

On the equations $\operatorname{rot} v = g$ and $\operatorname{div} u = f$ with zero boundary conditions

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1. Introduction

Let Ω be a bounded or an exterior domain in \mathbf{R}^n . We consider the equations

$$(1.1) \quad \operatorname{rot} v = g, \quad v|_{\partial\Omega} = 0$$

with $n=3$, $g=(g_1, g_2, g_3) \in L^r(\Omega)^3$, and

$$(1.2) \quad \operatorname{div} u = f, \quad u|_{\partial\Omega} = 0$$

with $n \geq 2$, $f \in L^r(\Omega)$. $L^r(\Omega)$ with $1 < r < \infty$ denotes the usual Lebesgue space with norm $\|\cdot\|_r$, and $L^r(\Omega)^n$ is the corresponding space of vector fields; $\partial\Omega$ denotes the boundary of Ω . These equations play an important role in the theory of Navier-Stokes equations (see [9, 10, 13, 14, 19, 23, 24, 26]). The problem is to find solutions $v=(v_1, v_2, v_3)$ and $u=(u_1, u_2, \dots, u_n)$, respectively, with the best possible regularity corresponding to the regularity on the right hand sides. Our aim is to give essentially complete existence and regularity results in Sobolev spaces; we are interested in necessary and sufficient conditions on g, f for solvability in these spaces. There are various special results on the above equations. Grenz [11] treated the case $g \in L^2(\Omega)^3$ in (1.1) if Ω is simply connected. Griesinger [12] proved a result on (1.1) for bounded starlike domains. Recently von Wahl [27] gave an integral equation method for (1.1) and (1.2) in \mathbf{R}^3 which yields L^r -results for the lowest regularity level. Our approach to (1.1) is completely different and has been inspired by Martensen [16]; we get also results in higher order Sobolev spaces and we characterize the corresponding null spaces of the operator rot . As it turns out, the theory for (1.1) depends on topological properties of the manifold $\partial\Omega$, e.g. on the genus of $\partial\Omega$ (the number of "handles"). Concerning (1.2) the most complete result up to now has been given by Bogovski [1, 2]. He developed the regularity theory for (1.2) in bounded domains and he proved the result for the lowest regularity level in exterior domains. One of our purposes is to give a complete proof of the higher regularity results in this

case. The higher regularity in exterior domains is a crucial part of the theory for both equations. For the sake of completeness we will also present an outline of Bogovski's proof on (1.2) in bounded domains. We extend Bogovski's result in some directions. So we give a result for distributions $f \in H^{-1,r}(\Omega)$ and we prove some additional properties on the operator $f \rightarrow u$ which chooses a certain solution u (Theorem 2.10).

Ladyzhenskaja and Solonnikov [14, 15] considered (1.2) in L^2 -spaces. Von Wahl's integral equation approach [27] is a completely different way to solve (1.2) in \mathbf{R}^3 . Another approach to (1.2) in L^2 -spaces can be developed from Necas [20, 7, Lemma 7.1] by using the gradient ∇ and duality arguments.

In order to establish higher regularity results for (1.1) and (1.2) in exterior domains we have to use the homogeneous Sobolev space $\hat{H}_0^{m,r}(\Omega)$ being the completion of $C_0^\infty(\Omega)$ under the highest order norm $\|\nabla^m u\|_r = (\sum_{|\alpha|=m} \|\partial^\alpha u\|_r^r)^{1/r}$. Here a polynomial growth of the elements $u \in \hat{H}_0^{m,r}(\Omega)$ is possible for $|x| \rightarrow \infty$, and if $m \geq 1$, i. e. for higher regularities, we have to admit such a growth for g and f in our equations. In order to develop a theory for (1.1) and (1.2) in these homogeneous spaces, we prove first a result on the entire space and then we argue by localization. For this purpose we give a criterion for $u \in \hat{H}_0^{m,r}(\Omega)$ using decay properties for $|x| \rightarrow \infty$, and we study the Laplacian Δ in homogeneous Sobolev spaces.

Using duality arguments, the theory on the equation (1.2) yields immediately estimates for the gradient ∇ . Bogovski [2, p. 38] gave a result for the lowest regularity level $m=0$. The general estimates on ∇ given here seem to be partially new. Applications can be found in [4] and [10].

NOTATIONS Let $1 < r < \infty$ and $m \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$. Then $H^{m,r}(\Omega)$ denotes the usual Sobolev space with norm $\|u\|_{m,r} = (\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_r^r)^{1/r}$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N}_0^n$, $\partial_i = \partial/\partial x_i$, $i=1, 2, \dots, n$, $x = (x_1, x_2, \dots, x_n) \in \Omega$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$, and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. For $m=0$ we obtain the usual Lebesgue space $L^r(\Omega) = H^{0,r}(\Omega)$ with norm $\|u\|_r = \|u\|_{0,r}$. Let $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$. $C_0^\infty(\Omega)$ denotes the space of smooth functions with compact support contained in Ω , and $H_0^{m,r}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to $\|u\|_{m,r}$. All spaces are real or complex.

Let $\bar{\Omega}$ be the closure of Ω . Then $u \in L_{loc}^r(\bar{\Omega})$ means that $u \in L^r(\Omega \cap B)$ for every ball $B \subseteq \mathbf{R}^n$. Similarly, $u \in H_{loc}^{m,r}(\bar{\Omega})$ means that $u \in H^{m,r}(\Omega \cap B)$ for every ball $B \subseteq \mathbf{R}^n$.

The seminorm $\|\nabla^m u\|_r = (\sum_{|\alpha|=m} \|\partial^\alpha u\|_r^r)^{1/r}$ is defined by the highest order derivatives; $\|\nabla^m u\|_r$ is a norm on $C_0^\infty(\Omega)$. Completing $C_0^\infty(\Omega)$ under the

norm $\|\nabla^m u\|_r$ we get the homogeneous Sobolev space $\widehat{H}_0^{m,r}(\Omega)$. It holds $\widehat{H}_0^{m,r}(\Omega) \subseteq H_{\text{loc}}^{m,r}(\overline{\Omega})$ if $\partial\Omega$ is nonempty. We obtain $\widehat{H}_0^{m,r}(\Omega) = H_0^{m,r}(\Omega)$ for bounded domains. Therefore we use the notation $\widehat{H}_0^{m,r}(\Omega)$ only for exterior domains.

For the entire space, $\widehat{H}_0^{m,r}(\mathbf{R}^n)$ is in general an abstract space of Cauchy sequences since we cannot apply Poincaré's inequality. Therefore, in view of localization procedures we choose on \mathbf{R}^n another norm instead of $\|\nabla^m u\|_r$. We fix an arbitrary open ball $K \subseteq \mathbf{R}^n$ and let $H_K^{m,r}(\mathbf{R}^n)$ be the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|u\|_{K,m,r} = \|\nabla^m u\|_{L^r(\mathbf{R}^n)} + \|u\|_{L^r(K)}$.

Let r' be defined by $\frac{1}{r} + \frac{1}{r'} = 1$. By $H^{-m,r}(\Omega) = H_0^{m,r'}(\Omega)^*$ and $\widehat{H}^{-m,r}(\Omega) = \widehat{H}_0^{m,r'}(\Omega)^*$ we understand the corresponding dual spaces; if Ω is bounded we obtain $\widehat{H}^{-m,r}(\Omega) = H^{-m,r}(\Omega)$ with equivalent norms.

Let $L^r(\Omega)^n$, $H^{m,r}(\Omega)^n, \dots$ be the corresponding spaces of vector fields $u = (u_1, u_2, \dots, u_n)$.

We set $\langle u, v \rangle = \int_\Omega uv \, dx$ or $\langle u, v \rangle = \int_\Omega u \cdot v \, dx$ with $u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$ if u, v are vector fields. By $\langle u, v \rangle$ we denote also the value of a distribution u at v .

Using the gradient $\nabla = (\partial_1, \partial_2, \dots, \partial_n)$ we set $\operatorname{div} u = \nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \dots + \partial_n u_n$ for $u = (u_1, u_2, \dots, u_n)$ and $\operatorname{rot} v = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$ for $v = (v_1, v_2, v_3)$.

