On minimal points of Riemann surfaces, II.

By Zenjiro KURAMOCHI

Dedicated to Prof. Yukinari Tôki on his 60th birthday

This paper is the continuation of the paper with the same title [1]. Definitions and terminologies in the previous paper will be used here also. Let R be a Riemann surface with positive boundary and let G be a domain in R. We suppose Martin's topologies M and M' are defined over $R+\Delta(R, M)$ and $G+\Delta(G,M')$, where $\Delta(R, M)$ and $\Delta(G, M')$ are sets of all Martin's boundary points of R and G respectively. Let $\underset{1}{\Delta}(R, M)$ (resp. $\underset{1}{\Delta}(G, M')$) be the set of all minimal points of $\Delta(R, M)$ (resp. $\Delta(G, M')$). Let $G(z, p)$ and $G'(z, p)$ be Green's functions of R and G respectively and let p^* be a fixed point in G. Put $G_\delta=\left\{z\in G: \dfrac{G'(z,p^*)}{G(z,p^*)}>\delta\right\}$. Then

THEOREM 1. (M. Brelot) [2]. *Let p be a point on ∂G. If p is irregular for the Dirichlet problem in G, the set of points in $\Delta(G, M')$ lying on p consists of only one point which is minimal.*

THEOREM 2. (M. Brelot) [3]. *Let $p\in\underset{1}{\Delta}(R, M)$. Then there exists a path Γ in R M-tending to p.*

THEOREM 3. (L. Naïm) [4]. *Let $\{p_i\}$ be a sequence in G_δ: $\delta>0$ such that $p_i\overset{M}{\longrightarrow}p\in\underset{1}{\Delta}(R, M)$. Then $\{p_i\}$ M'-tends to a point $q\in\underset{1}{\Delta}(G, M')$.*

We shall consider extensions of the above theorems. In this paper we use I and E operations. Let A and B be two hyperbolic domains in R such that $A\subset B$. Let $U(z)$ be a positive harmonic function in B. We denote by $\overset{B}{\underset{A}{I}}[U(z)]$ the upper envelope of continuous subharmonic functions in A smaller than $U(z)$ and vanishing on ∂A except a set of capacity zero. Let $V(z)$ be a positive harmonic function in A vanishing on ∂A except a set of capacity zero. We denote by $\overset{B}{\underset{A}{E}}[V(z)]$ the lower envelope of continuous superharmonic functions larger than $V(z)$. Then I and E have following properties:

1). *E and I are positive linear operators.*

2). *$I\ E[V(z)]=V(z)$.*

3). *If $U(z)$ is minimal in G and $I[U(z)]>0$, $EI[U(z)]=U(z)$.*

4). *If* $I[U(z)]>0$, $U(z)$ *preserves minimality of* $U(z)$ *and if* $E[V(z)]<\infty$, E *preserves minimality of* $V(z)$.

Assertions except 3) are clear. We shall show only 3). Clearly $E\,I[U(z)]\leqq U(z)$. Hence $E\,I[U(z)]=a\,U(z)$: $0<a\leqq 1$ by 4). Hence $E\,I\,E\,I[U(z)]=a^2\,U(z)$. On the other hand by 2) $a\,U(z)=E(IE)I[U(z)]=a^2\,U(z)$, whence $a=1$ and $E\,I[U(z)]=U(z)$.

Let $U(z)$ be a positive superharmonic function in R. Let F be a closed set in R. We denote the lower envelope of superharmonic functions larger than $U(z)$ on F by ${}_FU(z)$. Then ${}_FU(z)=H_U^{CF}(z)$ in CF, where $H_U^{CF}(z)$ is the solution of Dirichlet problem in CF with boundary value $U(z)$ on ∂F and $=0$ on the ideal boundary of R. If $G(z,p)\neq{}_{CG}G(z,p)$: $p\in R$, we say CG *is thin at* p. For $p\in\partial G$, it is known that p is *irregular* for the Dirichlet problem in G *if and only if* CG *is thin at* p.

1. The mapping $f(p)$. Let $K(z,p)$ and $K'(z,p)$ be Martin's kernels in $R+\Delta(R,M)$ and $G+\Delta(G,M')$ respectively such that $K(p,p^*)=1=K'(p,p^*)$. Regard $R-p$ as a Riemann surface, then we can define $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]$ for $p\in R+\Delta(R,M)$. Then we shall prove the following

PROPOSITION 1.

1). *If* $p\in R-\bar{G}$ *or* $p\in\partial G$ *and* p *is regular,*

$$K(z,p)-{}_{CG}K(z,p)=\overset{R-p}{\underset{G-p}{I}}[K(z,p)]=0\,.$$

2). *If* $p\in\partial G$ *and* p *is irregular,* $p\in\bar{G}_\delta^M$ *for a const.* $\delta>0$, *where—M means the closure relative to M-top.*

If $p\in\partial G$ *and* p *is irregular or* $p\in G$, $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]>0$.

3). $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]>0$ for $p\in\bar{G}_\delta^M\cap\Delta(R,M)$.

4). $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]=K(z,p)-{}_{CG}K(z,p)$ in G for $p\in R+\Delta(R,M)$.

PROOF OF 1). If $p\in\partial G$ and p is regular or $p\in R-\bar{G}$, CG is not thin at p, i.e. $G(z,p)={}_{CG}G(z,p)=H^G_{G(\cdot,p)}(z)$ in G. Hence ${}_{CG}G(z,p)$ is quasibounded in G(if $U(z)$ is a limit of a increasing sequence of bounded positive harmonic functions, $U(z)$ is called quasibounded). Now $K(z,p)=\dfrac{G(z,p)}{G(p^*,p)}$ and $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]$ is clearly a singular function in G or $=0$ (if a positive harmonic function $U(z)$ has no positive bounded harmonic function smaller than $U(z)$, $U(z)$ is called singular). Evidently $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]\leqq K(z,p)$. Since $K(z,p)$ is quasibounded in G, $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]=0$. We have

$$K(z,p)-{}_{CG}K(z,p)=0=\overset{R-p}{\underset{G-p}{I}}[K(z,p)].$$

PROOF OF 2). Suppose p is irregular. Then we can find a sequence $\{p_i\}$ and a const. $\delta>0$ such that $p_i\to p$ and $\lim\limits_{i=\infty} G'(p_i, p^*)>\delta>0$. Hence

$$p\in\overline{G}_{\delta'},\quad \text{where } \delta'=\frac{\delta}{G(p,p^*)}.$$

Suppose $p\in\partial G$ and p is irregular. then $K(z,p)-{}_{CG}K(z,p)>0$ (because CG is thin at p). Clearly $K(z,p)={}_{CG}K(z,p)=H^{G}_{K(.,p)}(z)$ on ∂G except a set of capacity zero. Hence we have at once

$$\overset{R-p}{\underset{G-p}{I}}\Big[K(z,p)\Big]\geqq K(z,p)-{}_{CG}K(z,p)>0.$$

Let $p\in G$. Then evidently $K(z,p)-{}_{CG}K(z,p)>0$ and $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]>0$.

PROOF OF 3). Let $p_i\in G_\delta$. Then

$$K(z,p_i)=\frac{G(z,p_i)}{G(p^*,p_i)}\geqq\frac{G(z,p_i)}{G'(p^*,p_i)}\frac{G'(p^*,p_i)}{G(p^*,p_i)}\geqq\delta K'(z,p_i)\ \text{ in } G.$$

Let $\{p_i\}\subset G_\delta$ be a sequence and $\{p_{i'}\}$ be its subsequence such that $p_i\xrightarrow{M}p$ and $p_{i'}\xrightarrow{M'}r\in\Delta(G,M')$ respectively. Then

$$K(z,p)=\lim_{i'=\infty}K(z,p_{i'})\geqq\delta K'(z,r)>0.$$

Now $K'(z,p)=0$ on ∂G except a set of capacity zero. Hence $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]\geqq\delta K'(z,r)>0$.

PROOF OF 4). Since $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]\geqq K(z,p)-{}_{CG}K(z,p)$, we have to prove the inverse inequality. Let $v_n(p)$ be a neighbourhood of p (if $p\notin R$ put $v_n(p)=0$). Let $U_n(z)$ be a harmonic function in $(G\cap R_n)-v_n(p)$ such that $U_n(z)=K(z,p)$ on $\partial G\cap R_n-v_n(p)$, $=0$ on $\partial v_n(p)+(\partial R_n\cap G)$. Then $U_n(z)\nearrow{}_{CG}K(z,p)$ as $n\to\infty$, where $\{R_n\}$ is an exhaustion of R. $K(z,p)-U_n(z)$ is superharmonic in $G\cap R_n-v_n(p)$ and $=K(z,p)$ on $\partial v_n(p)+\partial R_n\cap G$, whence

$$\overset{R-p}{\underset{G-p}{I}}[K(z,p)]\leqq K(z,p)-{}_{CG}K(z,p),$$

and we have 4).

Let $\qquad G(M, \varDelta)=\left\{p\in R-G+\varDelta_{1}(R, M): \overset{R-p}{\underset{G-p}{I}}\left[K(z,p)\right]>0\right\}.$

We shall define the mapping $f(p)$: $p\in G+G(M, \varDelta)$ as follows: Since $\overset{R-p}{\underset{G-p}{I}}$ preserves the minimality in $R-p$ of $K(z,p)$, $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]>0$ is minimal in $G-p$ and there exists a uniquely determined point $f(p)$ in $G+\varDelta_{1}(G, M')$ and a const. $a(p)>0$ depending on p such that

$$\overset{R-p}{\underset{G-p}{I}}\left[K(z,p)\right]=a(p)\ K'\left(z,f(p)\right),$$

where $K'(z,p)$ is Martin's kernel in G relative to M'-top.

If $p\in G$, $\sup_{z\in CG} K(z,p)<\infty$ and $K(z,p)=\infty$ at p. Hence $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]=a(p)\ K'(z,p)$ and $f(p)=p$ in G. Hence $f(p)$ is defined in $G+G(M, \varDelta)$ and $f(p)$ is continuous in G. Denote by $f(G+G(M, \varDelta))$ the set of point q of $G+\varDelta_{1}(G, M')$ such that there exists at least a point $p\in G+G(M, \varDelta)$ with $q=f(p)$. Then we shall prove.

THEOREM 4.

1). *$f(p)=p$ in G and $f(p)$ is univalent, i.e. $f(p_1)\neq f(p_2)$ for $p_1\neq p_2$. Hence $f(p)$ is one to one mapping from $G+G(M, \varDelta)$ onto $f(G+G(M, \varDelta))$.*

2). *As for $a(p)$: $p\in G+G(M, \varDelta)$ such that $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]\doteq a(p)\ K'(z,f(p))$.*

a). *$a(p)$ is continuous in G.*

b). *$a(p)$ is upper semicontinuous in $G+G(M, \varDelta)$.*

c). *Let $G^*_\delta=\{p\in G+G(M, \varDelta): a(p)\geqq\delta>0\}$. Then G^*_δ is M-closed in $G+G(M, \varDelta)$*

and $$G^*_\delta\supset\bar{G}^{M}_\delta\cap\left(G+G(M, \varDelta)\right).$$

3). *If $p_i\overset{M}{\longrightarrow}p$ in G^*_δ, $f(p_i)\overset{M'}{\longrightarrow}f(p)\in G+\varDelta_{1}(G, M')$. Hence $f(p)$ is continuous at p in G^*_δ.*

4). *By definition of $G+G(M, \varDelta)$ we have at once*

$$G+G(M, \varDelta)=\bigcup_{\delta>0}G^*_\delta.$$

PROOF OF 1). $f(p)=p$ in G is clear. We show $f(p_1)\neq f(p_2)$ for $p_1\neq p_2$.
CASE 1. $p_1\in G$ and $p_2\in G+G(M, \varDelta)$. Then $\overset{R-p_1}{\underset{G-p_1}{I}}[K(z,p_1)]=a(p_1)K'(z,p_1)$: $a(p_1)>0$ and $\overset{R-p_1}{\underset{G-p_1}{I}}[K(z,p_1)]=\infty$ at p_1. On the other hand, $\overset{R-p_2}{\underset{G-p_2}{I}}[K(z,p_2)]$ is harmonic at p_1, whence $K'(z,f(p_1))$ and $K'(z,f(p_2))$ are linearly independent

and $f(p_1)\neq f(p_2)$.

CASE 2. p_1 and $p_2\in G(M, \Delta)$. $K(z, p_1)$ and $K(z, p_2)$ are harmonic and linearly independent in G. Hence by $\overset{R-p_i}{\underset{G-p_i}{E}}[K'(z, f(p_i))]=a_i K(z, p_i) a_i>0 : i=1, 2$ $K'(z, f(p_1))$ and $K'(z, f(p_2))$ are linearly independent and $f(p_1)\neq f(p_2)$. Hence we have 1).

PROOF OF 2. CASE 1. $p\in G$. In this case, let $p_i\overset{M}{\longrightarrow}p$. Then $p_i\in G$ for $i\geqq i_0$ and $p_i=f(p_i)\to f(p)=p$. Now $p_i\in \bar{G}^M_\delta$ for a const. $\delta>0$ for $i\geqq i_0$ and $K(z, p_i)\geqq \delta K'(z, p_i)$: $i\geqq i_0$. Hence

$$\overset{R-p_i}{\underset{G-p_i}{I}}\left[K(z, p_i)\right]=K(z, p_i)-{}_{CG}K(z, p_i)=a(p_i)K'(z, p_i): \quad i\geqq i_0 .$$

Since $K(z, p_i)\leqq M<\infty$ on CG : $i\geqq i_0$, ${}_{CG}K(z, p)=\lim_{i=\infty}{}_{CG}K(z, p_i)$. Hence

$$\begin{aligned}a(p)K'(z, p)&=\overset{R-p}{\underset{G-p}{I}}\left[K(z, p)\right]=\lim_{i=\infty}\left(K(z, p_i)-{}_{CG}K(z, p_i)\right)\\&=\lim_{i=\infty}a(p_i)K'(z, p_i)=\lim_{i=\infty}a(p_i)K'(z, p)\end{aligned}$$

and $a(p)=\lim_{i=\infty}a(p_i)$ and $a(p)$ is continuous in G.

CASE 2. a). $p\in G(M, \Delta)$ and $\overline{\lim}_{\substack{r\overset{M}{\longrightarrow}p\\ r\in(G+G(M,\Delta))}}a(r)>0$. In this case, we can find a sequence $\{p_i\}$ in $G+G(M, \Delta)$ such that $p_i\overset{M}{\longrightarrow}p$ and $\lim_{i=\infty}a(p_i)=\overline{\lim}_{\substack{r\to p\\ r\in(G+G(M,\Delta))}}a(r)$. Let $\{p_{i'}\}$ be a subsequence of $\{p_i\}$ such that $f(p_{i'})\to r_0\in G+\Delta(G, M')$, since $G+\Delta(G, M')$ is compact.

Then $$\overset{R-p_{i'}}{\underset{G-p_{i'}}{I}}\left[K(z, p_{i'})\right]=K(z, p_{i'})-{}_{CG}K(z, p_{i'})=a(p_{i'})K'\left(z, f(p_{i'})\right).$$

Let $i'\to\infty$. Then by $K'(z, f(p_{i'}))\to K'(z, r_0)$ and ${}_{CG}K(z, p)\leqq\lim_{i'=\infty}{}_{CG}K(z, p_{i'})$, $\overset{R}{\underset{G}{I}}[K(z, p)]=K(z, p)-{}_{CG}K(z, p)\geqq\lim_{i'=\infty}K(z, p_{i'})-\underline{\lim}_{i'=\infty}{}_{CG}K(z, p_{i'})=\overline{\lim}_{i'=\infty}a(p_{i'})K'(z, r_0)>0$. Now $K(z, p)$ is minimal in R, whence $K'(z, r_0)$ is minimal and $r_0=f(p)$, i. e. $f(p_{i'})\overset{M'}{\longrightarrow}f(p)$. Hence $f(p_i)\overset{M'}{\longrightarrow}f(p)$.

As above by $f(p_i)\overset{M'}{\longrightarrow}f(p)$ we have

$$\begin{aligned}a(p)K'\left(z, f(p)\right)&=\overset{R-p}{\underset{G-p}{I}}\left[K(z, p)\right]\geqq\overline{\lim_{i=\infty}}\overset{R-p_i}{\underset{G-p_i}{I}}\left[K(z, p_i)\right]\\&=\overline{\lim_{i=\infty}}\left(a(p_i)K'\left(z, f(p_i)\right)\right)=\overline{\lim}\,a(p_i)K'\left(z, f(p)\right).\end{aligned}$$

Hence $$a(p)\geqq\overline{\lim}_{\substack{r\overset{M}{\longrightarrow}p\\ r\in(G+G(M,\Delta))}}a(r).$$

CASE 2. b). $p \in G(M, \Delta)$ and $\overline{\lim}_{\substack{r \to p \\ r \in (G+G(M,\Delta))}} a(r)=0$. In this case, by $p \in G(M, \Delta)$ $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]>0$ and $=a(p)K'(z,f(p))$. Hence we have at once $a(p) \geqq \lim_{\substack{r \to p \\ r \in (G+G(M,\Delta))}} a(r) =0$. Thus $a(p)$ is upper semicontinuous in $G+G(M, \Delta)$ with respect to M-top. Since $K(z,p) \geqq \delta K'(z,f(p))$ for $p \in G_\delta^* \cap (G+G(M,\Delta))$, $a(p) \geqq \delta$. By the upper semicontinuity of $a(p)$ we have $a(p) \geqq \delta$ in $\bar{G}_\delta^M \cap (G+G(M,\Delta))$ and we have c).

PROOF OF 3. Without loss of generality we can suppose $p_i \in G+G(M, \Delta)$. $p_i \in G_\delta^*$ implies $K(z,p_i) \geqq a(p_i)K'(z,f(p_i)) \geqq \delta K'(z,f(p_i))$. Let $\{p_{i'}\}$ be a subsequence of $\{p_i\}$ such that $f(p_{i'}) \xrightarrow{M'} r \in \Delta(G,M)$. Then

$$K(z,p)=\lim_{i'=\infty} K(z,p_{i'}) \geqq \delta \lim_{i'=\infty} K\left(z,f(p_{i'})\right)=\delta K'(z,r)>0\,.$$

Since $\overset{R-p}{\underset{G-p}{I}}[K(z,p)]$ ($\geqq \delta K'(z,r)>0$) is minimal in G, $f(p)=r \in \underset{1}{\Delta}(G,M')$ and by 2), b)

$$\overset{R-p}{\underset{G-p}{I}}\left[K(z,p)\right]=a(p)K'(z,r):\ a(p) \geqq \delta\,.$$

Hence $f(p_i) \xrightarrow{M'} f(p)$.

