 ($|b|=1,\ b_2\neq 0$).

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ADDED IN PROOF: We can remove the condition (II) in Theorem 2 as follows: Since $\partial R/\partial\tau(x_0', \xi_0', \sigma_0')\neq 0$ where $R(x_0', \xi_0', \sigma_0')=0$, there exists a C^∞-function $\alpha(x', \sigma')$ in $U(x_0', \sigma_0')$ such that $R(x', \xi', \sigma')=S(x', \xi', \sigma')$ $(\tau'-\alpha(x', \sigma'))$ and $S\neq 0$. Then Im $\alpha\geqq 0$ follows from that $R(x', \tau', \sigma')\neq 0$ if Im $\tau'<0$. This was pointed out by T. Shirota. Hence we have from Corollary 3.2

$$\langle\!\langle iS^{-1}R\Lambda\beta u\rangle\!\rangle_{0,\gamma}\geqq\gamma\langle\!\langle\beta u\rangle\!\rangle_{0,\gamma}-c\langle\!\langle u\rangle\!\rangle_{0,\gamma},$$

where $\beta=\phi_0\phi_j\psi_k$ and $u\in H_{1,\gamma}(\boldsymbol{R}^n)$. This is a key point in §4 (see (4. 11)). Moreover, in (i)-(iv) of §4, we omitted such error terms as $-c\langle\!\langle u\rangle\!\rangle_{0,\gamma}$ by considering u as βu, because terms arising from such ones are absorbed in the left hand side of (1. 4), taking γ large.

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