If the boundary $\partial\Omega$ is sufficiently smooth (e. g. of class C^1) we denote by $N = N(x)$ the outward normal vector at $x \in \partial\Omega$. Then we define $L_\sigma^r(\Omega) = \{v \in L^r(\Omega)^n : \operatorname{div} v = 0, N \cdot v|_{\partial\Omega} = 0\}$ and $H_{0,\sigma}^{m,r}(\Omega) = H_0^{m,r}(\Omega)^n \cap L_\sigma^r(\Omega)$ for $m \in \mathbf{N}_0$. We get $H_{0,\sigma}^{m,r}(\Omega) = \{v \in H_0^{m,r}(\Omega)^n : \operatorname{div} v = 0\}$ for $m \in \mathbf{N} = \{1, 2, \dots\}$ and $H_{0,\sigma}^{0,r}(\Omega) = L_\sigma^r(\Omega)$ for $m = 0$.

In this paper, an exterior domain is a domain whose complement is the closure of some nonempty bounded domain. Throughout the paper, c, c_1, c_2, \dots are positive constants whose values may change from line to line.

2. Bounded domains

First we investigate the properties of the operator rot in Sobolev spaces; here we assume that $\Omega \subseteq \mathbf{R}^3$ is at least of class C^2 . We need the spaces $F(\Omega) = \{v \in L^r(\Omega)^3 : \operatorname{div} v = 0, \operatorname{rot} v = 0, N \cdot v|_{\partial\Omega} = 0\}$ and $M^m(\Omega) = \{\nabla p \in H_0^{m+1,r}(\Omega)^3 : p \in H^{m+2,r}(\Omega), \Delta^{m+2} p = 0\}$ for $m \in \mathbf{N}_0$. $F(\Omega)$ and $M^m(\Omega)$ are finite dimensional and do not depend on r . We will show that $F(\Omega) = \{0\}$ if Ω is simply connected and $M^m(\Omega) = \{0\}$ if $\partial\Omega$ is connected. Then our main result concerning (1.1) reads as follows:

THEOREM 2.1. Let $m \in \mathbf{N}_0$, $1 < r < \infty$, and let $\Omega \subseteq \mathbf{R}^3$ be a bounded domain of class C^{m+2} . Then for every $g \in H_{0,\sigma}^{m,r}(\Omega)$ such that $\langle g, w \rangle = 0$ for all $w \in F(\Omega)$, there exists some $v \in H_0^{m+1,r}(\Omega)^3$ with the following properties :

- a) $\operatorname{rot} v = g, \Delta^{m+1} \operatorname{div} v = 0$ in Ω ,
- b) $\inf_{h \in M^m(\Omega)} \|\nabla^{m+1}(v+h)\|_r \leq c \|\nabla^m g\|_r$

where $c = c(\Omega, m, r) > 0$ is a constant ; v is unique modulo $M^m(\Omega)$.

Since $M^m(\Omega)$ has finite dimension we can easily construct a linear operator $g \rightarrow v$ for (1.1) which selects a certain solution.

COROLLARY 2.2. Let m, r , and Ω be as above. Then there exists a linear operator $S = S_{\Omega}^{m,r}$ from $F^{m,r}(\Omega) = \{g \in H_{0,\sigma}^{m,r}(\Omega) : \langle g, w \rangle = 0 \text{ for all } w \in F(\Omega)\}$ into $H_0^{m+1,r}(\Omega)^3$ with the following properties :

- a) $\operatorname{rot} Sg = g, \Delta^{m+1} \operatorname{div} Sg = 0$,
- b) $\|\nabla^{m+1} Sg\|_r \leq c \|\nabla^m g\|_r$

for all $g \in F^{m,r}(\Omega)$, where $c = c(\Omega, m, r) > 0$ is a constant. Sg is uniquely determined modulo $M^m(\Omega)$ and unique if $\partial\Omega$ is connected.

It is not difficult to see that every $u \in H_0^{m+1,r}(\Omega)^3$ has a unique decomposition $u = \nabla p + u_0$ with $p \in H_0^{m+2,r}(\Omega)$, $u_0 \in H_0^{m+1,r}(\Omega)^3$, and $\Delta^{m+1} \operatorname{div} u_0 = 0$; this follows from well known properties on Dirichlet's boundary value problem for Δ^{m+2} [22]. Recall that $\Delta^{m+1} \operatorname{div}$ is understood in the distribution sense, and that the norm on the left in 2.1 b) is that of the quotient space $H_0^{m+1,r}(\Omega)^3 / M^m(\Omega)$. We see that the operator

$$\operatorname{rot} : \{v \in H_0^{m+1,r}(\Omega)^3 : \Delta^{m+1} \operatorname{div} v = 0\} \rightarrow H_{0,\sigma}^{m,r}(\Omega)$$

has the closed range $R(\operatorname{rot}) = F^{m,r}(\Omega)$ and the null space $M^m(\Omega)$. It holds $F^{m,r}(\Omega) = H_{0,\sigma}^{m,r}(\Omega)$ if Ω is simply connected and the bounded inverse operator rot^{-1} exists if $\partial\Omega$ is connected.

We need some topological properties of the two-dimensional manifold $\partial\Omega$. Let $\partial_1\Omega, \partial_2\Omega, \dots, \partial_b\Omega$ ($b \in \mathbf{N}$) be the connected components of $\partial\Omega$; thus we get $b=1$ if $\partial\Omega$ is connected. These are connected two-dimensional manifolds in \mathbf{R}^3 which can be completely classified (see [25, 3]). Each such manifold is homeomorphic to a sphere with "handles" added in the sense of [3, p. 14]. The number h_i of handles of the manifold $\partial_i\Omega$ is the so-called genus of $\partial_i\Omega$, $i=1, 2, \dots, b$ [25, p. 58]. If Ω is simply connected, then there are no handles at each of $\partial_1\Omega, \partial_2\Omega, \dots, \partial_b\Omega$.

Using Dirichlet's boundary value problem for Δ^{m+2} [22] we see that $M^m(\Omega)$ is the set of all linear combinations $c_1 p_1 + c_2 p_2 + \dots + c_b p_b$ where $p_1, p_2, \dots, p_b \in H^{m+2,r}(\Omega)$ are defined by $\nabla p_i \in H_0^{m+1,r}(\Omega)^3$, $\Delta^{m+2} p_i = 0$, and $p_i|_{\partial\Omega} = \delta_{ij}$, $i, j = 1, 2, \dots, b$. The dimension of this space is $b-1$ such that $M^m(\Omega) = \{0\}$ if $\partial\Omega$ is connected.

Next we consider the space $F(\Omega)$ of so-called Neumann vector fields and show that the dimension of $F(\Omega)$ is equal to $h = h_1 + h_2 + \dots + h_b$ which is the total number of handles of $\partial\Omega$. To prove this we assume for simplicity that $b=1$ and $h=h_1$. Then we consider h disjoint curves k_1, k_2, \dots, k_h inside Ω surrounding the holes of the handles and also h curves $k_1^*, k_2^*, \dots, k_h^*$ inside the complement $\mathbf{R}^3 \setminus \bar{\Omega}$ surrounding the handles (complementary curves). Let $T = T(y)$ be the unit tangent vector to k_i^* and $\int \dots ds$ (y) the usual line integral. Then we define

$$A_i(x) = (1/4\pi) \int_{k_i^*} |x-y|^{-1} T(y) ds(y), \quad J_i = \operatorname{rot} A_i, \quad \theta_i = P_r J_i$$

for $i=1, 2, \dots, h$ where P_r denotes the usual projection from $L^r(\Omega)^3$ onto $L^r(\Omega)$ associated with the Helmholtz decomposition (in the sense of [8, 18, 23]). An elementary calculation (see [16, p. 33]) shows that $\int_{k_i} J_j \cdot T ds = 0$ if $i \neq j$ and $= \pm 1$ if $i=j$. We choose the orientation of the curves k_i in such a way that $\int_{k_i} J_j \cdot T ds = 1$ for $i=j$, $i=1, 2, \dots, h$.