PROOF OF 4). We have at once by 2) c) and the definition of G_δ.

Let $r \in \Delta(R,M)$ and μ be the canonical measure of $K(z,r)$: $K(z,r)=\int_{R+\underset{1}{\Delta}(R,M)} K(z,p_\alpha)d\mu(p_\alpha)$. Clearly $\mu=0$ in R. If μ has a measure ρ $(0<\rho \leqq 1)$ at $p \in \underset{1}{\Delta}(R,M)$, we say that r has activity ρ at p. It is easily verified that $r=p$ if and only if r has activity $\rho=1$ at p. Let $\mathfrak{G}=f(G+G(M,\Delta))$ and $\mathfrak{G}_\delta=f(G_\delta^*)$. Then $\mathfrak{G}=\bigcup_{\delta>0}\mathfrak{G}_\delta$ and $f^{-1}(q)$ is uniquely determined in $\mathfrak{G}$ by the univalency of $f(p)$. We shall prove the following.

THEOREM 5. 1). *Let* $\{q_i\}$ *be a sequence in* $\mathfrak{G}_\delta$ *such that* $q_i \xrightarrow{M'} q \in (G+\underset{1}{\Delta}(G,M'))$. *Then* $f^{-1}(q)$ *is defined. If* $f^{-1}(q_i) \xrightarrow{M} r \in R+\underset{1}{\Delta}(R,M)$, *then* $f(r)=q$.

2). *Let* $\{q_i\}$ *be a sequence in* 1). *If* $f^{-1}(q_i) \xrightarrow{M} r \in (\Delta(R,M)-\underset{1}{\Delta}(R,M))$, *there exists* $f^{-1}(q)$ *in* $G+G(M,\Delta)$ *with the following properties*: $f^{-1}(q) \neq r$, $K(z,r)$ *is non minimal and* r *has activity* $\geqq \dfrac{\delta}{a(f^{-1}(q))}$ *at* $f^{-1}(q)$, *where*

$a(f^{-1}(q))$ is the const. such that $I[K(z,f^{-1}(q))]=a(f^{-1}(q))\,K'(z,q)$.

3). *$\mathfrak{G}_\delta$ is M'-closed in $G+\underset{1}{\Delta}(G,M')$ and $f(G+G(M,\Delta))$ is an F_σ set in $G+\underset{1}{\Delta}(G,M')$.*

4). *Let $\{q_i\}$ be a sequence in 1). Let F be the set of limiting points of $\{f^{-1}(q_i)\}$. Put $A=F\cap(\Delta(R,M)-\underset{1}{\Delta}(R,M))$ (A may be empty). Then*

$$F\subset f^{-1}(q)+A \qquad \textit{and}$$

any point s in A is not so far from $f^{-1}(q)$ that s has activity $\geqq\dfrac{\delta}{a(f^{-1}(q))}$ at $f^{-1}(q)$.

5). *Let $\{q_i\}$ be a sequence in 1) and let A be in 4). Then if $A\neq 0$, $f^{-1}(q)\in\underset{1}{\Delta}(R,M)$.*

5'). *Let $\{q_i\}$ be a sequence of 1). If $f^{-1}(q)\in R$, $A=0$, i.e. $f^{-1}(q_i)\overset{M}{\longrightarrow}f^{-1}(q)$ and $f^{-1}(q)$ is continuous at q in $\mathfrak{G}_\delta$ ($\delta>0$) relative to M'-top.*

6). *Under what condition $f^{-1}(q_i)\to f^{-1}(q)$? As a sufficient condition we have the following: Let $\{q_i\}$ be a sequence in $\mathfrak{G}$ such that $q_i\overset{M'}{\longrightarrow}q\in\mathfrak{G}$ and $\lim\limits_{i=\infty}a(f^{-1}(q_i))=a(f^{-1}(q))$. Then $f^{-1}(q_i)\overset{M}{\longrightarrow}f^{-1}(q)$.*

PROOF OF 1). By definition, $f^{-1}(q_i)$ is in G_δ^*. Let $p_i=f^{-1}(q_i)$. Then

$$K(z,p_i)\geqq \overset{R-p_i}{\underset{G-p_i}{I}}\left[K(z,p_i)\right]\geqq\delta K'(z,p_i)\,.$$

Let $i\to\infty$. Then $K(z,r)\geqq\delta K'(z,q)>0$. Since $K(z,r)$ is minimal in $R-r$, $\overset{R-r}{\underset{G-r}{I}}[K(z,r)]=\delta' K'(z,q):\ \delta'\geqq\delta>0$. Hence $q=f(r)$ and we have 1).

PROOF OF 2). Similarly as 1)

$$K(z,r)\geqq\delta K'(z,q)\,. \tag{1}$$

Now since $K(z,r)$ is harmonic in R, $\overset{R}{\underset{G}{E}}[K'(z,q)]$ ($<\infty$ by (1)) is harmonic and minimal in R. Hence there exists a uniquely determined point p ($=f^{-1}(q)$) in $G+\underset{1}{\Delta}(R,M)$ such that

$$\overset{R-p}{\underset{G-p}{I}}\left[K(z,p)\right]=a(p)K'(z,q)\,. \tag{2}$$

By (1) and (2) we have $K(z,r)\geqq\dfrac{\delta\overset{R}{\underset{G}{I}}[K(z,p)]}{a(p)}$ and since $K(z,p)$ is minimal,

$$K(z, r) \geqq \frac{\delta \overset{R-p}{\underset{G-p}{E}} \overset{R-p}{\underset{G-p}{I}} [K(z, p)]}{a(p)} = \frac{\delta K(z, p)}{a(p)}. \qquad (3)$$

Let μ be the canonical measure of $K(z, r)$. Then by (3) μ has measure $\geqq \frac{\delta}{a(p)}$ at p.

PROOF OF 3). Let $\{q_i\}$ be a sequence in $\mathfrak{G}_\delta$ such that $q_i \xrightarrow{M'} q \in (G+\underset{1}{\Delta}(G, M))$. Let $p_i = f^{-1}(q_i)$. Since $(R+\Delta(R, M))$ is compact, there exists a subsequence $\{p_{i'}\}$ of $\{p_i\}$ and a point $r \in (R+\Delta(R, M))$ such that $p_{i'} \xrightarrow{M} r$. Then two cases occur: CASE 1. $r \in (R+\underset{1}{\Delta}(R, M))$ and CASE 2. $r \in (\Delta(R, M) - \underset{1}{\Delta}(R, M))$.

CASE 1. In this case, by 1) $r = f(q)$ and $K(z, r) \geqq \delta K'(z, q)$ and $q \in \mathfrak{G}_\delta$.

CASE 2. In this case by 2), there exists a point $p \in G+\underset{1}{\Delta}(R, M)$ and $r \in \Delta(R, M) - \underset{1}{\Delta}(R, M)$ and $f(p) = q$ and $K(z, r) \geqq \frac{\delta K(z, p)}{a(p)}$.

On the other hand, $K(p^*, r) = 1 = K(p^*, p)$. Hence $a(p) \geqq \delta$. Thus $p \in G^*_\delta$ and $f(p) = q$. Hence $\mathfrak{G}_\delta$ is M'-closed in $G+\underset{1}{\Delta}(G, M')$. Next by Theorem 4, 2) $\mathfrak{G} = f(G+G(M, \Delta))$ is an F_σ set in $G+\underset{1}{\Delta}(G, M')$.

By 1) and 2) we have at once 4).

PROOF OF 5). Assume $f^{-1}(q) \in R$ and $A \neq 0$. Then there exists a subsequence $\{q_{i'}\}$ of $\{q_i\}$ such that $f^{-1}(q_{i'}) \xrightarrow{M} r \in \Delta(R, M) - \underset{1}{\Delta}(R, M)$. Then by 3) r has activity $\geqq \frac{\delta}{a(f^{-1}(q))}$ at $f^{-1}(q)$. Since $K(z, r)$ is harmonic, the canonical measure of $K(z, r)$ has no measure in R. This is a contradiction. Hence $A = 0$ and we have 5). Next we have 5') by 5) at once.

PROOF OF 6). If $q \in G$, M and M'-top. s are homeomorphic to the original topology in G and $q = f(p)$ in G. Hence $q_i \xrightarrow{M'} q \in G$ implies $f^{-1}(q_i) \xrightarrow{M} f^{-1}(q)$. We can suppose without loss of generality $q \in \underset{1}{\Delta}(G, M')$ by $\mathfrak{G} \subset (G+\underset{1}{\Delta}(G, M))$. Let $q_i \xrightarrow{M'} q$ and $a(f^{-1}(q_i)) \to a(f^{-1}(q))$. Assume $f^{-1}(q_i)$ does not M-tend to $f^{-1}(q)$. Then since $R+\Delta(R, M)$ is compact, we can find a subsequence $\{q_{i'}\}$ of $\{q_i\}$ such that $f(q_{i'}) \xrightarrow{M} r \neq f^{-1}(q)$. Then

$$K(z, r) = \lim_{i'=\infty} K\left(z, f^{-1}(q_{i'})\right) \geqq \lim_{i'=\infty} a\left(f^{-1}(q_{i'})\right) K'(z, q_{i'}) \qquad (4)$$
$$= a\left(f^{-1}(q)\right) K'(z, q) > 0 .$$

Hence if $r \in R + \underset{1}{\Delta}(R, M)$, $f(r) = q$: $r = f^{-1}(q)$. This contradicts the assumption. Whence $r \in \Delta(R, M) - \underset{1}{\Delta}(R, M)$ and $K(z, r)$ is non minimal. We remark both $K(z, r)$ and $K(z, f^{-1}(q))$ are harmonic in R. Since $K'(z, q)$ is minimal in G, $\overset{R}{\underset{G}{E}}[K'(z, q)] = \dfrac{K(z, f^{-1}(q))}{a(f^{-1}(q))}$. Hence by (4) $K(z, r) \geqq K(z, f^{-1}(q))$. Now $K(p^*, r) = 1 = K(p^*, f^{-1}(q))$ implies $K(z, r) = K(z, f^{-1}(q))$ and $r = f(q)$. This is also a contradiction. Hence $f^{-1}(q_i) \to f^{-1}(q)$.

Let G be a domain in R and let $U(z)$ be a positive superharmonic function in R or in G. Let F be a closed set in G. We denote by ${}^{G}_{F}U(z)$ the lower envelope of superharmonic functions in G larger than $U(z)$ on F. Let v be a domain in R (resp. in G). If ${}_{Cv}K(z, p) < K(z, p)$: $p \in R + \underset{1}{\Delta}(R, M)$ (resp. ${}^{G}_{Cv}K'(z, q) < K'(z, q)$: $q \in G + \underset{1}{\Delta}(G, M')$), v is called a fine neighbourhood of p (resp of q). Let $v_n(p)$ and $v_n(q)$ be neighbourhoods of p and q such that

$$v_n(p) = \left\{z : M\text{-dist}(z, p) < \frac{1}{n}\right\}, \quad v_n(q) = \left\{z : M'\text{-dist}(z, q) < \frac{1}{n}\right\}$$

respectively. Then it is well known ${}_{v_n(p)}K(z, p) = K(z, p)$, ${}_{Cv_m(p)}K(z, p) < K(z, p)$ [5] (i. e. $v_m(p)$ is a fine neighbourhood) and $\lim_{n=\infty} {}_{v_n(p)}({}_{Cv_m(\ )}K(z, p)) = 0$. Consider G as a Riemann surface, then we have the same facts about $v_n(q)$ and $K'(z, q)$. Then we shall prove the following

THEOREM 6. 1). By *Theorem* 4. 3). *we have*: *For any* $v_n(f(p))$ *there exists a* $v_m(p)$ *such that* $v_m(p) \cap G^*_\delta \subset v_n(f(p)) \cap \mathfrak{G}_\delta \cap G$.

2). *By Theorem* 5. 2), *if* $f(p) \in \mathfrak{G}$,

$$\bigcap_n \overline{v_n(f(p)) \cap \mathfrak{G}_\delta \cap G}^{M} \subset p + A ,$$

where A *is a set of non minimal points with activity* $\geqq \dfrac{\delta}{a(p)}$ *at* p.

3). $G \cap v_n(p)$ *is a fine neighbourhood of* $f(p)$ *and* $v_n(f(p))$ *is a fine neighbourhood of* p, *where* $p \in G + G(M, \Delta)$ *and* $f(p) \in \mathfrak{G}$. Hence $\{v_n(p)\}$ *and* $\{v_n(f(p))\}$ *are almost equivalent.*

Since 1) and 2) are proved at once by Theorem 4 and 5, we have to prove only 3).

Case 1). $p \in G$. In this case since M and M'-top.s are homeomorphic to the original topology in R, our assertion is trivial.

Case 2). $p\in G(M, \varDelta)$. Suppose $p\in\partial G\cap G(M, \varDelta)$. Then p is irregular and there exists no continuum component of ∂G containing p and there exists only one point r of $G+\varDelta(G, M')$ and $r\in G+\underset{1}{\varDelta}(G, M')$ on p by Theorem 1 and $K(z, p)$ is harmonic in R except p, whence r must coincides with $f(p)$. On the other hand, M-top. is homeomorphic to the original topology, hence

$$\left\{v_n(p)\cap G\right\} \quad \textit{and} \quad \left\{v_n\left(f(p)\right)\right\} \quad \textit{are equivalent.} \qquad (5)$$

Let $p\in G(M, \varDelta)$ and $q=f(p)$. Then

$$\overset{R-p}{\underset{G-p}{I}}\left[K(z, p)\right] = K(z, p)-{}_{CG}K(z, p) = a(p)K'(z, q) \quad \text{in } G:\ a(p)>0. \qquad (6)$$

We show $v_m(p)\cap G$ is a fine neighbourhood of q. Assume $v_m(p)$ is not so. Then ${}_{Cv_m(p)}^{\ G}K'(z, q)=K'(z, q)$. Since ${}_{v_n(q)}^{\ G}K'(z, q)=K'(z, q)$ for any n, and since $G\subset R$ we have by (6)

$K(z, p)\geqq{}_{G\cap v_n(q)}({}_{Cv_m(p)\cap G}K(z,p))\geqq a(p)({}_{v_n(q)}^{\ G}({}_{Cv_m(p)}^{\ G}K'(z, q)))=a(p)({}_{v_n(q)}^{\ G}K'(z, q))=a(p)K'(z, q)>0.$

Let $n\to\infty$. Then $v_n(q)\to$ideal boundary of R or to p by (5) according as $p\in\underset{1}{\varDelta}(R, M)$ or $p\in\partial G\cap G(M, \varDelta)$. Put $U(z)=\lim\limits_{n=\infty}{}_{G\cap v_n(q)}({}_{Cv_m(p)}K(z, p))$. Then $K(z, p)\geqq U(z)$ and $K(z, p)$ and $U(z)$ are positive harmonic in R or $R-p$ and $K(z, p)$ is minimal in R or $R-p$. Hence

$$K(z, p)\geqq U(z)=\alpha K(z, p)>0:\quad 1\geqq\alpha>0. \qquad (7)$$

And

$$\lim_{m=\infty}{}_{v_m(p)}U(z)=\alpha K(z, p).$$

On the other hand,

$$0=\lim_{n=\infty}{}_{v_n(p)}({}_{Cv_m(p)}K(z, p))\geqq\lim_{n=\infty}{}_{v_n(p)}({}_{Cv_m(p)\cap G}K(z, p))\geqq\lim_{n=\infty}{}_{v_n(p)}U(z)>0.$$

This is a contradiction. Hence $v_m(p)\cap G$ is a fine neighbourhood of $f(p)$. Next we show $v_n(q)$ is a fine neighbourhood of p. By (6) we have

$${}_{Cv_n(q)}K(z, p)={}_{Cv_n(q)}\left({}_{CG}K(z, p)\right)+a(p)\left({}_{Cv_n(q)}^{\ G}K'(z, q)\right) \quad \text{in } v_n(q).$$

Since $v_n(q)$ is a fine neighbourhood of q, there exists a uniquely determined component $v_n^*(q)$ of $v_n(q)$ such that ${}_{Cv_n(q)}^{\ G}K'(z, q)<K'(z, q)$ in $v_n^*(q)$. Hence by ${}_{CG}K(z, p)\geqq{}_{Cv_n(q)}({}_{CG}K(z, p))$ we have ${}_{Cv_n(q)}K(z, p)<K(z, p)$ in $v_n^*(q)$. Hence $v_n(q)$ is a fine neighbourhood of p and we have 3).

2. Let R be a Riemann surface with null or positive boundary. If a non compact domain G has a compact relative boundary ∂G consisting of a finite number of analytic curves, we call G an end. G has not necessarily one ideal boundary component. In the following we denote by $G\in\mathcal{E}$, $G\in\mathcal{E}_0$ or $G\in\mathcal{E}_p$ according as G is an end of a Riemann surface, a Riemann surface with null or positive boundary. We suppose Kerékjártó-Stoïlow's topology is defined on $R+\beta(R)$, where $\beta(R)$ is the set of all boundary components. Let $\beta(G)$ be the set of all points of $\beta(R)$ such that G is a neighbourhood of relative to K-top. (Kerékjártó's top.). Let F_i $(i=1,2,\cdots)$ *be a compact continuum in* G *such that* 1). $G-F$ *is connected, where* $F=\sum F_i$. 2). $F_i\cap F_j=0$ *for* $i\neq j$. 3). $\partial G\cap F=0$. 4. $\{F_i\}$ *clusters at only* $\beta(G)$. 5). *There exists a determining sequence* $\{\mathfrak{B}_n(\mathfrak{p})\}$ *of* $\mathfrak{p}$ *such that*

$$\min_{z\in\partial\mathfrak{B}_n(\mathfrak{p})}\frac{G'(z,p^*)}{G(z,p^*)}\geqq\delta>0\quad \textit{for any } n,$$

where $\mathfrak{B}_n(\mathfrak{p})$ *has a compact relative boundary* $\partial\mathfrak{B}_n(\mathfrak{p})$ *in* G', $G(z,p^*)$ *and* $G'(z,p^*)$: $p^*\in G-F$ *are Green's functions of* G *and* $G'=G-F$ *respectively. Then we say that* F *is thin at a boundary component* $\mathfrak{p}$.

