A remark on permutation groups of degree 2*p*

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1. Introduction

Let Ω be the set of letters $1, 2, \cdots, 2p$, where p is an odd prime number. In this note we shall prove the following theorem.

THEOREM. *Let G be a permutation group on Ω. Then one of the following occurs.*

1) *G has a normal Sylow p-subgroup;*

2) *G has an irreducible complex character whose degree is divisible by p.*

In [4] N. Ito and the author proved the theorem in the case G is transitive. In this note we may assume that G is intransitive. The author thanks to Professor T. Tsuzuku and Professor H. Kimura who have given him valuable suggestions.

2. Proof of the theorem

Let Ω_i $(i=1, \cdots, r)$ be the orbit of G in Ω. If $|\Omega_i|<p$ for all $i=1, \cdots, r$, then Sylow p-subgroup of G is trivial. Therefore we may assume $|\Omega_1| \geqq p$. At first we assume that $|\Omega_1|=p$. Let π be the permutation representation of G on Ω_1. If G/Ker π is non-solvable, then 2) occurs by [3]. If G/Ker π is solvable, then G/Ker π is a Frobenius group whose kernel is Q/Ker π, where Q is the inverse image of the Frobenius kernel by the natural homomorphism G onto G/Ker π. If $r \geqq 3$, a Sylow p-subgroup of Q is normal in it. For Q=Ker $\pi \cdot$P for some Sylow p-subgroup P of G and every element of Ker π commutes with any element of P. Therefore 1) occurs. Assume that $r=2$, i.e. $\Omega=\Omega_1 \cup \Omega_2$, where $|\Omega_2|=p$. Let η be the permutation representation of Q on Ω_2. If Q/Ker η is intransitive on Ω_2, then a Sylow p-subgroup of Q is normal in it. Hence 1) occurs. So we may assume that Q/Ker η is transitive. Then it is easy to see that Ker η is a p-group. If Q/Ker η is non-solvable, then 2) occurs by [3] and the theorem of Clifford ([2], p. 565). If Q/Ker η is solvable, then a Sylow p-subgroup of Q is normal in it. Thus 1) occurs.

Next we assume that $|\Omega_1|=p+k$ $(0<k<p)$. Since a Sylow p-subgroup of G is not trivial, G/Ker π has an element of order p. It follows that

G/Ker π is primitive on Ω_1 ([5], Theorem 8. 4).

If G/Ker π contains the alternating group on Ω_1, then G has an irreducible character corresponding to the Young diagram $[p, 1^k]$ whose degree is divisible by p. Hence 2) occurs. If G/Ker π does not contain the alternating group, then $k=1$ or 2 by the theorem of Jordan ([5], Theorem 13. 9).

Assume $|\Omega_1|=p+1$. Since G/Ker π is doubly transitive on Ω_1, G/Ker π has the irreducible character of degree p which appears in the permutation character of G/Ker π on Ω_1. Thus 2) occurs.

Assume $|\Omega_1|=p+2$. G/Ker π is triply transitive on Ω_1 ([5], Theorem 13. 8). If the stabilizer of two letters a and b of G/Ker π is solvable, then it is a Frobenius group on $\Omega_1-\{a, b\}$. By [1] G/Ker π is one of the group PGL$(2, 2^m)$, where m is an integer, and PΓL$(2, 2^q)$, where q is prime, and $2^m+1=p+2$ or $2^q+1=p+2$ in each case (where PGL$(2, 2^m)$ is a two dimensional projective general linear group over the finite field GF(2^m), PΓL$(2, 2^q)$ is the extension of PGL$(2, 2^q)$ by the Galois group of GF(2^q)). It is well known that these groups satisfy 2). If the stabilizer of a and b of G/Ker π is non-solvable, then G/ker π is quadraply transitive on Ω_1 ([5], Theorem 11. 7). By a theorem of Frobenius ([2], p. 602) the restriction to G/Ker π of the irreducible character of the symmetric group on Ω_1 corresponding to the Young diagram $[p, 1^2]$ is also irreducible. Since its degree is $p(p+1)/2$, G/Ker π satisfies 2) and so does G. This completes the proof.

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