The well known properties of P_r [18] imply $\theta_i \in F(\Omega)$ for each $i=1, 2, \dots, h$. Conversely, let $\theta \in F(\Omega)$ and define $c_i = \int_{k_i} \theta \cdot T ds$ for $i=1, 2, \dots, h$ (so-called circulations). Then for $\hat{\theta} = \theta - \sum_{i=1}^h c_i \theta_i$ all circulations vanish; this leads to the representation $\hat{\theta} = \nabla p$ with some $p \in H^{1,r}(\Omega)$. Using the Helmholtz decomposition we conclude that $\hat{\theta} = 0$. The elements $\theta_1, \theta_2, \dots, \theta_h$ are linearly independent because of $\int_{k_i} J_j \cdot T ds = \delta_{ij}$ where $\delta_{ij} = 1$ for $i=j$ and $= 0$ for $i \neq j$. Thus we obtain that h is the dimension of $F(\Omega)$.

PROOF OF THEOREM 2.1. Let $g \in H_{0,\sigma}^{m,r}(\Omega)$ with $\langle g, \theta \rangle = 0$ for all $\theta \in F(\Omega)$. Extending g by zero we obtain $g \in H_{0,\sigma}^{m,r}(\mathbf{R}^3)$. Let F_3 be the fundamental solution of the Laplacian such that $\Delta F_3(x) = \delta(x)$ with Dirac's δ -distribution. We set $w = -\operatorname{rot} F_3 * g$ where $*$ means the convolution. Using $\operatorname{rot} \operatorname{rot} + \Delta = \nabla \operatorname{div}$ and $\operatorname{div} g = 0$ we get $\operatorname{rot} w = -\operatorname{rot} \operatorname{rot} F_3 * g = \Delta F_3 * g = g$, and the Calderon-Zygmund theorem yields $\|\nabla^{m+2} F_3 * g\|_r \leq c \|\nabla^m g\|_r$. It follows $w \in H_{\text{loc}}^{m+1,r}(\mathbf{R}^3)$. Since $g=0$ in $\mathbf{R}^3 \setminus \bar{\Omega}$ we get also $\operatorname{rot} w = 0$ in $\mathbf{R}^3 \setminus$

$\bar{\Omega}$. Using Fubini's theorem and $\langle g, \theta \rangle = 0$ for all $\theta \in F(\Omega)$ it follows $\int_{k^*} w \cdot T ds = 0$ for $i=1, 2, \dots, h$; therefore, w has no circulations within $\mathbf{R}^3 \setminus \bar{\Omega}$ and we get $w = \nabla p$ with some $p \in H_{\text{loc}}^{m+2, r}(\mathbf{R}^3 \setminus \bar{\Omega})$ (use 4.5 a) for example). Now we solve Dirichlet's boundary value problem

$$\Delta^{m+2} \hat{p} = 0 \text{ in } \Omega, \quad \partial^\alpha (p - \hat{p})|_{\partial\Omega} = 0 \text{ for all } |\alpha| \leq m+1$$

by Simader's theory [22] in the weak sense. This gives a solution $\hat{p} \in H^{m+2, r}(\Omega)$ with $w - \nabla \hat{p} \in H_0^{m+1, r}(\Omega)^3$. We set $v = w - \nabla \hat{p}$ and obtain $\text{rot } v = g$ and $\Delta^{m+1} \text{div } v = \Delta^{m+2} \hat{p} = 0$ since $\text{div } w = 0$.

To show 2.1 b) we let $v \in H_0^{m+1, r}(\Omega)^3$ be such that $\text{rot } v = 0$ and $\Delta^{m+1} \text{div } v = 0$. Extending v by zero we obtain $v \in H^{m+1, r}(\mathbf{R}^3)^3$. Since \mathbf{R}^3 is simply connected we get a representation $v = \nabla p$ with some $p \in H_{\text{loc}}^{m+2, r}(\mathbf{R}^3)$ such that $\nabla p \in H_0^{m+1, r}(\Omega)^3$. From $\Delta^{m+2} p = \Delta^{m+1} \text{div } v = 0$ it follows $\nabla p \in M^m(\Omega)$. Thus we get that $\text{rot} : \{v \in H_0^{m+1, r}(\Omega)^3 : \Delta^{m+1} \text{div } v = 0\} \rightarrow F^{m, r}(\Omega)$ is a bounded and surjective operator with null space $M^m(\Omega)$. The estimate in 2.1 b) now follows from the closed range theorem. This proves Theorem 2.1.

The assertion of Theorem 2.1 becomes very simple if Ω is simply connected and $\partial\Omega$ is connected.

COROLLARY 2.3. *Let $1 < r < \infty$.*

a) *If $\Omega \subseteq \mathbf{R}^3$ is a bounded simply connected domain of class C^2 with connected boundary $\partial\Omega$, then for every $g \in L^r(\Omega)$ there exists a unique $v \in H_0^{1, r}(\Omega)^3$ such that $\text{rot } v = g$, $\Delta \text{div } v = 0$, and $\|\nabla v\|_r \leq c \|g\|_r$ where $c = c(\Omega, r) > 0$ is a constant.*

b) *If $\Omega \subseteq \mathbf{R}^3$ is a bounded simply connected domain of class C^{m+2} ($m \in \mathbf{N}_0$) such that $\partial\Omega$ is connected, then for every $g \in H_{0, \delta}^{m, r}(\Omega)$ there exists a unique $v \in H_0^{m+1, r}(\Omega)^3$ such that $\text{rot } v = g$, $\Delta^{m+1} \text{div } v = 0$, and $\|\nabla^{m+1} v\|_r \leq c \|\nabla^m g\|_r$ where $c = c(\Omega, m, r) > 0$ is a constant.*

Following Bogovski [1, 2] we give now an outline of the theory on div in bounded domains. As before we study the linear operator $f \rightarrow u$ which selects a certain solution u to the equation (1.2). Apart from this formulation, the assertions a), b), and c) of the following main theorem are essentially due to Bogovski [2]. The assertion d) seems to be new as well as the next Theorem 2.10 which shows that the norm of the above operator $f \rightarrow u$ remains unchanged if Ω varies over a certain class of domains. This will be applied in Section 4.

THEOREM 2.4. *Let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be a bounded Lipschitz domain and $1 < r < \infty$, $m \in \mathbf{N}_0$. Then there exists a linear operator $R = R_\Omega^{m, r}$ from*

$H_0^{m,r}(\Omega)$ into $H_0^{m+1,r}(\Omega)^n$ with the following properties :

- a) $\operatorname{div} Rf = f$ for all $f \in H_0^{m,r}(\Omega)$ with $\int_{\Omega} f \, dx = 0$,
- b) $\|\nabla^{m+1} Rf\|_r \leq c \|\nabla^m f\|_r$ for all $f \in H_0^{m,r}(\Omega)$ where $c = c(\Omega, m, r) > 0$ is a constant,
- c) $Rf \in C_0^\infty(\Omega)^n$ if $f \in C_0^\infty(\Omega)$, and $Rf = R_{\Omega}^{m,r} f$ depends only on f and Ω if $f \in C_0^\infty(\Omega)$,
- d) $\|Rf\|_r \leq c \|f\|_{H^{-1,r}(\Omega)}$ for all $f \in H_0^{m,r}(\Omega)$ where $c = c(\Omega, r) > 0$ is a constant.

PROOF. Following [2] we give the proof first for a domain $\Omega \subseteq \mathbf{R}^n$ which is bounded and starlike with respect to some open ball B such that $\bar{B} \subseteq \Omega$. This means that $\Omega = \{tx + (1-t)y : x \in B, y \in \Omega, t \in [0, 1]\}$. We fix a function $h \in C_0^\infty(B)$ with $\int_B h \, dx = 1$, and we define Rf for each $f \in C_0^\infty(\Omega)$ by Bogovski's formula

$$(Rf)(x) = \int_{\Omega} G(x, y) f(y) \, dy, \quad G(x, y) = (x - y) \int_1^\infty h(y + t(x - y)) t^{n-1} dt.$$

Extending f by zero outside Ω , we conclude immediately that the support $\operatorname{supp} Rf$ of Rf is contained in the closure of $\{tx + (1-t)y : x \in B, y \in \operatorname{supp} f, t \in [0, 1]\}$. It follows $RC_0^\infty(\Omega) \subseteq C_0^\infty(\Omega)^n$ and 2.4 c).