It is easily seen that the thiness of F does not depend on p^*. If $\mathfrak{B}_{n_0}(\mathfrak{p})$ is conformally equivalent to $0<|z|<1$, F is thin at $\mathfrak{p}$ if and only if $z=0$ is irregular for the Dirichlet problem in $\{0<|z|<1\}-F$. Let $\{\mathfrak{B}_n(\mathfrak{p})\}$ be a determining sequence of $\mathfrak{p}$, i.e. a decreasing sequence of K-neighbourhoods relative to K-top. over G such that $\partial\mathfrak{B}_n(\mathfrak{p})\cap F=0$ for any n. Since F_i is compact, we can choose such $\{\mathfrak{B}_n(\mathfrak{p})\}$. $\mathfrak{B}_n(\mathfrak{p})\cap G'$ consists of at most a finite number of components: $\mathfrak{B}_n^1, \mathfrak{B}_n^2, \cdots, \mathfrak{B}_n^{i(n)}$, because $\partial\mathfrak{B}_n^i\subset\partial\mathfrak{B}_n(\mathfrak{p})$. A decreasing sequence $\mathfrak{B}_1^{j_1}\supset\mathfrak{B}_2^{j_2}\supset\mathfrak{B}_3^{j_3}\cdots$ determines a boundary component $\mathfrak{q}$ of $\beta(G')$. In this case we say that $\mathfrak{q}$ lies over $\mathfrak{p}$. We denote by $\mathfrak{S}'(\mathfrak{p})$ all points of $\beta(G')$ lying over $\mathfrak{p}$. $\mathfrak{S}'(\mathfrak{p})$ consists of many points generally. But if there exists a number n_0 such that $\mathfrak{B}_{n_0}(\mathfrak{p})\cap G'$ is connected and of planar character, $\mathfrak{B}_n(\mathfrak{p})\cap G'$ consists of only one component for $n\geqq n_0$ and $\mathfrak{S}'(\mathfrak{p})$ consists of only one point $\mathfrak{p}'\in\beta(G')$. Suppose Martin's topologies M and M' are defined on $G+\Delta(G,M)$ and $G'+\Delta(G',M')$ respectively. If there exists a sequence $\{p_i\}$ such that $p_i\xrightarrow{M}p$ and $p_i\xrightarrow{K}\mathfrak{p}$ (relative to Kerékjártó's top.), we say p lies over $\mathfrak{p}$. Let $\nabla(\mathfrak{p},G,M)$ be the set of points of $G+\Delta(G,M)$ (clearly of $\Delta(G,M)$) lying over $\mathfrak{p}$ and let $\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')$ be the set of points of $\Delta(G',M')$ lying over $\mathfrak{S}'(\mathfrak{p})$. Suppose F is thin at $\mathfrak{p}$. Then there exists a sequence $\{\mathfrak{B}_n(\mathfrak{p})\}$ and a const. $\delta_0>0$ such that

$$\min_{z\in\partial\mathfrak{B}_n(\mathfrak{p})}\frac{G'(z,p^*)}{G(z,p^*)}\geqq\delta_0\qquad \text{for}\quad n\geqq n_0.$$

Put $\{\gamma\}=\sum_{n\geqq n_0}\partial\mathfrak{B}_n(\mathfrak{p})$ and let $G_{\delta_0}=\left\{z\in G':\frac{G'(z,p^*)}{G(z,p^*)}\geqq\delta_0\right\}$ Then $\{\gamma\}\subset G_{\delta_0}$. Let $p\in\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$ (resp. $\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')\cap\underset{1}{\Delta}(G',M')$). Then by Theorem 1 there exists a path Γ in G (resp. G') M (resp. M')-tending to p. Γ intersects $\{\gamma\}$ and $p\in\bar{G}^M_{\delta_0}\cap\underset{1}{\Delta}(G,M)$. Hence the mapping $f(p)$ is defined in $\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$ and $f(p)\in\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')\cap(G+\underset{1}{\Delta}(G',M'))\subset\mathfrak{G}_{\delta_0}$. Till now we discussed the behaviour of $f(p)$. Theorem 4, 5, 6 are valied for $\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$. It is not necessary to quote them. In the following we consider only distinctive properties of $\nabla(\mathfrak{p},G,M)$. Then we have by Theorem 4, 5 and 6 following

THEOREM 7.

Suppose F is thin with const. δ_0. Then

1). *There exists a one to one mapping $f(p)$ from $\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$ onto $\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')\cap\underset{1}{\Delta}(G',M')$.*

2). *For any given $v_n(f(p))$: $p\in\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$, there exists a $v_m(p)$ such that $(v_m(p)\cap\{\gamma\})\subset(v_n(f(p))\cap\{\gamma\})$, where $v_m(p)$ and $v_n(f(p))$ are neighbourhoods relative to M and M'-top. s. Let Γ be a path M-terminating at p. Then any sequence $\{p_i\}$ on Γ M-tending to $\mathfrak{S}(\mathfrak{p})$ M'-tends to $f(p)$.*

3). *Let $q\in\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')\cap\underset{1}{\Delta}(G',M')$. Then $\bigcap_n\overline{v_n(q)\cap\{\gamma\}}^M\subset f^{-1}(q)+A$, where $A\subset\Delta(R,M)-\underset{1}{\Delta}(R,M)$ and any point of A has activity $\geqq\frac{\delta_0}{a(f(q))}$ at $f^{-1}(q)$. Let Γ be a path M'-tending q. Let $\{q_i\}$ be a sequence on $\Gamma\cap\{\gamma\}$ tending to $\mathfrak{S}'(\mathfrak{p})$. Then $\{f^{-1}(q_i)\}$ M-tends to $f^{-1}(q)$ or A.*

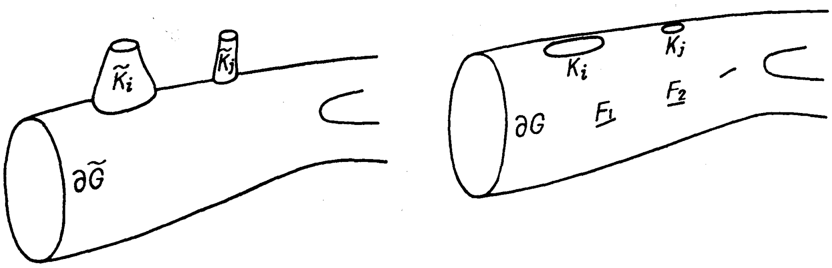
PROOF. 1) is clear 2) and 3) are direct consequences of Theorem 5 and 6.

REMARK. If F is thin at $\mathfrak{p}$, by removing F from G may be divided into some components in $\beta(G')$, though any point p in $\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$ is not divided into points in $\nabla(\mathfrak{S}'(\mathfrak{p}),G',M')\cap\underset{1}{\Delta}(G',M')$. In the other words, F may change superficial structure of p (relative to Kerékjártó's top.) but not potential theoretic structure so much. If p can be decomposed by removing a thin set, p consists of some points in the sense of potential theory by nature.

Let $\tilde{G}\in\mathcal{E}$ and let J_i $(i=1,2,\cdots)$ be a simple closed Jordan curve in $\tilde{G}$ such that J_i divides $\tilde{G}$ into two components. One of them which has not $\partial\tilde{G}$ in its relative boundary is denoted by $\tilde{K}_i$, i.e. $\partial\tilde{K}_i=J_i$, where $\tilde{K}_i$ may be compact or non compact. Let $G\in\mathcal{E}_0$ and let K_i $(i=1,2,\cdots)$ be a simply connected domain in G and F_i $(i=1,2,\cdots)$ be a continuum such that $K+F$ $(K=\sum_i K_i$ and $F=\sum_i F_i)$ is thin at a component $\mathfrak{p}\in\beta(G)$. Put $\widetilde{G}'=\widetilde{G}-\sum_i \widetilde{K}_i$, $G'=G-K$ and $G''=G'-F$. Suppose there exists a conformal mapping from $\widetilde{G}'$ onto G' so that $\partial\widetilde{G}\Longleftrightarrow\partial G$, $\partial\widetilde{K}_i\Longleftrightarrow\partial K_i$. Identify $\widetilde{G}'$ and G'. Then we can consider that G' is prolonged to $\widetilde{G}$ by $\sum_i \widetilde{K}_i$ through $\sum_i \partial K_i$ and we have

$$\begin{matrix}\widetilde{G} \supset \\ G \supset\end{matrix} G'=G-K\supset G''=G-K-F.$$

Let $\{\mathfrak{V}_n(\mathfrak{p})\}$ be a determining sequence of $\mathfrak{p}\in\beta(G)$ such that $\partial\mathfrak{V}_n(\mathfrak{p})\cap(K+F)=0$ for any n. Consider $\partial\mathfrak{V}_n(\mathfrak{p})$ in $\widetilde{G}$. Then $\partial\mathfrak{V}_n(\mathfrak{p})$ divides $\widetilde{G}$ into two components, because ∂K_i is a dividing cut in $\widetilde{G}$. One of the components not containing ∂G in its relative boundary is denoted by $\widetilde{\mathfrak{V}}_n$. Let $\mathfrak{V}'_n=\mathfrak{V}_n(\mathfrak{p})\cap G'$ and $\mathfrak{V}''=\mathfrak{V}_n(\mathfrak{p})\cap G''$. Then $\widetilde{\mathfrak{V}}_n$ and $\mathfrak{V}'_n$ consist of only one component respectively. $\mathfrak{V}''$ may consist of some components.



$\{\widetilde{\mathfrak{V}}_n\}$, $\{\mathfrak{V}'_n\}$ and $\{\mathfrak{V}''_n\}$ determine $\widetilde{\mathfrak{p}}\in\beta(\widetilde{G})$, $\mathfrak{p}'\in\beta(G')$ and a set of boundary components $\mathfrak{S}''(\mathfrak{p})$ respectively. Let $\widetilde{M}, M, M'$ and M'' be Martin's top.s over $\widetilde{G}, G, G'$ and G'' respectively. In the following, if there exists a one to one mapping f between A and B, we denote it by $A\approx B$. Then we have

Corollary 1. *Let $G\in\mathcal{E}_0$ and $F+K$ is thin at $\mathfrak{p}\in\beta(G)$, where K_i is a simply connected domain in G. Then*

$$V(\widetilde{\mathfrak{p}}, \widetilde{G}, \widetilde{M})\cap \underset{1}{\Delta}(\widetilde{G}, \widetilde{M})\approx V(\mathfrak{p}', G', M')\cap\underset{1}{\Delta}(G', M')$$
$$\approx V(\mathfrak{p}, M, G)\cap\underset{1}{\Delta}(G, M)\approx V(\mathfrak{S}''(\mathfrak{p}), G'', M'')\cap\underset{1}{\Delta}(G'', M'').$$

Proof. Since $F+K$ is thin at $\mathfrak{p}$, there exists a determining sequence $\{\mathfrak{V}'_n(\mathfrak{p})\}$ such that

$$\min_{z\in\partial\mathfrak{V}'_n(p)}\frac{G''(z,p^*)}{G(z,p^*)}>\delta>0:\ p^*\in G''-\mathfrak{V}'_1(\mathfrak{p}), \qquad (7)$$

where $G''(z,p^*)$ and $G(z,p^*)$ are Green's functions of G'' and G respectively. Without loss of generality we can suppose $\mathfrak{V}_n(\mathfrak{p})=\mathfrak{V}'_n(\mathfrak{p}')$. Let $G'(z,p)$ be a Green's function of G'. Then $G'(z, p^*) \geqq G''(z, p^*)$. Hence F is thin at $\mathfrak{p}$ in G and $F+K$ is thin at $\mathfrak{p}$, whence by Theorem 7. 1) we have

$$\nabla(\mathfrak{p}, G, M) \cap \underset{1}{\Delta}(G, M) \approx \nabla(\mathfrak{p}', G', M') \cap \underset{1}{\Delta}(G', M')$$
$$\approx \nabla(\mathfrak{S}''(\mathfrak{p}), G'', M'') \cap \underset{1}{\Delta}(G'', M'') .$$

Since $G \in \mathcal{E}_0$, $\inf_{z \in \mathfrak{B}_1(p)} G(z, p^*) = \min_{z \in \partial\mathfrak{B}_1(p)} G(z, p^*) = N > 0$, we have by (7) $\min_{z \in \partial\mathfrak{B}_n(p)} G''(z, p^*) > \delta_1 > 0$ for any n. Let $\widetilde{G}(z, p^*)$ be a Green's function of $\widetilde{G}$, then $\sup_{z \in \widetilde{\mathfrak{B}}_1} \widetilde{G}(z, p^*) \leqq M < \infty$ and we have

$$\min_{z \in \partial\mathfrak{B}_n(p)} \frac{G(z, p^*)}{\widetilde{G}(z, p^*)} \geqq \frac{\delta_1}{M} > 0 \qquad \text{for any } n .$$

Hence $\sum \widetilde{K}_i$ is thin at $\widetilde{\mathfrak{p}}$ in $\widetilde{G}$ and by Theorem 7, 1)

$$\nabla(\widetilde{\mathfrak{p}}, \widetilde{G}, \widetilde{M}) \cap \underset{1}{\Delta}(\widetilde{G}, \widetilde{M}) \approx \nabla(\mathfrak{p}', G', M') \cap \underset{1}{\Delta}(G', M') .$$

Thus we have the corollary.

REMARK. This corollary means, under the condition that $F+K$ is so small that $F+K$ may be thin at $\mathfrak{p}$, that the structure of $\nabla(\mathfrak{p}, G, M) \cap \underset{1}{\Delta}(G, M)$ of G does not change, however much G may increase to $\widetilde{G}$ or decrease to G'' through $\sum_i \partial K_i$ by $\sum_i \widetilde{K}_i$ or $\sum F_i$.

If a compact set F with ∂F consisting of at most a finite number of analytic curves, we call F a regular set. Let $G \in \mathcal{E}_0$ and $F = \sum_i F_i$ thin at $\mathfrak{p} \in \beta(G)$ such that F_i is regular. Let $\widetilde{G}'$ be the doubled surface of G' relative to $\sum_i \partial F_i$, i.e. $\widetilde{G}' = G' + \widehat{G}' + \partial F$, where $\widehat{G}'$ is the symmetric image of $G' = G - F$ relative to ∂F. Then $\widetilde{G}' \in \mathcal{E}_0$ [6]. In G we can find a determining sequence $\{\mathfrak{B}_n(\mathfrak{p})\}$ such that $\partial\mathfrak{B}_n(\mathfrak{p}) \cap F = 0$ and $\min_{z \in \partial\mathfrak{B}_n(p)} \frac{G'(z, p^*)}{G(z, p^*)} > \delta > 0$ for any n, where $G(z, p^*)$ and $G'(z, p^*)$ are Green's functions of G and $G' = G - F$: $p^* \in G' - \mathfrak{B}_1(\mathfrak{p})$. Now $G \in \mathcal{E}_0$, whence

$$\min_{z \in \partial\mathfrak{B}_n(p)} G'(z, p^*) \geqq \delta_1 > 0 \qquad \text{for any } n .$$

Let $\widehat{\mathfrak{B}}_n$ be the set obtained from $\mathfrak{B}_n(\mathfrak{p}) \cap G'$ and $\widehat{\mathfrak{B}}_n(\mathfrak{p}) \cap \widehat{G}'$ by identifying $\sum \partial F_i$ and $\sum \partial \widehat{F}_i$, where the summation is over F_i contained in $\mathfrak{B}_n(\mathfrak{p}) \cap G$. Then $\{\widetilde{\mathfrak{B}}_n\}$ determines a $\widetilde{\mathfrak{S}}(\mathfrak{p})$, set of boundary components of $\beta(\widetilde{G})$ lying over $\mathfrak{p}$. Analogously $\{\mathfrak{B}_n(\mathfrak{p}) \cap G'\}$ and $\{\widetilde{\mathfrak{B}}_n \cap G'\}$ determine $\mathfrak{S}'(\mathfrak{p})$ and $\widehat{\mathfrak{S}}'(\mathfrak{p})$ respectively. Let M, M' and $\widetilde{M}$ be Martin's top.s over G, G' and $\widetilde{G}'$ respectively. Then we have

COROLLARY 2. *Let $G\in\mathcal{E}_0$ and F be thin at $\mathfrak{p}\in\beta(G)$. Then*

$$\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), \tilde{G}', \tilde{M})\cap\underset{1}{\Delta}(\tilde{G}', \tilde{M}) = A^1+A^2, \quad A^1\cap A^2=0,$$

$A^1\approx\nabla(\mathfrak{S}'(\mathfrak{p}), G', M')\cap\underset{1}{\Delta}(G', M')\approx\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G, M)$ and A^2 is the symmetric image of A^1.

PROOF. Let $\tilde{G}^0=\tilde{G}'-\sum_{i\geqq 2}(\partial F_i+\partial\hat{F}_i)$. Then $\tilde{G}^0$ is obtained from G' and $\hat{G}'$ by identifying only ∂F_1 and $\partial\hat{F}_1$. Without loss of generality we can suppose $\mathfrak{V}_1(\mathfrak{p})\cap F_1=0$. Then

$$\tilde{\mathfrak{V}}_n\cap\tilde{G}^0=\mathfrak{V}_n(\mathfrak{p})\cap G'+\hat{\mathfrak{V}}_n\cap\hat{G}' \quad\text{and}\quad (\mathfrak{V}_n(\mathfrak{p})\cap G')\cap(\hat{\mathfrak{V}}_n\cap\hat{G}')=0.$$

Whence $\tilde{\mathfrak{S}}(\mathfrak{p})=\mathfrak{S}'(\mathfrak{p})+\hat{\mathfrak{S}}'(\mathfrak{p})$. Let $\tilde{M}^0$ be Martin's top. over $\tilde{G}^0$. Then

$$\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), \tilde{G}^0, \tilde{M}^0)\cap\underset{1}{\Delta}(\tilde{G}^0, \tilde{M}^0)=A^{1,0}+A^{2,0}, \quad A^{1,0}\cap A^{2,0}=0, \tag{8}$$

where $A^{1,0}=\nabla(\mathfrak{S}'(\mathfrak{p}), \tilde{G}^0, \tilde{M}^0)\cap\underset{1}{\Delta}(\tilde{G}^0, \tilde{M}^0)$ and $A^{2,0}$ is symmetric to $A^{1,0}$.