First we prove 2.4 b) for $m=0$. In the following calculations we need each of the integral transformations $t = |x - y|^{-1} \tau$, $t = 1 + |x - y|^{-1} \tau$, $z = x - y$, and $t = \tau + 1$. Let $\partial_i = \partial / \partial x_i$ for $i = 1, 2, \dots, n$. For $\varepsilon > 0$ we set

$$\begin{aligned} (R_\varepsilon f)(x) &= \int_{|x-y| \geq \varepsilon} G(x, y) f(y) \, dy \\ &= \int_{|z| \geq \varepsilon} z \int_1^\infty h(x + (t-1)z) t^{n-1} f(x-z) \, dt \, dz \end{aligned}$$

and we obtain the decomposition $\partial_i R_\varepsilon f = V_\varepsilon^i f + S_\varepsilon^i f$, $i = 1, 2, \dots, n$, where

$$(V_\varepsilon^i f)(x) = \int_{|x-y| \geq \varepsilon} \partial_i G(x, y) f(y) \, dy$$

and

$$(S_\varepsilon^i f)(x) = \int_{|x-y| = \varepsilon} (x_i - y_i) |x - y|^{-1} G(x, y) f(y) \, do_y;$$

$\int \dots do_y$ means the usual surface integral. Using the transformation $x - y = \varepsilon z$, $do_y = \varepsilon^{n-1} do_z$ we obtain

$$(S_\varepsilon^i f)(x) = \int_{|z|=1} z z_i \int_0^\infty h(x + \tau z) (\tau + \varepsilon)^{n-1} f(x - \varepsilon z) d\tau d o_z.$$

Setting $d = \text{diam } \Omega = \sup\{|x - y| : x, y \in \Omega\}$ we obtain the estimate

$$(2.5) \quad \|S_\varepsilon^i f\|_r \leq c(\varepsilon + d)^n \|h\|_\infty \|f\|_r$$

where $\|h\|_\infty = \sup_x |h(x)|$, and $c > 0$ is a constant. We also obtain $\lim_{\varepsilon \rightarrow 0} S_\varepsilon^i f = fg^i$, $i = 1, 2, \dots, n$, with

$$\begin{aligned} g^i(x) &= \int_{|z|=1} z z_i \int_0^\infty h(x + \tau z) \tau^{n-1} d\tau d o_z \\ &= \int_\Omega (x - y)(x_i - y_i) |x - y|^{-2} h(y) dy. \end{aligned}$$

Using the notation $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ we get the decomposition

$$\partial_i G(x, y) = M_i(x, x - y) + N_i(x, x - y)$$

where

$$\begin{aligned} M_i(x, x - y) &= e_i |x - y|^{-n} \int_0^d h(x + \tau(x - y)) |x - y|^{-1} \\ &\quad \cdot (\sum_{\nu=0}^{n-2} \binom{n-2}{\nu} \tau^\nu |x - y|^{n-1-\nu}) d\tau \\ &\quad + (x - y) |x - y|^{-n-1} \int_0^d (\partial_i h)(x + \tau(x - y)) |x - y|^{-1} \\ &\quad \cdot (\sum_{\nu=0}^{n-1} \binom{n}{\nu} \tau^\nu |x - y|^{n-\nu}) d\tau \end{aligned}$$

and

$$\begin{aligned} N_i(x, x - y) &= e_i |x - y|^{-n} \int_0^d h(x + \tau(x - y)) |x - y|^{-1} \tau^{n-1} d\tau \\ &\quad + (x - y) |x - y|^{-n-1} \int_0^d (\partial_i h)(x + \tau(x - y)) |x - y|^{-1} \tau^n d\tau. \end{aligned}$$

Here $N_i(x, x - y)$ is strongly and $M_i(x, x - y)$ weakly singular at $x = y$. We obtain $|M_i(x, z)| \leq c(\sum_{\nu=0}^{n-2} \|h\|_\infty d^{\nu+1} |z|^{-\nu-1} + \sum_{\nu=0}^{n-1} \|\partial_i h\|_\infty d^{\nu+1} |z|^{-\nu})$ and

$$(2.6) \quad \|M_i f\|_r \leq c(\|h\|_\infty d^n + \|\partial_i h\|_\infty d^{n+1}) \|f\|_r.$$

Since $\int_{|z|=1} N_i(x, z) d o_z = \int (\partial/\partial y_i)(y h(x + y)) dy = 0$, $\sup_x \sup_{|z|=1} |N_i(x, z)| \leq c(\|h\|_\infty d^n + \|\partial_i h\|_\infty d^{n+1})$, we can apply the Calderon-Zygmund theorem [5, 21] to $N_i(x, x - y)$ to obtain the existence of $N_i f = \lim_{\varepsilon \rightarrow 0} N_{i,\varepsilon} f$ where $N_{i,\varepsilon} f$ is

defined by $(N_{i,\varepsilon} f)(x) = \int_{|x-y| \geq \varepsilon} N_i(x, x - y) f(y) dy$. We also obtain the estimate

$$(2.7) \quad \|N_{i,\varepsilon} f\|_r \leq c(\|h\|_\infty d^n + \|\partial_i h\|_\infty d^{n+1}) \|f\|_r.$$

Thus the limit $\partial_i Rf = \lim_{\varepsilon \rightarrow 0} \partial_i R_\varepsilon f$ exists in $L^r(\Omega)^n$ and from (2.5), (2.6), (2.7) we obtain

$$(2.8) \quad \|\nabla Rf\|_r \leq c(\|h\|_\infty d^n + \|\nabla h\|_\infty d^{n+1})\|f\|_r$$

which leads to 2.4 b) for $m=0$ using a closure argument. The assertion 2.4 a) follows from

$$\operatorname{div} Rf = \sum_{i=1}^n \lim_{\varepsilon \rightarrow 0} \partial_i (R_\varepsilon f)_i = \sum_{i=1}^n \lim_{\varepsilon \rightarrow 0} e_i \cdot V_\varepsilon^i f + \sum_{i=1}^n f e_i \cdot g^i = -h \int_\Omega f \, dx + f.$$

To prove 2.4 d) we consider some $f \in H^{-1,r}(\Omega)$ and we choose $g_i \in L^r(\Omega)$, $i=1, 2, \dots, n$, with $\langle f, v \rangle = \langle g_1, \partial_1 v \rangle + \langle g_2, \partial_2 v \rangle + \dots + \langle g_n, \partial_n v \rangle = -\langle \operatorname{div} g, v \rangle$ for all $v \in H_0^{1,r'}(\Omega)$, and with $\|f\|_{H^{-1,r}} = \|g\|_r$; every $f \in H^{-1,r}(\Omega)$ has the form $f = \operatorname{div} g$ with some $g \in L^r(\Omega)^n$. Thus the set of all $f = \operatorname{div} g$ with $g \in C_0^\infty(\Omega)^n$ is a dense subspace of $H^{-1,r}(\Omega)$. With $g \in C_0^\infty(\Omega)^n$ we obtain the representation

$$\begin{aligned} (\partial_i Rg_i)(x) &= \int z |z|^{-n} \int_{|z|}^\infty \{(\partial_i h)(x-z + \tau z |z|^{-1}) \tau^{n-1} g_i(x-z) \\ &\quad + h(x-z + \tau z |z|^{-1}) \tau^{n-1} \partial_i g_i(x-z)\} d\tau dz, \end{aligned}$$

and $R\partial_i g_i = \partial_i Rg_i + W_i$, where W_i has a weakly singular kernel. Using (2.8) and the same argument as for (2.6), we get $\|R\partial_i g_i\|_r \leq c\|g\|_r$ and 2.4 d) follows.

The proof of 2.4 b) for arbitrary $m \in \mathbb{N}$ can be reduced to the case $m=0$. For $f \in C_0^\infty(\Omega)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha| = m$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$, we obtain

$$\begin{aligned} (\partial^\alpha Rf)(x) &= \int z |z|^{-n} \int_{|z|}^\infty \partial_x^\alpha \{h(x-z + \tau z |z|^{-1}) \tau^{n-1} f(x-z)\} d\tau dz \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int z |z|^{-n} \int_{|z|}^\infty (\partial^\beta h)(x-z + \tau z |z|^{-1}) \tau^{n-1} \\ &\quad \cdot (\partial^{\alpha-\beta} f)(x-z) d\tau dz \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_\Omega G^\beta(x, y) (\partial^{\alpha-\beta} f)(y) dy \end{aligned}$$

where $G^\beta(x, y) = (x-y) \int_1^\infty (\partial^\beta h)(y+t(x-y)) t^{n-1} dt$. Applying the estimate from above now to $G^\beta(x, y)$, $\partial^{\alpha-\beta} f$ instead of $G(x, y)$, f , we obtain

$$(2.9) \quad \|\nabla \partial^\alpha Rf\|_r \leq c \sum_{\beta \leq \alpha} (\|\partial^\beta h\|_\infty d^n + \|\nabla \partial^\beta h\|_\infty d^{n+1}) \|\partial^{\alpha-\beta} f\|_r$$

for all α with $|\alpha| = m$. This proves 2.4 b).

Now we consider the general domain Ω as in 2.4. This case can be reduced to the case above by localization. Using the Lipschitz regularity

of $\partial\Omega$ we can choose bounded domains U_i and functions $\phi_i \in C_0^\infty(U_i)$, $i = 1, 2, \dots, k$, with following properties: $0 \leq \phi_i \leq 1$, the $\Omega_i = \Omega \cap U_i$ are starlike, $\Omega \subseteq \bigcup_{i=1}^k U_i$, $\sum_{i=1}^k \phi_i(x) = 1$ for $x \in \Omega$. Then we construct a decomposition $f = \Gamma_1(f) + \Gamma_2(f) + \dots + \Gamma_k(f)$ such that $\int_{\Omega_i} \Gamma_i(f) dx = 0$ if $f \in C_0^\infty(\Omega)$ and $\int_{\Omega} f dx = 0$. For simplicity we assume $k=2$. Now we choose $\psi \in C_0^\infty(\Omega_1 \cap \Omega_2)$ with $\int \psi dx = 1$ and set $\Gamma_1(f) = \phi_1 f - (\int \phi_1 f dx) \psi$. Then it follows that the Γ_i are linear operators from $C_0^\infty(\Omega)$ into $C_0^\infty(\Omega_i)$ with the desired property. Let R_i be the operator R from above with Ω_i instead of Ω . Then we define $Rf = R_1 \Gamma_1(f) + R_2 \Gamma_2(f)$ for $f \in C_0^\infty(\Omega)$. Using Poincaré's inequality we get $\|\nabla^m \Gamma_i(f)\|_r \leq c \|\nabla^m f\|_r$ and this leads to 2.4 a), b), and c). A similar argument leads also to 2.4 d). This proves Theorem 2.4.

The next theorem extends Bogovski's result and shows that the constant c in 2.4 b) does not depend on Ω if Ω varies over a certain class of domains.

THEOREM 2.10. *Let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be a bounded Lipschitz domain, $1 < r < \infty$, and $m \in \mathbf{N}_0$. Consider $y \in \mathbf{R}^n$, $0 \neq \tau \in \mathbf{R}$ and set $\Omega(y, \tau) = (1 - \tau)y + \tau\Omega = \{(1 - \tau)y + \tau x : x \in \Omega\}$. Then there exists a linear operator $R_{y, \tau} = R_{\Omega(y, \tau)}^{m, r}$ from $H_0^{m, r}(\Omega(y, \tau))$ into $H_0^{m+1, r}(\Omega(y, \tau))^n$ with the properties $\operatorname{div} R_{y, \tau} f = f$ for all $f \in H_0^{m, r}(\Omega(y, \tau))$ with $\int_{\Omega(y, \tau)} f dx = 0$, $R_{y, \tau} C_0^\infty(\Omega(y, \tau)) \subseteq C_0^\infty(\Omega(y, \tau))^n$, and*

$$\|\nabla^{m+1} R_{y, \tau} f\|_{L^r(\Omega(y, \tau))^n} \leq c \|\nabla^m f\|_{L^r(\Omega(y, \tau))}, \quad f \in H_0^{m, r}(\Omega(y, \tau)),$$

where the constant $c = c(\Omega, m, r) > 0$ does not depend on f, y , and τ .

PROOF. We may assume that $y=0$ and we set $\Omega(\tau) = \Omega(0, \tau) = \tau\Omega$. We assume first that Ω is starlike and we use the notations $h, d, R, \Omega_i, \phi_i, \psi, \dots$ as in the foregoing proof. Let h_τ be defined by $h_\tau(x) = |\tau|^{-n} h(\tau^{-1}x)$, and let R_τ be the operator R constructed with $h_\tau, \Omega(\tau)$ instead of h, Ω in Bogovski's formula. Let ϕ_i^τ, ψ^τ be defined by $\phi_i^\tau(x) = \phi_i(\tau^{-1}x)$, $\psi^\tau(x) = |\tau|^{-n} \psi(\tau^{-1}x)$, let $\Omega_i(\tau) = \tau\Omega_i$, and let Γ_i^τ be defined in the same way as Γ_i with ϕ_i^τ, ψ^τ instead of ϕ_i, ψ . Let $d_\tau = |\tau|d$.

Using $\|h_\tau\|_\infty \leq |\tau|^{-n} \|h\|_\infty$ and $\|\nabla h_\tau\|_\infty \leq |\tau|^{-n-1} \|\nabla h\|_\infty$ we get from (2.8) the estimate

$$\begin{aligned} \|\nabla R_\tau f\|_{L^r(\Omega(\tau))^n} &\leq c_1 (\|h_\tau\|_\infty d_\tau^n + \|\nabla h_\tau\|_\infty d_\tau^{n+1}) \|f\|_{L^r(\Omega(\tau))} \\ &\leq c_2 (\|h\|_\infty d^n + \|\nabla h\|_\infty d^{n+1}) \|f\|_{L^r(\Omega(\tau))}, \quad f \in C_0^\infty(\Omega(\tau)), \end{aligned}$$

which proves the case $m=0$ in the estimate of 2.10. In case $m \in \mathbf{N}$ we get from (2.9) the estimate

$$\|\nabla \partial^\alpha R_\tau f\|_{L^r(\Omega(\tau))} \leq c \sum_{\beta \leq \alpha} (\|\partial^\beta h_\tau\|_\infty d_\tau^n + \|\nabla \partial^\beta h_\tau\|_\infty d_\tau^{n+1}) \|\partial^{\alpha-\beta} f\|_{L^r(\Omega(\tau))},$$

and using

$$\|\partial^\beta h_\tau\|_\infty \leq |\tau|^{-n-|\beta|} \|\partial^\beta h_\tau\|_\infty, \quad \|\nabla \partial^\beta h_\tau\|_\infty \leq |\tau|^{-n-|\beta|-1} \|\nabla \partial^\beta h_\tau\|_\infty,$$

and Poincaré's inequality

$$\|\partial^{\alpha-\beta} f\|_{L^r(\Omega(\tau))} \leq c d_\tau^{|\beta|} \|\partial^\alpha f\|_{L^r(\Omega(\tau))}$$

we obtain the estimate

$$\|\nabla^{m+1} R_\tau f\|_{L^r(\Omega(\tau))} \leq c \|\nabla^m f\|_{L^r(\Omega(\tau))}$$

which proves 2.10 for starlike Ω . For general Ω we obtain

$$\begin{aligned} \|\partial^\alpha \Gamma_i^\tau(f)\|_r &\leq c_1 (\sum_{\beta \leq \alpha} \|(\partial^\beta \phi_i^\tau)(\partial^{\alpha-\beta} f)\|_r) + c_2 \|f\|_r \|\partial^\alpha \psi^\tau\|_r |\tau|^{n/r'} \|\phi_1\|_{r'} \\ &\leq c_3 (\sum_{\beta \leq \alpha} |\tau|^{-|\beta|} |\tau|^{|\beta|} \|\partial^\beta f\|_r) + c_4 \|\partial^\alpha f\|_r |\tau|^{|\alpha|-|\alpha|-n+n(1/r+1/r')} \\ &\leq c_5 \|\partial^\alpha f\|_r, \quad |\alpha| = m, \\ \|\nabla^m \Gamma_i^\tau(f)\|_{L^r(\Omega(\tau))} &\leq c \|\nabla^m f\|_{L^r(\Omega(\tau))}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla^{m+1} R_\tau f\|_{L^r(\Omega(\tau))} &\leq c_1 (\|\nabla^m \Gamma_1^\tau(f)\|_{L^r(\Omega_1(\tau))} + \|\nabla^m \Gamma_2^\tau(f)\|_{L^r(\Omega_2(\tau))}) \\ &\leq c_2 \|\nabla^m f\|_{L^r(\Omega(\tau))}. \end{aligned}$$

This proves the theorem.