The structure of $\underset{1}{\Delta}(\tilde{G}^0, \tilde{M}^0)$ does not change in a neighbourhood of $\mathfrak{S}(\mathfrak{p})$ by removing a compact set $\partial F_1+\partial\hat{F}_1$. Extract $\partial F_1+\partial\hat{F}_1$ from $\tilde{G}^0$, then $\tilde{G}^0$ is decomposed into G' and $\hat{G}'$. Hence

$$A^{1,0}\approx\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), G', M')\cap\underset{1}{\Delta}(G', M').$$

By Corollary 1

$$A^{1,0}\approx\nabla(\mathfrak{S}(\mathfrak{p}'), G', M')\cap\underset{1}{\Delta}(G', M')\approx\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G, M).$$

Next consider $\tilde{G}'$ and $\tilde{G}^0$. Let $\tilde{G}^0(z, p^*)$: $p^*\in G'-\mathfrak{V}_1(\mathfrak{p})$ be a Green's function of $\tilde{G}^0$. Then

$$\tilde{G}^0(z, p^*)\geqq G'(z, p^*) \qquad \text{in} \quad G'. \tag{9}$$

Let $\hat{G}'(z, \hat{p}^*)$ be a Green's function of $\hat{G}'$, then $\hat{G}'(\hat{z}, \hat{p}^*)=G'(z, p^*)$, where $\hat{p}^*$ and $\hat{z}$ are symmetric to p^* and z respectively. Consider $\tilde{G}^0(z, p^*)$ in $\hat{G}'$. Let $v(\hat{p}^*)$ be a neighbourhood of $\hat{p}^*$ in $\hat{G}'-\hat{\mathfrak{V}}_1$. Then $\tilde{G}^0(z, p^*)\geqq\delta>0$ and $\hat{G}'(z, \hat{p}^*)<M_1<\infty$ on $\partial v(\hat{p}^*)$. Hence

$$\tilde{G}^0(z, p^*)\geqq\frac{\delta}{M_1}\hat{G}'(z, \hat{p}^*) \qquad \text{in} \quad \hat{\mathfrak{V}}_n. \tag{10}$$

Let $\tilde{G}'(z, p^*)$ be a Green's function of $\tilde{G}'$. Then $\tilde{G}'(z, p^*)\leqq M_2<\infty$ in $\tilde{\mathfrak{V}}_1$. Hence by (9) and $\min_{z\in\partial\mathfrak{V}_n(p)} G'(z, p^*)\geqq\delta_1$ and (10) we have

$$\min_{z\in\partial\mathfrak{V}_n}\frac{\tilde{G}^0(z, p^*)}{\tilde{G}'(z, p^*)}\geqq\min_{z\in\partial\mathfrak{V}_n(p)}\frac{(G'(z, p^*)}{M_2} \quad\text{and}\quad \frac{\tilde{G}^0(z, p^*)}{\tilde{G}'(z, p^*)}\geqq\frac{\delta\delta_1}{M_1M_2}>0 \text{ on } \partial\hat{\mathfrak{V}}_n. \tag{11}$$

Hence $\sum_{i\geqq 2}\partial F_i+\partial\hat{F}_i$ is a thin set in $\tilde{G}'$ at any boundary component in $\tilde{\mathfrak{S}}(\mathfrak{p})$. Hence $\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), \tilde{G}', \tilde{M})\cap\underset{1}{\Delta}(\tilde{G}', \tilde{M})\approx\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), \tilde{G}^0, \tilde{M}^0)\cap\underset{1}{\Delta}(\tilde{G}^0, \tilde{M}^0)$ and by (8) we have the corollary.

Let $G\in\mathcal{E}_0$ and F be thin at $\mathfrak{p}\in\beta(G)$ such that $\min_{z\in\partial\mathfrak{V}_n(p)}\dfrac{G'(z,p^*)}{G(z,p^*)}\geqq\delta$, where $G'(z.p^*)$ is a Green's function of $G'=G-F$. If there exists a number n_0 such that $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ is connected and is of planar character (for instance, if $\mathfrak{V}_{n_0}(\mathfrak{p})$ is of planar character, $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ is also of planar character), $\mathfrak{S}'(\mathfrak{p})$ consists of only one component $\mathfrak{p}'$. Map $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ conformally by $\xi=g(z)$ onto a domain Ω in $|\xi|<1$ so that $\gamma\to|\xi|=1$, where γ is a component of $\partial\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$. Let $A=\bigcap_n\overline{g(\mathfrak{V}_n(\mathfrak{p})\cap G')}$. Assume A is a continuum. Then $G'(z,p^*)\to 0$ as $z\to\mathfrak{p}$. This contradicts the thiness of F. Hence $A=$one point ξ_0 and ξ_0 is an irregular point for Ω. Otherwise, $G'(z,p^*)\to 0$ as $z\to\mathfrak{p}$. Consider $K'(z,p)$: $p\in\nabla(\mathfrak{p}', G', M')\cap\underset{1}{\Delta}(G', M')$. Assume $\sup_{z\in G'}K'(z,p)<\infty$. Then $K'(z,p)=0$ by $K'(z,p)=0$ on $\partial G+\partial F$ and $G\in\mathcal{E}_0$. Hence $\sup_{z\in G'}K'(z,p)=\infty$. By $p\in\bar{G}_\delta$ clearly $K'(z,p)\leqq\dfrac{1}{\delta}K(z,f^{-1}(p))$: $f^{-1}(p)\in\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G,M)$. Since $\sup_{G-\mathfrak{V}_n(p)}K(z,f^{-1}(q))<\infty$ for any $\mathfrak{V}_n(\mathfrak{p})$,

$$\overline{\lim_{z\to p}}\,K'(z,p)=\infty\,.$$

Since $\partial\mathfrak{V}_{n_0}(\mathfrak{p})$ is compact, $\max_{z\in\partial\mathfrak{V}n_0(p)}K'(z,p)=M<\infty$ and since $K'(z,p)$ is minimal in G',

$$\overset{G'}{\underset{\mathfrak{V}n_0(p)\cap G'}{I}}\left[K'(z,p)\right]=K'(z,p)-H(z)>0 \text{ and is minimal in } \mathfrak{V}_{n_0}(\mathfrak{p})\cap G',$$

where $H(z)$ is the solution of Dirichlet problem in $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ with the boundary value $K'(z,p)$ and $H(z)\leqq M$.

By Theorem 1 there exists only one positive harmonic function $U(z)$ in Ω vanishing on $\partial\Omega-\xi_0$ except its multiples. Hence $\overset{G'}{\underset{\mathfrak{V}n_0(p)\cap G'}{I}}[K'(z,p)]=aU(z)$: $a>0$. Now $U(z)$ is expressed by $\lim_{i=\infty}G''(z,q_i)$, where $\{q_i\}$ is a sequence in Ω tending to ξ_0 such that $\{G''\{z,q_i)\}$ converges to a positive harmonic function in Ω and $G''(z,q_i)$ in a Green's function of $G'\cap\mathfrak{V}_{n_0}(\mathfrak{p})$. Such sequence can be chosen by the irregularity of ξ_0. By $K'(z,p)=\overset{G'}{\underset{\mathfrak{V}n_0(p)\cap G'}{E}}\ \overset{G'}{\underset{\mathfrak{V}n_0(p)\cap G'}{I}}[K'(z,p)]$ there exists only one point in $\nabla(\mathfrak{p}', G', M')\cap\underset{1}{\Delta}(G', M')$. Hence by Theorem 7 $\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G,M)$ consists of only one point. Let $\mathfrak{q}\in\nabla(\mathfrak{p}, G, M)\cap$

$\Delta(G, M)$. Then $K(z, q)=\int K(z, p_\alpha)d\mu(p_\alpha)$, μ is a positive measure on $\nabla(\mathfrak{p}, G, M)\cap\Delta(G, M)$. Hence $q=p$ and $\nabla(\mathfrak{p}, G, M)\cap\Delta(G, M)=\nabla(\mathfrak{p}, G, M)\cap\Delta_1(G, M)$ $=$one point. Similarly $\nabla(\mathfrak{p}', G, M)\cap\Delta(G', M')=\nabla(\mathfrak{p}', G', M')\cap\Delta_1(G' M')=$one point. Hence we have

COROLLARY 3. *Let $G\in\mathcal{E}_0$ and F be thin at $\mathfrak{p}\in\beta(G)$. If there exists a number n_0 such that $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ is connected and is of planar character, then*

$$\nabla(\mathfrak{p}', G', M')\cap\Delta(G' M')=\nabla(\mathfrak{p}', G' M')\cap\Delta_1(G' M')=\textit{one point},$$

$$\nabla(\mathfrak{p}, G, M)\cap\Delta(G, M)=\nabla(\mathfrak{p}, G, M)\cap\Delta_1(G, M)=\textit{one point}.$$

This means the following: Let $\{p_i\}$ be a sequence tending to $\mathfrak{p}$ (res. $\mathfrak{p}'$). Then $p_i\xrightarrow{M}$ one point (resp $p_i\xrightarrow{M'}$ one point) which is minimal.

As a special case of Corollary 1 we have similarly as Corollary 3

COROLLARY 4. *Let $G\in\mathcal{E}_0$ and $F+K$ be thin at $\mathfrak{p}\in\beta(G)$. If there exists a number n_0 such that $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G''$ is connected and is of planar character,*

$$\nabla(\tilde{\mathfrak{p}}, \tilde{G}, \widetilde{M})\cap\Delta(\tilde{G}, \widetilde{M})=\nabla(\tilde{\mathfrak{p}}, \tilde{G}, \widetilde{M})\cap\Delta_1(\tilde{G}, \widetilde{M})=\textit{one point},$$

$$\nabla(\mathfrak{p}, G, M)\cap\Delta(G, M)=\nabla(\mathfrak{p}, G, M)\cap\Delta_1(G, M)=\textit{one point},$$

$$\nabla(\mathfrak{p}', G', M')\cap\Delta(G', M')=\nabla(\mathfrak{p}', G', M')\cap\Delta_1(G', M')=\textit{one point},$$

$$\nabla(\mathfrak{p}'', G'', M'')\cap\Delta(G'', M'')=\nabla(\mathfrak{p}'', G'', M'')\cap\Delta_1(G'', M'')=\textit{one point}.$$

As a special case of Corollary 2 we have

COROLLARY 5. *Let $G\in\mathcal{E}_0$ and F be thin at $\mathfrak{p}\in\beta(G)$. Let $\tilde{G}'$ be the doubled surface. If there exists a number n_0 such that $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$: $G'=G-F$ is connected and is of planar character, and $\mathfrak{V}_n(\mathfrak{p})\cap F\neq 0$ for any n, then $\tilde{\mathfrak{S}}(\mathfrak{p})$ consists of one component $\tilde{\mathfrak{p}}$ and $\nabla(\tilde{\mathfrak{p}}, \tilde{G}', \widetilde{M})\cap\Delta_1(\tilde{G}', \widetilde{M})$ consists of two points p_1 and p_2 and any point $q\in\nabla(\tilde{\mathfrak{p}}, \tilde{G}', \widetilde{M})\cap\Delta(\tilde{G}', \widetilde{M})$ is expressed by $K(z, q)=\alpha K(z, p_1)+\beta K(z, p_2)$, where $\alpha\geqq 0$, $\beta\geqq 0$ and $\alpha+\beta=1$.*

3. Let $\Omega\in\mathcal{E}$ and let F be a closed set in Ω such that $F\cap\partial\Omega=0$ and $\Omega'=\Omega-F$ is connected. Let $U(z)$ be a positive continuous superharmonic function in Ω' such that $U(z)=0$ on $\partial\Omega$ and $D(\min(M, U(z)))<\infty$ for any $M<\infty$. Let D be a regular compact set in Ω'. Let ${}_DU(z)$ be a function such that ${}_DU(z)=U(z)$ in D and ${}_DU(z)=W(z)$ in $\Omega'-D$, where $W(z)$ has M. D. I. (minimal Dirichlet integral) over $\Omega'-D$. If for any regular compact set D, ${}_DU(z)\leqq U(z)$, $U(z)$ is called a *fullsuperharmonic function* in Ω'. Let $U(z)$ be a full superharmonic function in Ω. If $V(z)=\alpha U(z)$ for any full superharmonic function $V(z)$ such that both $U(z)-V(z)$ and $V(z)$ are full superharmonic in Ω', $U(z)$ *is called N-minimal in* Ω'.

Let $G\in\mathcal{E}_0$ and let F_i $(i=1,2,\cdots)$ be a regular compact set such that $F_i\cap F_j=0$ for $i\neq j$ and $F=\sum_i F_i$ cluster only at $\beta(G)$. Let $G'=G-F$. Let $L(z,p)$: $p\in G'$ be an N-Green's function [7] of G' such that $L(z,p)=0$ on ∂G, $L(z,p)$ has a logarithmic singularity at p and $L(z,p)$ has $M.D.I.$ over $G-V_M(p)$: $V_M(p)=[z\in G'\colon L(z,p)>M]$. Then clearly $\frac{\partial}{\partial n}L(z,p)=0$ on ∂F_i and

$$L(z,p)=\frac{1}{2}\left(\tilde{G}(z,p)+\tilde{G}(z,\hat{p})\right) \qquad \text{in } G',$$

where $\tilde{G}(z,p)$ is a Green's function of the doubled surface $\tilde{G}'$ of G' with respect to $\sum\partial F_i$ and $\hat{p}$ is the symmetric point of p.

Let $\Delta(G',L)$ be the boundary points of G' relative to N-Martin's topology [8] L. If there exists a sequence $\{p_i\}$ in G' tending to $\mathfrak{p}\in\beta(G)$ such that $p_i\xrightarrow{L}p$, i.e. $L(z,p_i)\to L(z,p)$, we say p lies over $\mathfrak{p}$, we denote by $\nabla(\mathfrak{p},G',L)$ be the set of all points lying over $\mathfrak{p}$. Let $p\in\nabla(\mathfrak{p},G',L)$. Then we see easily the following

Lemma 1. *Let* $p\in\nabla(\mathfrak{p},G'L)$. *Then* $\frac{\partial}{\partial n}L(z,p)=0$ *on* ∂F *and* $L(z,p)$ *can be continued harmonically into* $\hat{G}'$ *across* ∂F *so that* $L(z,p)=L(\hat{z},p)$, *where* $\hat{z}$ *is the symmetric point of* z.

We shall prove

Lemma 2. *Let* $H(z)$ *be a positive harmonic function in* $G'\subset G\in\mathcal{E}_0$ *such that* $H(z)=0$ *on* ∂G, $H(z)$ *is continuous on* ∂F *and* $\frac{\partial}{\partial n}H(z)=0$ *on* ∂F. *Then* $H(z)$ *is full superharmonic in* G'.

Proof. $H(z)$ can be continued into $\hat{G}'$ across ∂F and $H(z)$ is harmonic in $\tilde{G}'$ by putting $H(\hat{z})=H(z)$. Suppose G is an end of a Riemann surface R and let $\{R_n\}$ be its exhaustion and let R'_n be the symmetric image of R'_n relative to ∂F, where $R'_n=R_n-F$. Put $\tilde{R}_n=R'_n+\hat{R}'_n+\sum'\partial F_i$, where the summation is over F_i contained in R_n. Let $\omega_n(z)$ be a harmonic function in $\tilde{G}'\cap\tilde{R}_n$ such that $\omega_n(z)=0$ on $\partial\tilde{G}=\partial G+\partial\hat{G}=1$ on $\partial R_n+\partial\hat{R}_n$. Since $G\in\mathcal{E}_0$, $\tilde{G}'\in\mathcal{E}_0$ and $\lim_n\omega_n(z)=0$. Let $\Omega=\{z\in G'\colon H(z)>M\}$. Put $\tilde{\Omega}=\Omega+\hat{\Omega}+\partial F$, where $\hat{\Omega}$ is the symmetric image of Ω. Let $H'_n(z)$, $H_n(z)$ and $H''_n(z)$ be harmonic functions in $\tilde{G}'\cap\tilde{R}_n-\tilde{\Omega}$ such that $H'_n(z)=H_n(z)=H''_n(z)=0$ on $\partial\tilde{G}$,

$H'_n(z)=H_n(z)=H''_n(z)=M$ on $\partial\tilde{\Omega}\cap R_n$ and $H'_n(z)=\frac{\partial}{\partial n}H_n(z)=0$, $H''_n(z)=M$ on $\partial\tilde{R}_n-\tilde{\Omega}$. Then $H'_n(z)\leqq H_n(z)\leqq H''_n(z)$, $H'_n(z)\leqq H(z)\leqq H''_n(z)$ and $0\leqq H''_n(z)-H'_n(z)\leqq M\ \omega_n(z)$. Let $n\to\infty$. Then $\lim_n H_n(z)=H(z)$ in $\tilde{G}-\tilde{\Omega}$. Hence

$$D\left(\min\left(M, H(z)\right)\right)\leqq \varliminf_{n=\infty} D\left(H_n(z)\right)=\frac{1}{2}M\int_{\partial\tilde{G}}\frac{\partial}{\partial n}H(z)ds<\infty\,.$$

Let D be a regularly compact set in G', then there exists a const. M such that $H(z)<M$ on D. Let $H'_n(z)$, $H_n(z)$ and $H''_n(z)$ be harmonic function in $\tilde{G}'\cap\tilde{R}_n-\tilde{D}:\tilde{D}=D+\hat{D}$ such that $H'_n(z)=H_n(z)=H''_n(z)=H(z)$ on $\partial\tilde{G}+\partial\tilde{D}$, $H'_n(z)=\frac{\partial}{\partial n}H_n(z)=0$, $H''_n(z)=M$ on $\partial\tilde{R}_n-\tilde{D}$. Then ${}_{\tilde{D}}H(z)=\lim_{n=\infty}H_n(z)=\lim_{n=\infty}H''_n(z)=\lim_{n=\infty}H'_n(z)\leqq H(z)$. Hence $H(z)$ is full superharmonic in G'.

LEMMA 3. *Let $L(z)$ be a positive harmonic function in $\tilde{G}'$ such that $L(z)=0$ on $\partial\tilde{G}$ and $L(z)=L(\hat{z})$. Then by Lemma $L(z)$ is full superharmonic in G'. Let $L'(z)$ be a positive harmonic and full superharmonic function in G' such that $L(z)\geqq L'(z)$, $L(z)-L'(z)$ is full superharmonic in G'. Then $\frac{\partial}{\partial n}L'(z)=0$ on ∂F and $L'(z)$ can be continued harmonically into $\hat{G}'$ across ∂F by putting $L'(\hat{z})=L'(z)$.*

PROOF. Since $L(z)$ is harmonic on ∂F, $\frac{\partial}{\partial n}L(z)=0$ on ∂F. Let $v(F_i)$ be a neighbourhood of F_i with compact relative boundary $\partial v(F_i)$. Then we see at once $L(z)$ has $M.D.I.$ over $v(F_i)$ among all harmonic functions in $v(F_i)$ with the same value as $L(z)$ on $\partial v(F_i)$. Hence ${}_{\widetilde{G'-v(F_i)}}L(z)=L(z)$, where $L(z)$ is regarded as a function only in G'. Also by the full superharmonicity ${}_{\widetilde{G'-v(F_i)}}L'(z)\leqq L'(z)$, ${}_{\widetilde{G'-v(F_i)}}(L(z)-L'(z))\leqq L(z)-L'(z)$ in $v(F_i)$. Hence ${}_{\widetilde{G'-v(F_i)}}L'(z)=L'(z)$. This implies $\frac{\partial}{\partial n}L'(z)=0$ on ∂F and $L'(z)$ can be continued into $\hat{G}'$ by putting $L'(\hat{z})=L'(z)$.

THEOREM 8. *Let $G\in\mathcal{E}_0$ and let F_i $(i=1,2,\cdots)$ be a compact regular set such that $F=\sum_i F_i$ is thin at a boundary component $\mathfrak{p}\in\beta(G)$. Then N-Green's function: $p\in\nabla(\mathfrak{p}, G', L)$ is N-minimal if and only if*

$$L(z,p)=a\left(\tilde{K}(z,q)+\tilde{K}(z,\hat{q})\right),$$

where q and $\hat{q}\in\nabla(\tilde{\mathfrak{S}}(\mathfrak{p}), \tilde{G}', \tilde{M})\cap\underset{1}{\Delta}(\tilde{G}', \tilde{M})$, $\hat{q}$ is symmetric to q, $q\in A^1$ and $\hat{q}\in A^2$ of Corollary 2 of Theorem 7 and a is given by $2\pi/\int_{\partial\tilde{G}}\frac{\partial}{\partial n}\tilde{K}(z,q)ds$. Hence *by the same corollary*

$$\Delta\left(\mathfrak{S}'(\mathfrak{p}), G', L\right) \cap \underset{1}{\Delta}(G', L) \approx \nabla(\mathfrak{p}, G, M) \cap \underset{1}{\Delta}(G, M).$$

PROOF. Let $G(z, p^*)$ and $G'(z, p^*)$: $p^* \in G'$ be Green's functions of G and G' respectively. Then by $G \in \mathcal{E}_0$, $\inf_{z \in \mathfrak{B}_1(p)} G(z, p^*) > \delta_0 > 0$. Since F is thin at $\mathfrak{p}$,

$$\min_{z \in \partial \mathfrak{B}_n(p)} \frac{G'(z, p^*)}{G(z, p^*)} \geqq \delta_1 > 0 \qquad \text{for any } n.$$

Whence $\min_{z \in \partial \mathfrak{B}_n(\ )} G'(z, p) \geqq \delta_2 > 0$. Let $\{p_i\} \subset G'$ be a sequence on $\sum_n \partial \mathfrak{B}_n(\mathfrak{p})$ such
M'
that $p_i \xrightarrow{\quad} p$ in G' and that $G'(z, p_i)$ converges uniformly to a harmonic function $G'(z, \{p_i\})$. Then $G'(p^*, \{p_i\}) \geqq \delta_3$, $G'(z, \{p_i\}) = 0$ on ∂G and $G'(z, \{p_i\})$ is a positive harmonic function in G'. Let $L(z, p)$: $p \in \nabla(\mathfrak{S}'(\mathfrak{p}), G', L) \cap \underset{1}{\Delta}(G', L)$. Then $L(z, p)$ is N-minimal and by Theorem 1 there exists an L-tending path Γ in G'. Γ intersects $\partial \mathfrak{B}_n(\mathfrak{p})$ for $n \geqq n(\Gamma)$. Let $\{p_i\}$ be a sequence on $\Gamma \cap \sum_n \partial \mathfrak{B}_n(\mathfrak{p})$ such that $p_i \to \mathfrak{p}$ and $G'(z, p_i)$ converges to a positive harmonic function $G'(z, \{p_i\})$. Then by $L(z, p_i) \geqq G'(z, p_i)$.

$$L(z, p) \geqq G'\left(z, \{p_i\}\right) > 0. \tag{12}$$

By Lemma 1 $L(z, p)$ can be continued harmonically into $\hat{G}'$ so that $\underset{\tilde{G}'}{L}(\hat{z}, p) = L(z, p)$. In the following we suppose $L(z, p)$ is defined in $\tilde{G}'$. Let $\overset{\tilde{G}'}{\underset{G'}{I}} = \underset{G'}{I}$, $\overset{\tilde{G}'}{\underset{\hat{G}'}{I}} = \underset{\hat{G}'}{I}$, $\overset{\tilde{G}'}{\underset{G'}{E}} = \underset{G'}{E}$ and $\overset{\tilde{G}'}{\underset{\hat{G}'}{E}} = \underset{\hat{G}'}{E}$. Then by (12) and by the symmetry of $L(z, p)$ we have

$$\underset{G'}{I}\left[L(z, p)\right] > 0 \quad \text{and} \quad \underset{\hat{G}'}{I}\left[L(z, p)\right] > 0.$$

Put $U(z) = \underset{G'}{E}\,\underset{G'}{I}[L(z, p)]$. Then $0 < U(z) \leqq L(z, p)$. By $G' \cap \hat{G}' = 0$ we have

$$\underset{\hat{G}'}{I}\left[U(z)\right] = 0. \tag{13}$$

Put $V(z) = L(z, p) - U(z)$ $(\geqq 0)$. Then by $\underset{G'}{I}[U(z)] = \underset{G'}{I}\,\underset{G'}{E}\,\underset{G'}{I}[L(z, p)] = \underset{G'}{I}[L(z, p)]$ we have

$$\underset{G'}{I}\left[L(z, p) - U(z)\right] = \underset{G'}{I}\left[V(z)\right] = 0. \tag{14}$$

Let $V^*(z) = \underset{\hat{G}'}{E}\,\underset{\hat{G}'}{I}[V(z)]$. Then

$$V^*(z) \leqq V(z). \tag{15}$$

We shall show $V(z)=V^*(z)$. Now by (13)

$$V^*(z) = \underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[V(z)\right] = \underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[L(z,p)-U(z)\right]$$
$$= \underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[L(z,p)\right] - \underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[U(z)\right] = \underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[L(z,p)\right],$$

Hence by the structure of $V^*(z)$, $V^*(z)$ is symmetric to $U(z)$, whence by (13)

$$\underset{G'}{I}\left[V^*(z)\right] = 0. \tag{16}$$

Since $U(z)=V^*(\hat{z})$, $\frac{\partial}{\partial n}(U(z)+V^*(z))=\frac{\partial}{\partial n}(U(z)+V(z))=\frac{\partial}{\partial n}L(z,p)=0$ on ∂F, whence

$$\frac{\partial}{\partial n}\left(L(z,p)-\left(U(z)+V^*(z)\right)\right)=0 \qquad \text{on } \partial F.$$

By Lemma 2, $U(z)+V^*(z)$ and $L(z,p)-(U(z)+V^*(z))$ are full superharmonic in G'. By the N-minimality of $L(z,p)$ we have by (15)

$$U(z)+V^*(z) = aL(z,p): \quad 0<a\leqq 1.$$

On the other hand, by (16) $\underset{G'}{I}[U(z)]=a\underset{G'}{I}[L(z,p)]=a\underset{G'}{I}[U(z)]$, whence $a=1$ and

$$V^*(z) = V(z) \qquad \text{and}$$

$$L(z,p) = U(z)+V(z) = \underset{G'}{E}\underset{G'}{I}\left[L(z,p)\right]+V^*(z) \tag{17}$$
$$= \underset{G'}{E}\underset{G'}{I}\left[L(z,p)\right]+\underset{\hat{G}'}{E}\underset{\hat{G}'}{I}\left[L(z,p)\right] \qquad \text{in } \tilde{G}'.$$

We shall show $U(z)=\underset{G'}{E}\underset{G'}{I}[L(z,p)]$ is a minimal function in $\tilde{G}'$. Let $U'(z)$ be a positive harmonic function in $\tilde{G}'$ such that $0<U'(z)\leqq U(z)$. Put $V'(z)=U'(\hat{z})$. Then $\frac{\partial}{\partial n}(U'(z)+V'(z))=0$ on ∂F. Since the function $V(z)$ is symmetric to $U(z)$ by $V^*(z)=V(z)$, $\frac{\partial}{\partial n}((U(z)+V(z))=0$ on ∂F. Hence by Lemma 2 $U'(z)+V'(z)$, $(U(z)+V(z))-(U'(z)+V'(z))$ are full superharmonic in G'. By the N-minimality of $L(z,p)$ we have

$$U'(z)+V'(z) = aL(z,p) = a\left(U(z)+V(z)\right): \quad 0<a\leqq 1. \tag{18}$$

By $U(z)-U'(z)\geqq 0$ we have $U(z)-U'(z)\geqq \underset{G'}{E}\underset{G'}{I}[U(z)-U'(z)]$, because EI are positive linear operators. Whence $U(z)-\underset{G'}{E}\,\underset{G'}{I}[U(z)]\geqq U'(z)-\underset{G'}{E}\,\underset{G'}{I}[U'(z)]$. By $U(z)=\underset{G'}{E}\,\underset{G'}{I}[L(z,p)]=\underset{G'}{E}\,\underset{G'}{I}\,\underset{G'}{E}\,\underset{G'}{I}[L(z,p)]=\underset{G'}{E}\,\underset{G'}{I}[U(z)]$ we have

$$U'(z)=\underset{G'}{E}\,\underset{G'}{I}\Big[U'(z)\Big]. \tag{19}$$

On the other hand, by $U'(z)\leqq U(z)$ and $\underset{\hat{G}'}{E}\,\underset{\hat{G}'}{I}[U(z)]=0$ (by (13)) we have

$$\underset{\hat{G}'}{E}\,\underset{\hat{G}'}{I}\Big[U'(z)\Big]=0 \quad \text{and} \quad \underset{G'}{E}\,\underset{G'}{I}\Big[V'(z)\Big]=0. \tag{20}$$

Hence by (19), (20), (18) and (16) we have

$$U'(z)=\underset{G'}{E}\,\underset{G'}{I}\Big[U'(z)+V'(z)\Big]=a\Big(\underset{G'}{E}\,\underset{G'}{I}\Big[U(z)+V(z)\Big]\Big)=aU(z).$$

Thus $U(z)$ ($V(z)$ is symmetric to $U(z)$) and $V(z)$ are minimal in $\widetilde{G}'$ and there exists a uniquely determined point $q\in\underset{1}{\Delta}(\widetilde{G}',\widetilde{M})$ such that $U(z)=a\widetilde{K}(z,q)$: $a>0$. By $\widetilde{G}'\in\mathcal{E}_0$ $\sup_{z\in\widetilde{G}'}\widetilde{K}(z,q)=\infty$. Let $\mathfrak{q}$ be a component $\in\beta(\widetilde{G}')$ such that q lies over $\mathfrak{q}$. Then it is well known

$$\overline{\lim_{z\to\mathfrak{q}}}\,K(z,q)=\infty.$$

Let $\widetilde{\mathfrak{V}}_n$ be the doubled open set of $\mathfrak{V}_n(\mathfrak{p})\cap G'$. Then $\{\widetilde{\mathfrak{V}}_n\}$ determines $\widetilde{\mathfrak{S}}(\mathfrak{p})\subset\beta(\widetilde{G})$. At the top of the proof it was shown the following: for $p\in\nabla(\mathfrak{S}'(\mathfrak{p}),G'L)\cap\underset{1}{\Delta}(G',L)$

$$L(z,p)=\lim_{i=\infty}L(z,p_i):\quad p_i\in\partial\mathfrak{V}_{n(i)}(\mathfrak{p}).$$

By $L(z,p_i)={}_{\mathfrak{V}_n\widetilde{(p)\cap G_1}}L(z,p)$ for $i\geqq i(n)$ we have $\sup_{z\in C\widetilde{\mathfrak{V}}_n}L(z,p)\leqq\max_{z\in\partial\mathfrak{V}_n(p)\cap G'}L(z,p)$ $<\infty$ for any n, because $L(z,p)$ is symmetric relative to ∂F. Assume $\mathfrak{q}\notin\widetilde{\mathfrak{S}}(\mathfrak{p})$. Then there exists $\mathfrak{V}(\mathfrak{q})$ such that $\mathfrak{V}(\mathfrak{q})\subset C\widetilde{\mathfrak{V}}_n$, where $\mathfrak{V}(\mathfrak{q})$ is a neighbourhood of $\mathfrak{q}$ relative to Kerékjártó's top. over $\widetilde{G}'$. Hence by $a\widetilde{K}(z,q)=U(z)\leqq L(z,p)$ $\overline{\lim_{z\to\mathfrak{q}}}\,\widetilde{K}(z,q)<\infty$. This is a contradiction. Hence $\mathfrak{q}\in\widetilde{\mathfrak{S}}(\mathfrak{p})$ and $q\in\nabla(\mathfrak{S}(\mathfrak{p}),\widetilde{G}',\widetilde{M})\cap\underset{1}{\Delta}(\widetilde{G}',\widetilde{M})$. By $V(z)=U(\hat{z})$, $V(z)=a\widetilde{K}(z,\hat{q})$ and $\hat{\mathfrak{q}}\in\nabla(\widetilde{\mathfrak{S}}(\mathfrak{p}),\widetilde{G}',\widetilde{M})\cap\underset{1}{\Delta}(\widetilde{G},\widetilde{M})$ and $q\in A^1$, $\hat{q}\in A^2$. Since $\int_{\partial\widetilde{G}}\frac{\partial}{\partial n}L(z,p)ds=4\pi$, a is given by $2\pi\Big/\int_{\partial\widetilde{G}}\frac{\partial}{\partial n}\widetilde{K}(z,q)ds$. Thus, if $L(z,p)$: $p\in\nabla(\mathfrak{S}'(\mathfrak{p}),G',L)\cap\underset{1}{\Delta}(G',L)$, $L(z,p)=a(\widetilde{K}(z,q)+K(z,\hat{q}))$.

Let q and $\hat{q}\in\nabla(\widetilde{\mathfrak{S}}(\mathfrak{p}),\widetilde{G}',\widetilde{M})\cap\underset{1}{\Delta}(\widetilde{G}',\widetilde{M})$ and $\widetilde{K}(\hat{z},\hat{q})=\widetilde{K}(z,q)$. Put $L(z)=\widetilde{K}(z,q)+\widetilde{K}(z,\hat{q})$. We shall show $L(z)$ is N-minimal. Let $q\in\nabla(\widetilde{\mathfrak{S}}(\mathfrak{p}),$

$\widetilde{G}', \widetilde{M}) \cap \underset{1}{\Delta}(\widetilde{G}', \widetilde{M})$. Then by Theorem 1 there exists an $\widetilde{M}$-tending path Γ to q. Γ intersects $\partial\widetilde{\mathfrak{B}}_n$: $n \geqq n(\Gamma)$, hence we can find a sequence $\{q_i\}$ on $\Gamma \cap \sum_n \partial\mathfrak{B}_n(\mathfrak{p})$ or $\Gamma \cap \sum_n \hat{\mathfrak{B}}_n$ such that $q_i \xrightarrow{\widetilde{M}} q$. Without loss of generality we can suppose $\{q_i\} \subset \Gamma \cap \sum_n \mathfrak{B}_n(\mathfrak{p})$. Let $\{q_{i'}\}$ be a subsequence of $\{q_i\}$ such that $\widetilde{G}(z, q_{i'})$ and $G'(z, q_{i'})$ converge to positive harmonic functions $\widetilde{G}(z, \{q_{i'}\})$ and $G'(z, \{q_{i'}\})$ respectively. Because F is thin at $\mathfrak{p}$ and $\min_{z \in \partial\mathfrak{B}_n(p)} G'(z, p^*) \geqq \delta > 0$ for $n \geqq n_0$ and $G(p^*, \{q_{i'}\}) > 0$. Hence

$$\widetilde{K}(z, q) = \frac{\widetilde{G}(z, \{q_{i'}\})}{\widetilde{G}(p^*, \{q_{i'}\})} \geqq \frac{G'(z, \{q_{i'}\})}{\widetilde{G}(p^*, \{q_{i'}\})} \quad \text{and} \quad \underset{G'}{I}\left[K(z, q)\right] > 0 .$$

Since $\widetilde{K}(z, q)$ is minimal in $\widetilde{G}'$,

$$\underset{G'}{E}\,\underset{G'}{I}\left[\widetilde{K}(z, q)\right] = \widetilde{K}(z, q) . \tag{21}$$

By $G' \cap \hat{G}' = 0$, $0 = \underset{\hat{G}'}{I}\,\underset{G'}{E}\,\underset{G'}{I}[\widetilde{K}(z, q)] = \underset{\hat{G}'}{I}[\widetilde{K}(z, q)] = 0$. Since $\widetilde{K}(\hat{z}, \hat{q}) = \widetilde{K}(z, q)$, we have

$$\underset{\hat{G}'}{E}\,\underset{\hat{G}'}{I}\left[\widetilde{K}(z, \hat{q})\right] = \widetilde{K}(z, \hat{q}) \quad \text{and} \quad \underset{G'}{I}\left[\widetilde{K}(z, \hat{q})\right] = 0 . \tag{22}$$