3. Exterior domains

First we consider the operators rot and div in homogeneous Sobolev spaces for the entire space. Then we get our results for exterior domains by localization.

In view of this localization procedure it is convenient to use on \mathbf{R}^n the Sobolev space $H_K^{m,r}(\mathbf{R}^n)^n$ which is by definition the completion of $C_0^\infty(\mathbf{R}^n)^n$ under the norm $\|u\|_{K,m,r} = \|\nabla^m u\|_{L^r(\mathbf{R}^n)^n} + \|u\|_{L^r(K)}$; in the following $K \subseteq \mathbf{R}^n$ is a fixed open ball. Thus we get $H_K^{m,r}(\mathbf{R}^n)^n \subseteq H_{\text{loc}}^{m,r}(\mathbf{R}^n)^n$. Furthermore we use the subspace $H_{K,\sigma}^{m,r}(\mathbf{R}^n) = \{v \in H_K^{m,r}(\mathbf{R}^n)^n : \operatorname{div} v = 0\}$ with the same norm. We also use the notation $\widehat{H}_{0,\sigma}^{m,r}(\Omega) = \{v \in \widehat{H}_0^{m,r}(\Omega)^3 : \operatorname{div} v = 0\}$ for $m=1, 2, \dots$ and $\widehat{H}_{0,\sigma}^{0,r}(\Omega) = L_\sigma^r(\Omega) = \{v \in L^r(\Omega)^n : \operatorname{div} v = 0, N \cdot v|_{\partial\Omega} = 0\}$ for $m=0$.

Our main result on the entire space reads as follows:

THEOREM 3.1. *Let $m \in \mathbf{N}_0$ and $1 < r < \infty$.*

a) *There exists a bounded linear operator $M^{m,r}$ from $H_{K,\sigma}^{m,r}(\mathbf{R}^3)$ into $H_{K,\sigma}^{m+1,r}(\mathbf{R}^3)$ such that $\operatorname{rot} M^{m,r}g = g$ for all $g \in H_{K,\sigma}^{m,r}(\mathbf{R}^3)$.*

b) Let $n \geq 2$. There exists a bounded linear operator $J^{m,r}$ from $H_K^{m,r}(\mathbf{R}^n)$ into $H_K^{m+1,r}(\mathbf{R}^n)^n$ such that $\operatorname{div} J^{m,r}f = f$ for all $f \in H_K^{m,r}(\mathbf{R}^n)$.

This leads to our main result on rot for exterior domains; here we consider only simply connected domains.

THEOREM 3.2. Let $m \in \mathbf{N}_0$, $1 < r < \infty$, and let $\Omega \subseteq \mathbf{R}^3$ be a simply connected exterior domain of class C^{m+2} . Then there exists a linear operator $S = S_\Omega^{m,r}$ from $\hat{H}_{0,\sigma}^{m,r}(\Omega)$ into $\hat{H}_0^{m+1,r}(\Omega)^3$ such that $\operatorname{rot} Sg = g$ and $\|\nabla^{m+1} Sg\|_{L^r(\Omega)^3} \leq c \|\nabla^m g\|_{L^r(\Omega)^3}$ for all $g \in \hat{H}_{0,\sigma}^{m,r}(\Omega)$; $c = c(\Omega, m, r) > 0$ is a constant.

The following theorem is the main result on div for exterior domains. For the lowest regularity level $m=0$, a complete proof is contained in [2].

THEOREM 3.3. Let $m \in \mathbf{N}_0$, $1 < r < \infty$, and let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be an exterior Lipschitz domain. Then there exists a linear operator $R = R_\Omega^{m,r}$ from $\hat{H}_0^{m,r}(\Omega)$ into $\hat{H}_0^{m+1,r}(\Omega)^n$ such that $\operatorname{div} Rf = f$ and $\|\nabla^{m+1} Rf\|_{L^r(\Omega)^n} \leq c \|\nabla^m f\|_{L^r(\Omega)}$ for all $f \in \hat{H}_0^{m,r}(\Omega)$; $c = c(\Omega, m, r) > 0$ is a constant.

In order to prove these theorems we prepare two lemmas on the spaces $H_K^{m,r}(\mathbf{R}^n)$. First we observe that the norms $\|u\|_{K,m,r}$ and $\|u\|_{K',m,r}$ are equivalent if K' is another open ball in \mathbf{R}^n [17, p. 161]. Then we note that $H_K^{m,r}(\mathbf{R}^n) \subseteq H_{\text{loc}}^{m,r}(\mathbf{R}^n)$ and $\hat{H}_0^{m,r}(\Omega) \subseteq H_K^{m,r}(\mathbf{R}^n)$; thus the space $\hat{H}_0^{m,r}(\Omega)$ obtained by completion of $C_0^\infty(\Omega)$ under $\|\nabla^m u\|_r$ is embedded in $H_K^{m,r}(\mathbf{R}^n)$; this follows easily by applying Poincaré's inequality near the boundary $\partial\Omega$.

In our first lemma we give a criterion for $u \in H_K^{m,r}(\mathbf{R}^n)$ using decay properties.

LEMMA 3.4. Let $m \in \mathbf{N}_0$, $1 < r < \infty$, $n \geq 2$. Suppose $u \in H_{\text{loc}}^{m,r}(\mathbf{R}^n)$ and assume that for all $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbf{N}_0^n$ the following holds:

a) If $0 \leq m - \frac{n}{r}$ and $0 \leq |\beta| \leq m - \frac{n}{r}$, then $|\partial^\beta u(x)| \leq c|x|^{m - \frac{n}{r} - |\beta|}$ for all sufficiently large $|x|$ with some $c = c(u) > 0$.

b) If $m - \frac{n}{r} < |\beta| \leq m$, then $\partial^\beta u \in L^q(\mathbf{R}^n)$ where $q \geq r$ is defined by $m - \frac{n}{r} = |\beta| - \frac{n}{q}$.

Then it holds $u \in H_K^{m,r}(\mathbf{R}^n)$.

PROOF. Let u be as in 3.4. It suffices to construct $u_j \in H^{m,r}(\mathbf{R}^n)$ with compact supports such that $\|\nabla^m(u - u_j)\|_r \rightarrow 0$ and $\|u - u_j\|_{L^r(K)} \rightarrow 0$ as $j \rightarrow \infty$. Observe that $\|\nabla^m u\|_r < \infty$; this follows from b) with $|\beta| = m$ and

$q = r$. In order to construct the approximations u_j we choose a function $\phi \in C_0^\infty(\mathbf{R}^n)$ with $\phi = 1$ for $|x| \leq 1$, $0 \leq \phi \leq 1$, and $\phi(x) = 0$ for $|x| \geq 2$. Then we set $\phi_j(x) = \phi(j^{-1}x)$ and $v_j(x) = u(x)\phi_j(x)$. We get $|\partial^\beta \phi_j(x)| \leq c j^{-|\beta|}$ with some $c = c(\beta) > 0$, and for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| = m$ we obtain

$$\|\partial^\alpha v_j\|_r^r \leq c(\|\partial^\alpha u\|_r^r + \sum_{\beta < \alpha} \int_{j \leq |x| \leq 2j} |\partial^{\alpha-\beta} \phi_j|^r |\partial^\beta u|^r dx).$$

Using 3.4 a) we obtain for $0 \leq |\beta| \leq m - \frac{n}{r}$ and sufficiently large j the estimate

$$\int_{j \leq |x| \leq 2j} |\partial^{\alpha-\beta} \phi_j|^r |\partial^\beta u|^r dx \leq c_1 j^{-(|\alpha| - |\beta| + m - \frac{n}{r} - |\beta|)r} \int_j^{2j} \tau^{n-1} d\tau \leq c_2$$

where $c_1 = c_1(u) > 0$ and $c_2 = c_2(u) > 0$ are independent of j . Using 3.4 b) with $m - \frac{n}{r} < |\beta| \leq m$, we obtain with $\frac{1}{q} + \frac{m - |\beta|}{n} = \frac{1}{r}$ the estimate

$$\int_{j \leq |x| \leq 2j} |\partial^{\alpha-\beta} \phi_j|^r |\partial^\beta u|^r dx \leq c_1 j^{-(|\alpha| - |\beta|)r} \|\partial^\alpha u\|_q^r \left(\int_j^{2j} \tau^{n-1} d\tau \right)^{\frac{r}{n}(m - |\beta|)} \leq c_2$$

where $c_1 = c_1(m, r, n) > 0$ and $c_2 = c_2(u) > 0$ are constants. Thus we get $\|\partial^\alpha v_j\|_r \leq c$ where $c = c(u) > 0$ is independent of j . The properties of ϕ_j yield the weak convergence of $\partial^\alpha v_j$ to $\partial^\alpha u$ for all α with $|\alpha| = m$. Then Mazur's theorem [28, V, 1] implies the existence of a sequence (u_j) of convex combinations of the v_j such that $\|\nabla^m(u - u_j)\|_r \rightarrow 0$ as $j \rightarrow \infty$. Since $u(x) = u_j(x)$ for $x \in K$ if j is sufficiently large, we obtain also $\|\nabla^m(u - u_j)\|_r + \|u - u_j\|_{L^r(K)} \rightarrow 0$ as $j \rightarrow \infty$. This proves the lemma.

Our second lemma gives the properties of the Laplacian Δ in the spaces $H_K^{m,r}(\mathbf{R}^n)$; we set $\Delta = \partial_1^2 + \partial_2^2 + \dots + \partial_n^2$.

LEMMA 3.5. *Let $m \in \mathbf{N}_0$, $1 < r < \infty$, and $n \geq 2$. Then Δ is a bounded surjective operator from $H_K^{m+2,r}(\mathbf{R}^n)$ onto $H_K^{m,r}(\mathbf{R}^n)$ with finite dimensional null space $N^{m,r} = \{v \in H_K^{m+2,r}(\mathbf{R}^n) : \Delta v = 0\}$. In case $m + 2 - \frac{n}{r} < 0$, it holds $N^{m,r} = \{0\}$, and in case $m + 2 - \frac{n}{r} \geq 0$, $N^{m,r}$ is the space of harmonic polynomials P with degree $d(P) \leq m + 2 - \frac{n}{r}$. There is a bounded linear operator $V^{m,r} : H_K^{m,r}(\mathbf{R}^n) \rightarrow H_K^{m+2,r}(\mathbf{R}^n)$ such that $\Delta V^{m,r} f = f$ for $f \in H_K^{m,r}(\mathbf{R}^n)$. The operator $\Delta : H_K^{m+2,r}(\mathbf{R}^n) / N^{m,r} \rightarrow H_K^{m,r}(\mathbf{R}^n)$ is an isomorphism.*

PROOF. $V^{m,r}$ can be constructed as follows. Let F_n be the fundamental solution of Δ , i.e. it holds $\Delta F_n(x) = \delta(x)$ with Dirac's δ -

distribution. Consider any bounded linear projection $Q^{m,r}$ from $H_K^{m+2,r}(\mathbf{R}^n)$ onto the finite dimensional space $N^{m,r}$. Then we set $V^{m,r}f = (I - Q^{m,r})F_n^*f$ first for $f \in C_0^\infty(\mathbf{R}^n)$ and then by continuity for all $f \in H_K^{m,r}(\mathbf{R}^n)$. Here I means the identity and $*$ the convolution.

The continuity of $\Delta : H_K^{m+2,r}(\mathbf{R}^n) \rightarrow H_K^{m,r}(\mathbf{R}^n)$ follows from the estimates $\|\nabla^m \Delta v\|_r \leq c \|\nabla^{m+2} v\|_r$ and $\|\Delta v\|_{L^r(K)} \leq c(\|\nabla^{m-2} v\|_{L^r(K)} + \|v\|_{L^r(K)})$. To prove the surjectivity of Δ we consider some $f \in H_K^{m,r}(\mathbf{R}^n)$ and a sequence $f_j \in C_0^\infty(\mathbf{R}^n)$ such that $\|f - f_j\|_{K,m,r} \rightarrow 0$ as $j \rightarrow \infty$. Let w_j be defined by $w_j(x) = (F_n^* f_j)(x) = \int F_n(x-y) f_j(y) dy$. The Calderon-Zygmund theorem yields $\|\nabla^{m+2} w_j\|_r \leq c \|\nabla^m f_j\|_r$ with some $c = c(m, r) > 0$. Using the elementary properties $|F_n(x)| \leq c_1 |x|^{2-n}$ ($n \geq 3$) and $|F_2(x)| \leq c_2 |\log|x||$ we see that $\partial^\beta w_j \in L^q(\mathbf{R}^n)$ for $m+2 - \frac{n}{r} < |\beta| \leq m+2$, $m+2 - \frac{n}{r} = |\beta| - \frac{n}{q}$, and that $|\partial^\beta w_j(x)| \leq c|x|^{m+2 - \frac{n}{r} - |\beta|}$ for $0 \leq |\beta| \leq m+2 - \frac{n}{r}$ and large $|x|$ if $m+2 - \frac{n}{r} \geq 0$. Thus from 3.4 it follows $w_j \in H_K^{m+2,r}(\mathbf{R}^n)$, $j=1, 2, \dots$.

Now we consider the conditions $m+2 - \frac{n}{r} = \nu - \frac{n}{q}$ with $0 \leq \nu \leq m+2$, $q \geq r$. Then for $m+2 - \frac{n}{r} < \nu < m+2$ we obtain $\|\nabla^\nu w_j\|_q \leq c \|\nabla^{m+2} w_j\|_r$ by Sobolev's inequality [7, p. 24], and it follows $\|\nabla^\nu w_j\|_q \leq c \|\nabla^m f_j\|_r$ and $\|\nabla^\nu(w_j - w_i)\|_q \leq c \|\nabla^m(f_j - f_i)\|_r \rightarrow 0$ as $j, i \rightarrow \infty$. In case $m+2 - \frac{n}{r} < 0$ it follows $\|w_j - w_i\|_q \rightarrow 0$ and therefore $\|w_j - w_i\|_{K,m,r} \rightarrow 0$ as $j, i \rightarrow \infty$; we get some $w \in H_K^{m+2,r}(\mathbf{R}^n)$ with $\|w - w_j\|_{K,m,r} \rightarrow 0$. In case $m+2 - \frac{n}{r} \geq 0$ we let k be the largest integer such that $k \leq m+2 - \frac{n}{r}$. Let $q \geq r$ be defined by $m+2 - \frac{n}{r} = k+1 - \frac{n}{q}$. Then we get $\|\nabla^{k+1} w_j\|_q \leq c \|\nabla^{m+2} w_j\|_r$ as above since $m+2 - \frac{n}{r} < k+1 \leq m+2$, and it follows $\|\nabla^{k+1} w_j\|_q \leq c \|\nabla^m f_j\|_r$ and $\|\nabla^{k+1}(w_j - w_i)\|_q \leq c \|\nabla^m(f_j - f_i)\|_r$.