Hence

$$L(z) = \underset{G'}{E}\,\underset{G'}{I}\left[L(z)\right] + \underset{\hat{G}'}{E}\,\underset{\hat{G}'}{I}\left[L(z)\right]. \tag{23}$$

Clearly $\frac{\partial}{\partial n} L(z) = 0$ on ∂F and by Lemma 2 $L(z)$ is full superharmonic in G'. Let $L'(z)$ be a positive harmonic function such that $L(z) - L'(z)$ and $L'(z)$ are positive full superharmonic in G'. It is sufficient to show $L'(z) = cL(z)$: $0 \leqq c \leqq 1$. By Lemma 3 $L'(z)$ can be continued harmonically into $\hat{G}'$ across ∂F by putting $L'(\hat{z}) = L'(z)$. We denote the continued function in $\widetilde{G}'$ also by $L'(z)$. Now $L(z) \geqq L'(z)$ and

$$L(z) - L'(z) \geqq \underset{G'}{E}\,\underset{G'}{I}\left[L(z) - L'(z)\right] = \underset{G'}{E}\,\underset{G'}{I}\left[L(z)\right] - \underset{G'}{E}\,\underset{G'}{I}\left[L'(z)\right], \quad \text{whence}$$

$$L(z) - \underset{G'}{E}\,\underset{G'}{I}\left[L(z)\right] \geqq L'(z) - \underset{G'}{E}\,\underset{G'}{I}\left[L'(z)\right] \geqq 0 .$$

Hence by (23) and (22)

$$\widetilde{K}(z, \hat{q}) = L(z) - \underset{G'}{E}\,\underset{G'}{I}\left[L(z)\right] \geqq L'(z) - \underset{G'}{E}\,\underset{G'}{I}\left[L'(z)\right].$$

By the minimality of $\widetilde{K}(z, \hat{q})$

$$L'(z)-\underset{G'}{E}\,\underset{G'}{I}\left[L'(z)\right]=\alpha\widetilde{\widetilde{K}}(z,\hat{q}):\quad 0\leqq\alpha\leqq 1\,. \tag{24}$$

Also by (22) and (21)

$$\widetilde{\widetilde{K}}(z,q)=\underset{G'}{E}\,\underset{G'}{I}\left[\widetilde{\widetilde{K}}(z,q)\right]=\underset{G'}{E}\,\underset{G'}{I}\left[L(z)\right]\geqq\underset{G'}{E}\,\underset{G'}{I}\left[L'(z)\right].$$

Since $\widetilde{\widetilde{K}}(z,q)$ is minimal $\underset{G'}{E}\,\underset{G'}{I}[L'(z)]=\alpha'\widetilde{\widetilde{K}}(z,q)$: $0\leqq\alpha'\leqq 1$. Hence by (24)

$$L'(z)=\alpha'\widetilde{\widetilde{K}}(z,q)+\alpha'\widetilde{\widetilde{K}}(z,\hat{q})\,.$$

Since $L'(z)=L'(\hat{z})$, $\alpha\widetilde{\widetilde{K}}(z,q)+\alpha'\widetilde{\widetilde{K}}(z,\hat{q})=\alpha'\widetilde{\widetilde{K}}(z,q)+\alpha\widetilde{\widetilde{K}}(z,\hat{q})$. Now $\hat{q}\neq q$ implies $\alpha=\alpha'$. Hence $L'(z)=\alpha L(z)$. This means $L(z)$ is N-minimal. Hence there exists a uniquely determined point $p\in\underset{1}{\Delta}(G',L)$ such that $L(z)=aL(z,p)$ and a is given by $2\pi\Big/\int_{\partial\widetilde{G}}\frac{\partial}{\partial n}\widetilde{\widetilde{K}}(z,q)ds$. Since $\sup\widetilde{\widetilde{K}}(z,q)=\infty$ and $\overline{\lim_{z\to\mathfrak{q}}}K(z,q)<\infty$ for $\mathfrak{q}\in\beta(G)$ and $\mathfrak{p}\neq\mathfrak{q}$, $p\in\nabla(\mathfrak{S}'(\mathfrak{p}),G',L)\cap\underset{1}{\Delta}(G',L)$. Hence for any pair q and $\hat{q}$: $q\in A^1$, there exists a uniquely determined point p in $\nabla(\mathfrak{S}'(\mathfrak{p}),G',L)\cap\underset{1}{\Delta}(G'\,L)$. Conversely for any point p in $\nabla(\mathfrak{S}'(\mathfrak{p}),G',L)\cap\underset{1}{\Delta}(G',L)$ there exists a pair q and $\hat{q}$: $q\in A^1$. Hence by Corollary 2 of Theorem 7

$$\nabla\left(\mathfrak{S}'(\mathfrak{p}),G',L\right)\cap\underset{1}{\Delta}(G',L)\approxeq\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)\,.$$

Striclty thiness of F at $\mathfrak{p}\in\beta(G)$. Let $G\in\mathcal{E}$ and let $\mathfrak{p}\in\beta(G)$. Let F_i $(i=1,2,\cdots)$ be a compact set and let $F=\sum_i F_i$ such that $G-F$ is connected, $\{F_i\}$ clusters only at $\beta(G)$. If there exists a determining sequence $\{\mathfrak{V}_n(\mathfrak{p})\}$ such that $\min_{z\in\partial\mathfrak{V}_n(p)}G'(z,p^*)>\delta>0$: $n=1,2,\cdots$, we say F is strictly thin at $\mathfrak{p}$, where $G'(z,p^*)$ is a Green's function of $G'=G-F$: $p^*\in G'$. Clearly if $G\in\mathcal{E}_0$ and F is thin at $\mathfrak{p}$, F is strictly thin at $\mathfrak{p}$. Suppose F_i is regular compact set. Then the doubled surface $\widetilde{G}'$ of G' relative to ∂F can be considered. We see Lemma 1 is valid for $G\in\mathcal{E}$ not neccessarily $G\in\mathcal{E}_0$. Also we see Lemma 2 and Lemma 3 hold not only for $G\in\mathcal{E}_0$ but also for G' such that $\widetilde{G}'$ (of G')$\in\mathcal{E}_0$. We proved Theorem 8 under the condition $G\in\mathcal{E}_0$. But the proof of

$$\nabla\left(\mathfrak{S}'(\mathfrak{p}),G',L\right)\cap\underset{1}{\Delta}(G',L)\approxeq\nabla(\mathfrak{p},G,M)\cap\underset{1}{\Delta}(G,M)$$

depends following two facts:

a). $\overset{\widetilde{G}'}{\underset{G'}{I}}\left[L(z,p)\right]>0\,.$

b). *Any positive harmonic function $U(z)$ with $U(z)=0$ on $\partial\widetilde{G}$ and*

$\frac{\partial}{\partial n}U(z)=0$ *on* ∂F *is full superharmonic in* G'.

Now we see at once a) is satisfied under the condition that F is strictly thin at $\mathfrak{p}$ and b) is satisfied under the condition that $\widetilde{G}'\in\mathcal{E}_0$. Hence we have the following

COROLLARY 1. *Let* $G\in\mathcal{E}$ *and* F *be strictly thin at* $\mathfrak{p}\in\beta(G)$ *and* F_i *be regular compact. If* $\widetilde{G}'$ *(doubled surface of* $G'=G-F$*)*$\in\mathcal{E}_0$, *then*

$$\Delta\big(\mathfrak{S}'(\mathfrak{p}), G', L\big)\cap\underset{1}{\Delta}(G', L)\approx\nabla\big(\mathfrak{S}'(\mathfrak{p}), G', M'\big)\cap\underset{1}{\Delta}(G', M')$$
$$\approx\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G,M).$$

Let $G'\in\mathcal{E}$ and $G'=G-F$ be of planar character. Suppose F is strictly thin at $\mathfrak{p}\in\beta(G)$. Map G' conformally onto a domain Ω in $|\xi|<1$ by $\xi=g(z)$. Then similarly as Corollary 3 of Theorem 7 we can prove 1). $\bigcap_n\overline{g(\mathfrak{V}_n(\mathfrak{p})\cap G'}=$ one point, 2). $\mathfrak{S}'(\mathfrak{p})$, the set of boundary components of $\beta(G)$ lying over $\mathfrak{p}$ consists of only one component $\mathfrak{p}'$ and 3). $\nabla(\mathfrak{p}', G', M')\cap\underset{1}{\Delta}(G', M')=$ one point. Suppose F_i is compact set. Then $\widetilde{G}'$ can be considered. If $\widetilde{G}'\in\mathcal{E}_0$, then by Corollary 1 $\nabla(\mathfrak{p}', G', L)\cap\underset{1}{\Delta}(G', L)$ consists of only one point. Now $L(z,p)$: $p\in\nabla(\mathfrak{p}', G', L)\cap\Delta(G', L)$ is represented by a canonical measure on $\nabla(\mathfrak{p}', G', L)\cap\underset{1}{\Delta}(G, L)$. Hence $\nabla(\mathfrak{p}', G', L)\cap\underset{1}{\Delta}(G', L)=\nabla(\mathfrak{p}, G', L)\cap\Delta(G', L)$. Hence we have

COROLLARY 2. *Let* F_i *be a regular compact set and* $F=\sum_i F_i$ *be strictly thin at* $\mathfrak{p}\in\beta(G)$ *and* G' *be of planar character. If* $\widetilde{G}'\in\mathcal{E}_0$,

$$\nabla(\mathfrak{p}', G', L)\cap\Delta(G', L)=\nabla(\mathfrak{p}, G', L)\cap\underset{1}{\Delta}(G', L)=\textit{one point}.$$

4. Applications to conformal mappings. Let $G\in\mathcal{E}$ and ∂G consists of one component. Let F_i be a compact set such that $F=\sum_i F_i$ is strictly thin at $\mathfrak{p}\in\beta(G)$, $G'=G-F$ is of planar character and $\min_{z\in\partial\mathfrak{V}_n(p)} G'(z,\mathfrak{p}^*)>\delta>0$ for $n=1,2,\cdots$, where $\{\mathfrak{V}_n(\mathfrak{p})\}$ is a determining sequence of $\mathfrak{p}$. Then $\mathfrak{S}'(\mathfrak{p})$ consists of only one component $\mathfrak{p}'$. We shall prove

THEOREM 9. *Let* $G\in\mathcal{E}$ *and let* F *be strictly thin at* $\mathfrak{p}\in\beta(G)$ *such that* F_i *is a regular compact set,* ∂G *consists of only one component,* $G'=G-F$ *is of planar character and* $\widetilde{G}'$ *(doubled surface of* G'*)*$\in\mathcal{E}_0$. *Then we can map* G' *conformally onto a domain* Ω *in* $|w|<1$ *by* $w=g(z)$ *such that* F_i *is mapped onto a radial slit,* $\bigcap_n\overline{g(\mathfrak{V}_n(\mathfrak{p})\cap G')}=w_0$ *and* $\partial G\to|w|=1$. *Then* *such function $g(z)$ is uniquely determined except roation and Ω has the Gross's property.*

PROOF. By Corollary 2 of Theorem 8 $\nabla(\mathfrak{p}', G', L)\cap\underset{1}{\Delta}(G', L)=\nabla(\mathfrak{p}' G', L)\cap\underset{1}{\Delta}(G', L)=$one point p. This means $L(z, p_i)\rightarrow L(z, p)$ for any sequence $\{p_i\}$ tending to $\mathfrak{p}'$. Hence $\{G'\cap\mathfrak{B}_m(\mathfrak{p})\}$ and $\{v_n(p)\}$ are equivalent, where $v_n(p)=\left\{z\in G': L\text{-dist}(p, z)<\frac{1}{n}\right\}$. Let $v_n(\beta)$ be a neighbourhood of $\beta(G)$, i.e. $G'-v_n(\beta)$ is compact and $v_n(\beta)\searrow 0$ as $n\rightarrow\infty$. Let $\omega(T, z, G')$ be a harmonic function in $G'-T$ such that $\omega(T, z, G')=0$ on $\partial G=1$ on T and $\omega(T, z, G')$ has M.D.I over $G'-T$, where T is a closed set with $T\cap\partial G=0$. Put $\omega(\Delta, z, G')=\lim_{n=\infty}\omega(v_n(\beta)\cap G', z, G')$ and $\omega(p, z, G')=\lim_{n}\omega(v_n(p), z, G)$. Then by $\widetilde{G}'\in\mathcal{E}_0$ we have

$$0=\omega(\Delta, z, G')=\omega(p, z, G').$$

Hence p is not singular [9], i.e. $\sup_{z\in G'} L(z, p)=\infty$. Let $V_M(p)=\{z\in G': L(z, p)>M\}$. Then by $p\in\underset{1}{\Delta}(G', L)$, for any $V_M(p)$ there exists a $v_n(p)$ such that $V_M(p)\supset(v_n(p)\cap G')$ [10]. Hence $L(z, p)\rightarrow\infty$ as $z\rightarrow\mathfrak{p}'$. Next $\widehat{_{v_n(p)}L}(z, p)={}_{\mathfrak{B}_n(p)\cap G'}L(z, p)=L(z, p)$ implies $\sup_{z\in C\mathfrak{B}_n(p)} L(z, p)\leqq\max_{z\in\partial\mathfrak{B}_n(p)} L(z, p)<\infty$. Hence

1). *$L(z, p)=\infty$ at $\mathfrak{p}'$ and* $\sup L(z, p)<\infty$ *in $G'-\mathfrak{B}_n(\mathfrak{p}')$.*

It is known $L(z, p)$ has the following properties [11].

2). $\frac{\partial}{\partial n}L(z, p)=0$ *on* ∂F.

3). $\int_{\partial V_M(p)}\frac{\partial}{\partial n}L(z, p)ds=2\pi$ *for almost* M: $0<M<\infty$.

4). $\int_{\partial G}\frac{\partial}{\partial n}L(z, p)ds=2\pi$.

Hence

5). *Let γ be a smooth Jordan curve in G'. Then* $\int_{\gamma}\frac{\partial}{\partial n}L(z, p)ds=0$ *or 2π according as γ encloses p or not.*

Let $H(z, p)$ be the conjugate harmonic function of $L(z, p)$. Then $\exp(L(z, p)+iH(z, p))=g(z)$ maps G' onto Ω satisfing the conditions of Theorem. *We shall show such $g(z)$ is uniquely determined.* Suppose an analytic function $F(z)$ such that 1). $|F(z)|=1$ on ∂G, 2). $|F(z)|<1$ in G'. 3). $F(z)$ maps ∂F_i onto a radial slit. 4). $\inf_{z\in G'-\mathfrak{B}_n(p)}|F(z)|>0$ for any n. Then $U(z)=-\log|F(z)|$

is positive harmonic in G' and $U(z)=0$ on ∂G. 3) implies $\frac{\partial}{\partial n}U(z)=0$ on ∂F_i. Hence by $\widetilde{G}'\in\mathcal{E}_0$ (by Corollary 2 of Theorem 7) $U(z)$ is full superharmonic in G'. Then $U(z)=\int_{\underset{1}{\Delta}(G',L)} L(z,q)d\mu(q)$. By 1) $\int d\mu(q)=1$. Assume μ has a positive canonical measure on $\underset{1}{\Delta}(G',L)$ outside of $\mathfrak{V}_n(\mathfrak{p})$. Since $\mu=0$ on ∂F_i and since $\{F_i\}$ clusters at $\beta(G)$, we can find a closed set A in $\underset{1}{\Delta}(G',L)-\mathfrak{V}_n(\mathfrak{p})$ such that $A\subset\bigcap\overline{(v_n(\beta)\cap G')}^{L}$ and $\mu>0$ on A. By $\omega(\Delta,z,G')=0$ A is a set of capacity zero, whence

$$\infty=\sup_{z\in A}L(z,p)\leqq\sup_{z\in\partial\mathfrak{V}_n(p)\cap G'}L(z,p)>\infty \quad [12].$$

This is a contradiction. Hence $\mu>0$ on only at $V(\mathfrak{p}',G'L)\cap\underset{1}{\Delta}(G',L)=p$. Thus $U(z)=L(z,p)$ and the uniqueness of $g(z)$ is proved.

We show Ω has the Gross's property. Let F_i^W be the image of ∂F_i: $F_i^W=\{re^{i\theta}:\ r_i\leqq r\leqq r_i',\ \theta=\theta_i\}$. Let $R_n=1>|w|>\frac{1}{n}$ and Θ_n be the angular projection of $\sum^1 F_i^W$, where the summation is over F_i^W s contained in R_n. We show mes $\bar{\Theta}_n=0$ for any n. Put $\delta_n^M=\max_{z\in\partial\mathfrak{V}_n(p)}|g(z)|$, $\delta_n^N=\min_{z\in\partial\mathfrak{V}_n(p)}|g(z)|$. Then $\delta_n^M\geqq\delta_n^N>0$ and $\lim_{n=\infty}\delta_n^M=0$ by $\bigcap_n\overline{g(\mathfrak{V}_n(\mathfrak{p})\cap G')}=$one point $\{w=0\}$. For any given $\frac{1}{n}$, there exists a number n_0 such that $\delta_{n_0}^M<\frac{1}{n}$. We consider only $G'-\mathfrak{V}_{n_0}(\mathfrak{p})$. Then any F_i contained in $G-\mathfrak{V}_{n_0}(\mathfrak{p})$ is mapped onto a radial slit in $1>|w|\geqq\delta_{n_0}^N$. Let $\theta'_{n_0}=\bigcup_\theta\{\theta:\ re^{i\theta}\subset\sum' F_i^W\}$, where the summation is over F_i contained in $G-\mathfrak{V}_{n_0}(\mathfrak{p})$. Then $\Theta_n\subset\theta'_{n_0}$. By $\widetilde{G}'\in\mathcal{E}_0$ [13] and by Evans's theorem there exists a positive harmonic function $U(z)$ in $\widetilde{G}'$ such that $U(z)=0$ on $\partial\widetilde{G}$, $(U(z)\to\infty$ as $z\to\beta(\widetilde{G}')$ and $\int_{\gamma_M}\frac{\partial}{\partial n}U(z)ds=2\pi$: $\gamma_M=\{z\in\widetilde{G}':\ U(z)=M\}$. Consider $U(z)$ in G'. Then $\int_{\gamma_M\cap G'}\frac{\partial}{\partial n}U(z)ds\leqq 2\pi$. Now the area of $\Omega<\infty$. By the length and area's method we see that there exists a sequence $\{M_l\}$: $l=1,2,\cdots$, such that 1). $M_l\to\infty$ as $l\to\infty$. 2). the length of $g(\gamma_{M_l})=\varepsilon_l$, $\varepsilon_l\to 0$ as $l\to\infty$. Since $\max_{z\in\partial\mathfrak{V}_{n_0}(p)}U(z)<\infty$, there exists a number l_0 such that γ_{M_l} does not touch $\partial\mathfrak{V}_{n_0}(\mathfrak{p})$ for $l\geqq l_0$. Let γ'_{M_l} $(l\geqq l_0)$ be the part of γ_{M_l} in $G'-\mathfrak{V}_{n_0}(\mathfrak{p})$. Then $g(\gamma_{M_l})$ separates $\beta(G)-\overline{\mathfrak{V}_{n_0}(\mathfrak{p})}$ from $\mathfrak{V}_{n_0}(\mathfrak{p})\cap G'$ and $g(\gamma'_{M_l})$ is contained in $1>|w|\geqq\delta_{n_0}^N$ and $g(\gamma'_{M_l})$ separates the limiting points of $\overline{\sum' F_i^W}$ from $|w|=1$ and $g(\mathfrak{B}_{n_0}(\mathfrak{p})\cap G')$, where the summation is over F_i^W outside of $g(\mathfrak{B}_{n_0}(\mathfrak{p})\cap G')$. Because $|g(z)|\geqq\delta_{n_0}^N$ for $z\in G'-\mathfrak{B}_{n_0}(\mathfrak{p})$. Since the angular measure of $\sum g(F_i)=0$, $\varepsilon_l'\geqq$ length of $g(\gamma'_{M_l})\geqq \text{mes}\,\bar{\theta}_{n_0}\times\delta_{n_0}^N$. Let $l\to\infty$. Then

$$\text{mes}\,\bar{\Theta}_n\leqq\text{mes}\,\bar{\theta}_{n_0}=0\,.$$

Hence Ω has the Gross's property.

REMARK. If $G\in\mathcal{E}_0$ and F_i is a regular set, then $\widetilde{G}'\in\mathcal{E}_0$. Since in the case $G\in\mathcal{E}_0$ F is thin if and only if F is strictly thin. Hence Theorem 9 is valid under the condition $G\in\mathcal{E}_0$ and F is thin at $\mathfrak{p}$ instead of $\widetilde{G}'\in\mathcal{E}_0$ and F is strictly thin at $\mathfrak{p}$.

THEOREM 10. *Let E be a closed set in $|z|<1$ of capacity zero with $E\ni z_0$. Let $G=\{|z|<1\}-E$ and let $F=\sum_i F_i$ (F_i is a compact continuum in G) such that F clusters only at E, $F\cap\{|z|=1\}=0$, z_0 is irregular for the Dirichlet problem in $G'=G-F$. Then there exists a uniquely determined function $w=g(z)$ mapping G' onto a domain Ω in $|w|<1$ with radial slits such that $|z|=1\to|w|=1$, $z_0\to\{w=0\}$ except rotation and Ω has the Gross's property.*

Map G' conformally by $\xi=h(z)$ onto a domain D' with circular slits in $|\xi|<1$ such that $\{|z|=1\}\to\{|\xi|=1\}$. Let $v_n(z_0)$ be a neighbourhood in G' of z_0 such that $\partial v_n(z)\cap F=0$ and $v_n(z_0)\to z_0$ as $n\to\infty$. Then since z_0 is irregular, $\bigcap_n\overline{h(v_n(z_0))}=$one point ξ_0. Since z_0 is irregular, we can fined a sequence of curves $\{\gamma_m\}$ such that γ_m encloses z_0 in G' and $\gamma_m\to z_0$ as $m\to\infty$ and a const. $\delta>0$ such that $\min_{z\in\gamma_m}G'(z,p^*)\geqq\delta>0$, where $G'(z,p^*)$ is a Green's function of G'. Let $\mathfrak{B}_m(z_0)$ be a domain bounded by γ_m containing z_0 in its interior. Then $\{\mathfrak{B}_m(z)\}$ is a determining sequence of z_0 and

$$\bigcap_m\mathfrak{B}_m(z_0)\subset\bigcap_n v_n(z_0)=z_0\,.$$

Let $\{G_l\}$ be an increasing sequence of domains such that $G_l\nearrow G$, ∂G_l containes $\{|z|=1\}$ for any l, ∂G_l consists of a finite number of analytic curves and $\partial G_l\cap F=0$. Since $\{F_i\}$ clusters at only E, G_l containes a finite number of F_i in G_l. F_i is mapped onto a circular slit J_i. Let $D_l=h(G_l-F)+\sum^1 J_l$, where the summation is over J_i such that $h^{-1}(J_i)$ is contained in G_l. Then D_l is a domain and $D_l-\sum^1 J_i=h(G_l-F)$. Put $D_l'=D_l-\sum^1 J_i$ and $D=\bigcup_l D_l$. Then D is a domain. Consider $h(\partial\mathfrak{B}_m(z_0))$ in D. Then $h(\partial\mathfrak{B}_m(z_0))\to\xi_0$ as $m\to\infty$ and $\min_{\xi\in h(\partial\mathfrak{B}_m(z_0))}G'(\xi,h(p^*))\geqq\delta$. Hence

$\sum J_i$ *is strictly thin at* ξ_0 *in regarding* D *as a domain.* (25)

Let $\omega'_l(\xi)$ be a harmonic function in D'_l such that $\omega'_l(\xi)=0$ on $\{|\xi|=1\}$, $=1$ on $h(\partial G_l)$ and $\omega'_l(\xi)$ has M.D.I. over D'_l. Let $\omega_l(z)$ be a harmonic function in G_l such that $\omega_l(z)=0$ on $\{|z|=1\}$ and $\omega_l(z)=1$ on ∂G_l. Then

$$D\big(\omega'_l(\xi)\big) \leqq D\big(\omega_l(z)\big): \quad \xi=\xi(z).$$

Since $G\in\mathcal{E}_0$, $\lim_{l=\infty} D(\omega_l(z))=0$. Now J_i is a regular set, the doubled surface $\tilde{D}'_l$ of D'_l can be considered. Consider $\omega'_l(\xi)$ in D'_l. Then $\frac{\partial}{\partial n}\omega'_l(\xi)=0$ on J_i. Put $\omega_l(\hat{\xi})=\omega_l(\xi)$ in $\hat{D}'_l$, where $\hat{D}'_l$ is the symmetric image of D'_l relative to $\sum J_i$. Then

$$\bigcup_l \tilde{D}'_l \in \mathcal{E}_0, \tag{26}$$

where $\tilde{D}'_l$ *is the doubled surface of* D'_l*.*

Put $D'=\bigcup_l D'$. Then $D'=D-\sum_i J_i=h(G')$. By (25), (26) and by Theorem 9 D' is mapped conformally by a uniquely determined function onto a domain with radial slits. Hence G' is mapped uniquely determined function $g(z)$ onto a domain Ω with radial slits. Similarly as Theorem 9 it is proved that Ω has the Gross's property.

5. Let R be a Riemann surface and let Ω be a subdomain in R such that $\partial\Omega$ consists of enumerably number of analytic curves clustering nowhere in R. Let $N(z,p)$ be an N-Green's function [14] of Ω with $N(z,p)=0$ on $\partial\Omega$ (in case of $L(z,p)$, $L(z,p)=0$ on a compact relative boundary). Then N-Martin's topology can be defined with following metric

$$\delta(p,q)=\sup_{z\in\Omega_0}\left|\frac{N(z,p)}{1+N(z,p)}-\frac{N(z,q)}{1+N(z,q)}\right|: \ p \text{ and } q\in\Omega+\Delta(\Omega,N),$$

where Ω_0 is a compact disk in Ω and $\Delta(\Omega,N)$ is the boundary of Ω obtained by the compactification of Ω.

Let $\Omega\subset G\in\mathcal{E}$ and L and N be N-Martin's top. s over G and Ω induced by $\{L(z,p)\}$ and $\{N(z,q)\}$ respectively. Then clearly

$$L(z,p)-{}_{C\Omega}\tilde{L}(z,p)=N(z,p) \qquad \text{for } p\in\Omega.$$

If a sequence $p_i \xrightarrow{L} p$ and $p_i \xrightarrow{N} q$, we say q lies over p. Then it is known if $p\in\Delta_1(G,L)$ and $C\Omega$ is thin at p, there exists a uniquely determined point $f(p)\in\Delta_1(\Omega,N)$ [15] such that

$$L(z,p)-{}_{\widetilde{CQ}}L(z,p)=N\big(z,f(p)\big).$$

Condition K. Let $G\in\mathcal{E}_0$. If $\beta(G)$ *consists of only one component* $\mathfrak{p}$ *and* $\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G,M)=\nabla(\mathfrak{p}, G, M)\cap\Delta(G,M)=$*one point* p, we say G satisfies the condition K.

Problem. Suppose G satisfies the condition K. Then

$$\lim_{z\to\mathfrak{p}} G^{\Lambda}(z,p^*)=0\ ?$$

for any analytic curve Λ *tending to* $\mathfrak{p}$, *where* $G^{\Lambda}(z,p^*)$: $p^*\in G-\Lambda$ *is a Green's function of* $G-\Lambda$.

Condition H. (M. Heins) [16]. Let $G\in\mathcal{E}_0$ such that $\beta(G)$ consists of only one component. If there exists a sequence of disjoint annuli A_n ($n=1,2,\cdots$) with analytic Jordan boundaries on G satisfying the condition that for each n, A_{n+1} separates A_n from $\mathfrak{p}$ and A_1 separates ∂G from $\mathfrak{p}$ and $\sum_n 1/M(A_n)=\infty$, then we say G satisfies the condition H, where $M(A_n)=1/D(U_n(z))$ and $U_n(z)$ is a harmonic function in A_n with $U_n(z)=1$ on γ of A_n and $U_n(z)=0$ on γ' of A_n, γ and γ' are boundary components of ∂A_n.

M. Heins proved [17], if G satisfies the condition H. $\nabla(\mathfrak{p}, G, M)\cap\Delta(G, M)$ consists of one point. Suppose G satisfies the condition H. Then $D_{(G-v(p^*))}G^{\Lambda}(z,p^*))\leqq 2\pi\max_{z\in\partial v(p^*)}G^{\Lambda}(z,p^*)$, where $v(p^*)$ is a neighbourhood of p^*. Hence there exists a number n_0 and a const. M such that

$$D_{\sum_{n_0} A_n}(G^{\Lambda}(z,p^*))\leqq M \quad\text{and}\quad \sum_{n_0} 1/M_n=\infty\,.$$

By the length and area's method we see that there exists a sequence of dividing cuts $\{\gamma_i\}$: $i=1,2,\cdots$ such that γ_i is contained in som $A_{j(i)}$, $\gamma_i\to\mathfrak{p}$ and $\int_{\gamma_i} d|G^{\Lambda}(z,p^*)|=\varepsilon_i\to 0$ as $i\to\infty$. Since γ_i intersects Λ, $\max_{z\in\gamma_i} G^{\Lambda}(z,p^*)\leqq\varepsilon_i$. By $G\in\mathcal{E}_0$ and by the maximum principle $\max_{z\in\gamma_i}G^{\Lambda}(z,p^*)=\sup_{z\in G_i}G^{\Lambda}(z,p^*)$ and $G(z,p^*)\to 0$ as $z\to\mathfrak{p}$, where G_i is the domain divided by γ_i and containing a neighbourhood $\mathfrak{V}(\mathfrak{p})$ of $\mathfrak{p}$. Hence *the condition H is stronger than the condition K for this problem.*

REMARK. We shall show that the condition K is necessary for the problem. Let $U(z)$ be a positive harmonic function in G with $U(z)=0$ on ∂G. Then by $G\in\mathcal{E}_0$ $D(\min(M, U(z)))=M\int_{\partial G}\frac{\partial}{\partial n}U(z)ds$ and ${}_{\bar D}U(z)={}_{D}U(z)\leqq U^{\Lambda}(z)$ for any compact regular set D in G. Hence any positive harmonic function with $U(z)=0$ on ∂G is full superharmonic and $U(z)$ is N-minimal if and only if $U(z)$ is minimal. Suppose there exist two points p_1 and p_2

in $\nabla(\mathfrak{p}, G, M)\cap \underset{1}{\Delta}(G, M)$. Then there exitst two points q_1 and q_2 in $\nabla(\mathfrak{p}, G, L)\cap \underset{1}{\Delta}(G, L)$. Let $v(q_i)$ be a neighbourhood of q_i relative to L-top. such that $v(q_1)\cap v(q_2)=0$ and $\partial v(q_1)$ consists of analytic curves clustering nowhere in G. By $q_2\in \underset{1}{\Delta}(G, L)$ there exists an L-tending path Λ in $v(q_2)$ to q_2. By $q_1\in \underset{1}{\Delta}(G, L)$, $L(z, q_1)-{}_{\widetilde{Cvq(_1)}}L(z, q_1)>0$. Let $N(z, r)$ be an N-Green's function of $v(q_1)$ and suppose N-Martin's top. N is defined on $v(q_1)+\Delta(v(q_1), N)$. Then by $q_1\in \underset{1}{\Delta}(G, L)$, there exists a point $f(q_1)$ in $\underset{1}{\Delta}(v(q_1), N)$ and $f(q_1)$ lies over q_1 such that

$$0<L(z, q_1)-{}_{\widetilde{Cv(q_1)}}L(z, q)=N\big(z, f(q_1)\big).$$

Hence there exists a sequence $\{p_i\}$ in $v(q_1)$ N-tending to q_1 such that $\lim\limits_{i=\infty} N(p_i, z_0)>0$. On the other hand, by $v(p_1)\subset G\in\mathcal{E}_0$, $N(z, z_0)=G''(z, z_0)$, where $G''(z, z_0)$ is a Green's function of $v(q_1)$. Then $\overline{\lim\limits_{z\to q_1}} G^{\Lambda}(z, z_0)\geqq \overline{\lim\limits_{z\to q_1}} G''(z, z_0)>0$. This implies $\mathfrak{p}$ is not regular for $G-\Lambda$. Hence the condition K is necessary for the problem. The problem is plausible but difficult. As a condition that G is almost of planar character we shall prove the following.

THEOREM 11. *Let $G\in\mathcal{E}_0$ satisfying the condition K. Let F be a thin set at $\mathfrak{p}$ such that $G-F=G'$ is of planar character. Let Λ be a Jordan curve in G' tending to $\mathfrak{p}$. Then $\mathfrak{p}$ is regular for the domain $G-\Lambda$.*

PROOF. Since F is thin at $\mathfrak{p}$, there exists a sequence $\{p_i\}$ tending to $\mathfrak{p}$ such that $G(z, p_i)$ and $G'(z, p_i)$ converge to positive harmonic functions $G(z, \{p_i\})$ and $G'(z, \{p_i\})$, where $G(z, p_i)$ and $G'(z, p_i)$ are Green's functions of G and G' respectively. Now since $\nabla(\mathfrak{p}, G, M)\cap \underset{1}{\Delta}(G, M)=$one point, $G(z, \{p_i\})=aK(z, p)$: $a>0$. By ${}_F(\lim\limits_{i=\infty} G(z, p_i))\leqq \underline{\lim\limits_{i=\infty}}\, {}_FG(z, p_i)$ we have

$$G\big(z, \{p_i\}\big)-{}_FG\big(z, \{p_i\}\big)\geqq G'\big(z, \{p_i\}\big)>0\,.$$

Assume $\mathfrak{p}$ is not regular for $G-\Lambda$. Then there exists a sequence $\{p_j\}$ such that $\overline{\lim\limits_{j=\infty}} G^{\Lambda}(z, p_j)>0$ and similarly as above we have

$$G\big(z, \{p_j\}\big)-{}_{\Lambda}G\big(z, \{p_j\}\big)>0\,.$$

By $G\in\mathcal{E}_0$, $L(z, p_i)=G(z, p_i)$ and $G(z, \{p_i\})$ is not only minimal but also N-minimal and $\nabla(\mathfrak{p}, G, L)\cap \Delta(G, L)=\nabla(\mathfrak{p}, G, L)\cap \underset{1}{\Delta}(G, L)=$one point (we denote it by the same symbole p). Then $L(z, p)=G(z, \{p_i\})=G(z, \{p_j\})$, ${}_{\tilde F}L(z, p)={}_FG(z, \{p_i\})$, ${}_{\tilde\Lambda}L(z, p)={}_{\Lambda}G(z, \{p_i\})$. Since the sum of two thin sets is a thin set,

$$N\big(z, f(p)\big) = L(z, p) - {}_{\widehat{F+\Lambda}}L(z, p) = G\big(z, \{p_i\}\big) - {}_{F+\Lambda}G\big(z, \{p_i\}\big) > 0\,,$$

where $N(z, f(p))$ is an N-Green's function of $G-\Lambda-F$ vanishing on $\partial G+\Lambda+\partial F$ and $f(p)$ lies on p.

Hence there exists a sequence $\{s_i\}$ in $G-\Lambda-F$ such that $s_i \to p$ and

$$\lim_{i=\infty} N(z, s_i) = \lim_{i=\infty} G^{F+\Lambda}(z, s_i) > 0\,, \qquad z \in G-\Lambda-F\,, \tag{27}$$

where $G^{F+\Lambda}(z, s_i)$ is a Green's function of $G-\Lambda-F$. This means $\mathfrak{p}$ is not regular for the domain $G-\Lambda-F$. Map G' conformally by $w=g(z)$ onto a domain Ω in $|w|<1$. Then $g(z)$ maps $\mathfrak{p}$ onto a point w_0, because F is thin at $\mathfrak{p}$. By the assumption Λ_W, the image of Λ is a continuum and Λ_W tends to w_0 in the image of G. Because if Λ crosses F_i, Λ_W may be divided into many components, since two sides of ∂F_i (∂F_i may have one side) may be generally mapped two components. Hence w_0 is a regular point for the domain $g(G-\Lambda-F)$. This contradicts (27). Hence we have the theorem.

6. Let $G \in \mathcal{E}_0$ and F_i be a compact continuum such that $F_i \cap F_j = 0$ for $i \neq j$, $F = \sum_i F_i$ clusters at only the ideal boundary of G, $\partial G \cap F = 0$ and $G' = G-F$ is connected. If there exists a determining sequence $\{\mathfrak{V}_n(\mathfrak{p})\}$ of $\mathfrak{p}$ such that $\partial\mathfrak{V}_n(\mathfrak{p})$ is a dividing cut and

$$\min_{z \in \partial\mathfrak{V}_n(p)} G'(z, p^*) > \delta > 0 \qquad \text{for any } n,$$

we say *F is completely thin at* $\mathfrak{p}$, where $G'(z, p^*)$: $p^* \in G'$ is a Green's function of G'.

LEMMA 4. *Let $G \in \mathcal{E}_0$, $\mathfrak{p} \in \beta(G)$ and let F be a thin set at $\mathfrak{p}$. Put $G' = G-F$. Let $\mathfrak{S}'(\mathfrak{p})$ be the set of components $\in \beta(G')$ lying over $\mathfrak{p}$. Let $\mathfrak{p} \in \nabla(\mathfrak{S}'(\mathfrak{p}), G', M') \cap \underset{1}{\Delta}(G', M')$ and $v(p)$ be an M'-neighbourhood relative to M'-top. over G'. Then there exists a path Γ to $\mathfrak{p}$ in $v(p) \cap G'$ such that*

$$\varliminf_{\substack{z \to p \\ z \in \Gamma}} G''(z, p^*) > 0 : p^* \in v(p)\,,$$

where $G''(z, p^*)$ is a Green's function of $v(p)$.

PROOF. Since $p \in \nabla(\mathfrak{S}'(\mathfrak{p}), G', M') \cap \underset{1}{\Delta}(G' M')$, there exists a path Γ' in $v(p)$ M'-tending to p. Γ' intersects $\partial\mathfrak{V}_n(\mathfrak{p})$ for $n \geqq n(\Gamma')$ such that $G'(z, p^*) \geqq \delta > 0$ on $\partial\mathfrak{V}_n(\mathfrak{p})$, where $G'(z, p^*)$ is a Green's function of G'. Hence we can find an M'-tending sequence $\{p_i\}$ to p such that $\{G'(z, p_i)\}$ converges to a positive harmonic function $G'(z, \{p_i\})$ and

$$K'(z, p) = \frac{G'(z, \{p_i\})}{G'(p^*, \{p\})}\,.$$

Then $K'(z,p)>_{Cv(p)}K'(z,p)$. Hence there exists a uniquely determined component $v'(p)$ of $v(p)$ in which

$$G'\big(z,\{p_i\}\big)-_{Cv'(p)}G'\big(z,\{p_i\}\big)>0\,. \tag{28}$$

In the following we consider only $v'(p)$ and denote it by $v(p)$. Suppose G is an end of a Riemann surface R. Let $\{R_n\}$ be its exhaustion such that $\partial R_n\cap F=0$. Then

$$G'(z,p_i)-_{Cv(p)\cap R_n\cap G'}G'(z,p_i)=G''_n(z,p_i)\,,$$

where $G''_n(z,p_i)$ is a Green's function of $G'-R_n+v(p)$.

Since $G'(z,p_i)\leqq M$ for $i\geqq i_0$ on $\partial v(p)\cap R_n+\partial R_n$, we have by letting $i\to\infty$

$$\lim_{i=\infty}{}_{Cv(\ )\cap R_n\cap G'}G'(z,p_i)={}_{Cv(p)\cap R_n\cap G'}G'\big(z,\{p_i\}\big).\quad \text{Hence}$$

$$G'\big(z,\{p_i\}\big)-_{Cv(p)\cap R_n\cap G'}G'\big(z,\{p_i\}\big)=\lim_{n=\infty}G''_n(z,p_i)\,.$$

Let $n\to\infty$. Then ${}_{Cv(\ )\cap R_n\cap G'}G'(z,\{p_i\})\nearrow{}_{Cv(p)\cap G'}G'(z,\{p_i\})$ as $n\to\infty$ and

$$G'\big(z,\{p_i\}\big)-_{Cv(\ )\cap G'}G'\big(z,\{p_i\}\big)=\lim_{n=\infty}\lim_{i=\infty}G''_n(z,p_i)\,.$$

Now clearly $D(\min(M,G''_n(z,p_i)))=2\pi M$. Hence by Fatou's lemma

$$D\big(\min\big(M,U(z)\big)\big)\leqq 2\pi M:\ U(z)=G'\big(z,\{p_i\}\big)-_{Cv(p)\cap G'}G'\big(z,\{p_i\}\big). \tag{29}$$

Let D be a compact regular set in $v(p)$. Then ${}_{\bar D}U(z)$ and ${}_DU(z)$ in $v(p)$ can be considered, where ${}_{\bar D}U(z)=U(z)={}_DU(z)$ on $D+\partial v(p)+\partial F$ and ${}_{\bar D}U(z)$ has $M.D.I.$ over $v(p)-D$ and ${}_DU(z)$ is the least positive superharmonic function in $v(p)-D$. Then since $v(p)\subset G\in\mathcal{E}_0$, ${}_{\bar D}U(z)={}_DU(z)$. Evidently $0<U(z)$ $\overset{G'}{\underset{v(p)}{I}}[G'(z,\{p_i\})]$ and I preserves the minimality of $G'(z,\{p_i\})$, $U(z)$ is minimal in $v(p)$, whence $U(z)$ is N-minimal in $v(p)$. By (29) the function theoretic mass of $U(z)\leqq 1$ [18]. Hence

$$U(z)=\int_{\Delta_1(v(p),N)}N(z,r)d\mu(r)\,,\qquad \int d\mu(r)\leqq 1\,,$$

where $N(z,r)$ is an N-Green's function of $v(p)+\Delta(v(p),N)$ vanishing on $\partial v(p)+F$ and $\Delta_1(v(p),N)$ is the set of N-minimal boundary points of $v(p)$. Now $U(z)$ is N-minimal, and μ is a point measure at $q\in\Delta_1(v(p),N)$, i.e. $U(z)=aN(z,q)$: $a>0$. By $v(p)\subset G\in\mathcal{E}_0$, $\sup N(z,q)=\infty$ and $N(z,q)\leqq G'(z,\{p_i\})$, q lies over $\mathfrak{p}$, because $\sup\limits_{z\in C\mathfrak{B}_n(p)}G(z,\{p_i\})<\infty$ for any n. It is easily verified that Theorem 1 valied for the topology N over $v(p)$. Hence there exists a path Γ tending to $\mathfrak{p}$ in $v(p)$ on which $N(z, s)\to N(z, q)$ as $s\to\mathfrak{p}$ on Γ. Also by $v(p)\subset G\in\mathcal{E}_0$, $N(z, r)=G''(z, r)$ in $v(p)$ for $r\in v(p)$, where $G''(z, r)$ is a Green's function of $v(p)$. Hence

$$\lim_{\substack{z\to p\\ z\in\Gamma}} G''(z, z^*)=N(z^*, q)>0 \qquad \text{for } z^*\in v(p).$$

Hence we have the lemma.

THEOREM 12. *Let $G\in\mathcal{E}_0$ and F be a completely thin at $\mathfrak{p}\in\beta(G)$. Suppose $G'=G-F$ is represented as a covering surface over the w-sphere of at most m_0 number of sheets by an analytic function $g(z)$. Then $\nabla(\mathfrak{p}, G, M)\cap \underset{1}{\Delta}(G, M)$ consists of at most m_0 number of points.*

PROOF. Let G be an end of a Riemann surface R. Let $\{R_n\}$ be an exhaustion of R. Since the spherical area of $g(G)\leqq 4\pi m_0$, we can find a number n_0 such that the spherical area of $g((R-R_{n_0})\cap G')\leqq\pi$ for $n\geqq n_0$. Also it is easily seen $\min\limits_{z\in\partial\mathfrak{B}_n(p)} G^*(z, p^*)>\delta>0$ by the completely thiness of F, where $G^*(z, p^*)$ is a Green's function of a component of $(R-R_n)\cap G'$ containing a neighbourhood of $\mathfrak{p}$. Hence without loss of generality we can suppose

$$\text{spherical area of } g(G)\leqq\pi \text{ and } \min_{z\in\partial\mathfrak{B}_n(p)} G'(z, p^*)\geqq\delta>0 \quad \text{for any } n. \tag{30}$$

By Evans's [19] theorem there exists a positive harmonic function $U(z)$ in G' such that

1). *$U(z)=0$ on $\partial G+\partial F$, $D(\min(M, U(z)))=2\pi M$, $\int_{\partial\Omega_L}\frac{\partial}{\partial n}U(z)ds=2$ for almost all L, where $\Omega_L=\{z\in G' : U(z)>L\}$.*

2). *$U(z)\to\infty$ as $z\to\beta(G)$ in any $G_\delta=\{z\in G' : G'(z, p^*)>\delta\} : p^*\in G'$.*

Ω_L consists of at most enumerably number of domains. Let Ω'_L be one component of Ω_L. Then by $\Omega_L\subset G\in\mathcal{E}_0$, $\sup\limits_{z\in\Omega'_L} U(z)=\infty$. Since spherical area of $g(G)\leqq\pi$, by (1) we see by the length and area's method there exists a sequence $L_i : i=1, 2, \cdots$ such that

$$L_i\nearrow\infty \text{ and speherical length of } g(\partial\Omega_{L_i})=\varepsilon_i\to 0 \text{ as } i\to\infty.$$

Since $\partial\mathfrak{B}_n(\mathfrak{p})\to\mathfrak{p}$ as $n\to\infty$, $\min\limits_{z\in\partial\mathfrak{B}_n(p)} U(z)\to\infty$ as $n\to\infty$ by (2). Hence for any given Ω_{L_i}, there exists a number $n(L_i)$ such that

$$\Omega_{L_i}\supset\partial\mathfrak{B}_{n(L_i)}(\mathfrak{p}).$$

By Evans's theorem there exists a harmonic function $V(z)$ in G such that

1). $V(z)=0$ on ∂G, $D(\min(M, V(z)))=2\pi M$, $\int_{\partial D_M} \frac{\partial}{\partial n} V(z)ds=2\pi$ *for any* M, *where* $D_M=\{z\in G: V(z)<M\}$.

2). $V(z)\to\infty$ *as* $z\to\beta(G)$.

Similarly as $U(z)$, there exists a sequence M_j such that $M_j\to\infty$ and

$$\text{spherical length of } g(\partial D_{M_j}\cap G')=\varepsilon_j\to 0 \text{ as } j\to\infty .$$

Since $\Omega_{L_i}=\lim_{j=\infty}(\Omega_{L_i}\cap D_{M_j})$, there exists a number M_j such that $(\Omega_{L_j}\cap D_{M_j})\supset\partial\mathfrak{V}_{n(L_i)}(\mathfrak{p})$. Now $\Omega_{L_i}\cap D_{M_j}$ is compact in G' and since $\partial\mathfrak{V}_{n(L_i)}(\mathfrak{p})$ is a continuum, there exists only one component $\Omega_{i,j}$ of $\Omega_{L_i}\cap D_{M_j}$ containing $\partial\mathfrak{V}_{n(L_i)}(\mathfrak{p})$. By the theory of cluster sets

$$\text{boundary of } g(\Omega_{ij})\subset \text{ boundary of } g(\partial\Omega_{ij}).$$

$g(\partial\Omega_{i+j})$ divides the w-sphere into a number of domains $G_1, G_2, \cdots$. Since the spherical length of $g(\partial\Omega_{i+j})=\varepsilon_i+\varepsilon_j<\frac{1}{4}$, there exists only one domain with spherical area $\geqq 4\pi-\frac{(\varepsilon_i+\varepsilon_j)^2}{\pi}$. We denote such domain by G'. Then by (31) $g(\Omega_{i+j})=G_{i+j}$ or $g(\Omega_{i+j})\cap G'=0$. On the other hand, spehreical area of $g(\Omega_{i+j})<\pi$. Whence $g(\Omega_{i+j})\cap G'=0$ and $g(\Omega_{i+j})$ is contained in a semisphere. Hence we have spherical diameter of $g(\partial\mathfrak{V}_{n(L_i)}(\mathfrak{p}))\leqq$ spherical diameter of $g(\Omega_{i+j})\leqq$ speherical length of $g(\partial\Omega_{i+j})=\varepsilon_i+\varepsilon_j$. Hence we can find a subsequence $\{\mathfrak{V}_{n'}(\mathfrak{p})\}$ of $\{\mathfrak{V}_n(\mathfrak{p})\}$ such that the spherical diameter of $g(\partial\mathfrak{V}_{n'}(\mathfrak{p}))=\varepsilon_{n'}\to 0$ as $n'\to\infty$. Also we can find a subsequence $\{\mathfrak{V}_{n''}(\mathfrak{p})\}$ of $\{\mathfrak{V}_{n'}(\mathfrak{p})\}$ such that

$$g\big(\partial\mathfrak{V}_{n''}(\mathfrak{p})\big)\longrightarrow \text{ one point } w_0 \text{ as } n''\longrightarrow\infty . \qquad (31)$$

In the following we consider $\{\mathfrak{V}_{''m}(\mathfrak{p})\}$ only.

Since the spherical area of $g(G')<\pi$, there exists a closed set $\mathscr{F}$ of positive capacity in the complementary set of $g(G')$. Assume there exist $p_1, \cdots, p_{m_0+1}$ points in $\overline{V}(\mathfrak{S}'(\mathfrak{p}), G', M')\cap \underset{1}{\Delta}(G', M')$. Then there exist M'–neighbourhoods $v(p_i)$ such that $v(p_m)\cap v(p_{m'})=0$ for $m\neq m'$. Let p be of $\{p_m\}$. $v(p)$ consists of components but there exists only one component $v^*(p)$ of $v(p)$ such that

$$_{Cv(p)}K'(z, p)<K'(z, p) \text{ in } v^*(p).$$

Put $v=g(v^*(p))$. Then v is a domain, $v\cap\mathscr{F}=0$ and v is a hyperbolic. We shall show $w_0\in v$ or is an irregular point for the domain v. By Lemma

4 there exists a path Γ in $v^*(p)$ tending to p along which $\lim_{z\to p} G''(z, p^*)>0$, where $G''(z, p^*)$: $p^*\in v^*(p)$ is a Green's function of $v^*(p)$. Γ intersects $\partial\mathfrak{B}_{n''}(\mathfrak{p})$ for $n''>n(\Gamma)$. Hence we can find a sequence $\{q_j\}$ on $\Gamma\cap\sum_{n''}\partial\mathfrak{B}_{n''}(\mathfrak{p})$ such that $G''(q_j, p^*)\geqq\delta_0>0$ and $g(q_j)\to w_0$ as $j\to\infty$. Let $G(w, g(p^*))$ be a Green's function of V. Then

$$G\big(g(q_j), g(p^*)\big)\geqq G''(q_j, p^*)\geqq\delta_0>0 \quad \text{and}$$
$$g(q_j)\longrightarrow w_0 \quad \text{as} \quad j\longrightarrow\infty .$$

Hence w_0 is an inner point of V or an irregular point. By $V\subset g(v(p))$, $g(v(p))$ covers a neighbourhood $v(w_0)$ of w_0 except at most a thin set at w_0. Sum of a finite number of thin sets is also a thin set. Hence $g(G')$ covers a neighbourhood $v'(w_0)$ at least m_0+1 times except a thin set. This is a contradiction. Hence $\nabla(\mathfrak{S}'(\mathfrak{p}), G', M')\cap\underset{1}{\Delta}(G', M')$ consists of at most m_0 point. By Theorem 7 $\nabla(\mathfrak{p}, G, M)\cap\underset{1}{\Delta}(G, M)$ consists of at most m_0 points and we have the Theorem.

Department of Mathematics,
Hokkaido University

References

[1] Z. KURAMOCH: On minimul points of Riemann surfaces, I. J. Fac. Sci. Hokkaido Univ. I, 178-196 (1972).
[2] M. BRELOT: Sur le principe des singularités positives et la topologie de R. S. MARTIN, Ann. Univ. Grenoble. 23, 113-138 (1948).
[3] M. BRELOT: La probleme de Dirichlet. Axiomatique et frontière de MARTIN. J. Math. pures. Appl. 35, 297-335 (1956).
[4] L. NAÏM: Sur le rôle de la frontière de R. S. MARTIN dans la théorie du potentiel. Ann. Univ. Grenoble. 7, 181-281 (1957).
[5] R. S. MARTIN: Minimal positive harmonic function. Trans. Am. Math. Soc. 49, 137-172 (1941) or see [3].
[6] Z. KURAMOCHI: On covering surfaces. Osaka Math. J., 5, 154-201 (1953).
[7] Z. KURAMOCHI: Potentials on Riemann surfaces. J. Fac. Sci. Hokkaido Univ. 16, 5-79 (1962).
[8] It is easily verified that Theorem is valid for L-MARTIN's topology.
[9] See [6].
[10] See [6].
[11] See [6].
[12] See [6].
[13] Z. KURAMOCHI: Mass distributions on the ideal boundaries of abstract Riemann surfaces, 1. Osaka Math. J., 8, 119-138 (1956).

M. NAKAI: Green potential of Evans type on Royden's compactifications of a Riemann surfaces. Nagoya Math. J., 24, 205–239 (1964).

Z. KURAMOCHI: On the existence of functions of Evans's type. J. Fac. Sci. Hokkaido Univ., 19, 1–29 (1965).

[14] Z. KURAMOCHI: Surperharmonic functions in a domain of a Riemann surface. Nagoya Math. J., 331, 41–55 (1968).

[15] Z. KURAMOCHI: Relations between two MARTIN's topologies on Riemann surfaces. J. Fac. Sci. Hokkaido Univ., 19, 146–153 (1966).

[16] M. HEINS: Riemann surfaces of infinite genus. Ann. Math., 55, 296–317 (1950).

[17] See [15].

[18] See [14] or See [13].

[19] See [12].

(Received August 22, 1972)