In the next step we use the general Poincaré inequality [20, p. 112; see also 4.5 a)]

$$\inf_{d(P) \leq k} \|w + P\|_{L^r(K)} \leq c \|\nabla^{k+1} w\|_{L^r(K)}$$

and we conclude that (w_j) is a Cauchy sequence with respect to the norm on the left; here inf is taken over all polynomials P with degree $d(P) \leq k$. Therefore, there are polynomials P_j with $d(P_j) \leq k$ such that the sequence

$(w_j + P_j)$ converges in space $H^{k+1,r}(K)$. Since $d(P_j) \leq k \leq m+2 - \frac{n}{r}$, we obtain $P_j \in H_K^{m+2,r}(\mathbf{R}^n)$. Therefore, $(w_j + P_j)$ is a Cauchy sequence in $H_K^{m+2,r}(\mathbf{R}^n)$ and converges to some $w \in H_K^{m+2,r}(\mathbf{R}^n)$. From $\Delta w_j = f_j$ and $\|w - (w_j + P_j)\|_{H^{m+2,r}(K)} \rightarrow 0$ we obtain $0 = \lim_{j \rightarrow \infty} \|\Delta w - (f_j + \Delta P_j)\|_{L^r(K)} = \lim_{j \rightarrow \infty} \|\Delta w - f - \Delta P_j\|_{L^r(K)}$ and $\lim_{j,i \rightarrow \infty} \|\Delta(P_j - P_i)\|_{L^r(K)} = 0$. From this we conclude that there is a polynomial P such that $d(P) \leq k$ and $\lim_{j \rightarrow \infty} \|\Delta(P - P_j)\|_{L^r(K)} = 0$. This leads to $\Delta w = f + \Delta P$ and $\Delta(w - P) = f$ which proves the surjectivity assertion for Δ .

Let $v \in N^{m,r}$. For $m+2 - \frac{n}{r} < 0$ we obtain $v \in L^q(\mathbf{R}^n)$ where q is defined by $m+2 - \frac{n}{r} = -\frac{n}{q}$. Since v is harmonic, it follows $v = 0$ and therefore we get $N^{m,r} = \{0\}$. For $m+2 - \frac{n}{r} \geq 0$ we define k as above and obtain $\partial^\beta v \in L^r(\mathbf{R}^n)$ for $|\beta| = k+1$, $m+2 - \frac{n}{r} = k+1 - \frac{n}{q}$. Since $\partial^\beta v$ is also harmonic, it follows $\partial^\beta v = 0$ for $|\beta| = k+1$. Thus v is a harmonic polynomial with $d(v) \leq k$. Conversely, such a harmonic polynomial is contained in $N^{m,r}$; this follows from Lemma 3.4.

Since $\Delta : H_K^{m+2,r}(\mathbf{R}^n) \rightarrow H_K^{m,r}(\mathbf{R}^n)$ is bounded and surjective, $\Delta : H_K^{m+2,r}(\mathbf{R}^n)/N^{m,r} \rightarrow H_K^{m,r}(\mathbf{R}^n)$ is an isomorphism. Since $N^{m,r}$ is finite dimensional, we can construct a bounded linear projection $Q^{m,r}$ from $H_K^{m+2,r}(\mathbf{R}^n)$ onto $N^{m,r}$. For all $f \in C_0^\infty(\mathbf{R}^n)$ we obtain

$$\begin{aligned} \|f\|_{K,m,r} &= \|\Delta(F_n^* f)\|_{K,m,r} \geq c_1 \inf_{P \in N^{m,r}} \|(F_n^* f) + P\|_{K,m+2,r} \\ &= c_2 \inf_{P \in N^{m,r}} \|(I - Q^{m,r})\{(F_n^* f) + P\}\|_{K,m+2,r} \\ &= c_1 \|(I - Q^{m,r})(F_n^* f)\|_{K,m+2,r} \end{aligned}$$

and $\Delta(I - Q^{m,r})F_n^* f = \Delta F_n^* f = f$. The operator $f \rightarrow (I - Q^{m,r})F_n^* f$ can be extended by continuity from $C_0^\infty(\mathbf{R}^n)$ to $H_K^{m,r}(\mathbf{R}^n)$; we denote this extension by $V^{m,r}$ and get the desired properties. This proves the lemma.

PROOF OF THEOREM 3.1. Consider the operator $V^{m,r} : H_K^{m,r}(\mathbf{R}^n) \rightarrow H_K^{m+2,r}(\mathbf{R}^n)$ from the proof above. Then we set $J^{m,r} f = \nabla V^{m,r} f$ for all $f \in H_K^{m,r}(\mathbf{R}^n)$. From 3.5 it follows immediately that $J^{m,r}$ has the properties in 3.1 b); we have only to use $\Delta = \operatorname{div} \nabla$. To prove 3.1 a) we consider the space $H_K^{m,\sigma}(\mathbf{R}^n)$ instead of $H_K^{m,r}(\mathbf{R}^n)$. Using $\operatorname{rot} \operatorname{rot} + \Delta = \nabla \operatorname{div}$ we see that $\operatorname{rot} \operatorname{rot} v = -\Delta v$ for all $v \in H_K^{m+2,\sigma}(\mathbf{R}^3)$. The properties of $\operatorname{rot} \operatorname{rot}$ are completely analogous to those of Δ in 3.5. We set $M^{m,r} = \operatorname{rot} V^{m,r}$ and get

the properties in 3.1 a). This proves the theorem.

PROOF OF THEOREM 3.2. Let $g \in \widehat{H}_{0,\sigma}^{m,r}(\Omega)$. Extending g by zero we get $g \in \widehat{H}_{0,\sigma}^{m,r}(\mathbf{R}^3)$. Using $M^{m,r}$ from 3.1 a) we set $\widehat{v} = M^{m,r}g$ and obtain $\operatorname{rot} \widehat{v} = g = 0$ in $\mathbf{R}^3 \setminus \overline{\Omega}$. Since the bounded domain $\mathbf{R}^3 \setminus \overline{\Omega}$ is simply connected, there exists some $\widehat{p} \in H^{m+2,r}(\mathbf{R}^3 \setminus \overline{\Omega})$ such that $\widehat{v} = \nabla \widehat{p}$ in $\mathbf{R}^3 \setminus \overline{\Omega}$. Now we take an open ball B with $\partial\Omega \subseteq B$ and we solve the Dirichlet problem

$$\Delta^{m+2} \widehat{p} = 0, \quad \partial^\alpha (p - \widehat{p})|_{\partial\Omega} = 0, \quad \partial^\alpha \widehat{p}|_{\partial B} = 0, \quad |\alpha| \leq m+1$$

on $\Omega \cap B$ in the weak sense using Simader's theory [22]. We obtain a solution $\widehat{p} \in H^{m+2,r}(\Omega \cap B)$ with $\partial^\alpha (\widehat{v} - \nabla \widehat{p})|_{\partial\Omega} = 0$ for $|\alpha| \leq m+1$. Then we set $Sg = \widehat{v} - \nabla \widehat{p}$ and we obtain $Sg \in \widehat{H}_0^{m+1,r}(\Omega)^3$ and the estimate

$$\begin{aligned} \|\nabla^{m+1} Sg\|_r &\leq \|\nabla^{m+1} \widehat{v}\|_r + \|\nabla^{m+1} (\nabla \widehat{p})\|_r \\ &\leq c_1 \|\nabla^m g\|_r + c_2 \|\widehat{p}\|_{H^{m+2,r}(\Omega \cap B)} \leq c_3 \|\nabla^m g\|_r \end{aligned}$$

using the properties of $M^{m,r}$ in 3.1 a) and the well known trace theorem for $\partial\Omega$. This proves the theorem.

PROOF OF THEOREM 3.3. Consider open balls B_0 and B in \mathbf{R}^n with $\partial\Omega \subseteq B_0$ and $\overline{B_0} \subseteq B$ where $\overline{B_0}$ is the closure of B_0 . Let $\phi \in C_0^\infty(\mathbf{R}^n)$ be a function with $0 \leq \phi \leq 1$, $\phi = 0$ on B_0 and $\phi = 1$ on $\Omega \setminus B$. Let $f \in \widehat{H}_0^{m,r}(\Omega)$. Then we define $R = R_{\Omega}^{m,r}$ by $Rf = R_{\Omega \cap B}^{m,r}(\operatorname{div}(1-\phi)J^{m,r}f) + \phi J^{m,r}f$ with $R_{\Omega \cap B}^{m,r}$ from 2.4 and $J^{m,r}$ from 3.1 b). Here we suppress the operations of restricting to a subdomain and of extending by zero. In particular we set $f = 0$ outside Ω .

R is well defined since $\operatorname{div}(1-\phi)J^{m,r}f = (1-\phi)f - (\nabla\phi)J^{m,r}f \in H_0^{m,r}(\Omega \cap B)$; this follows from $f \in \widehat{H}_0^{m,r}(\Omega)$, $(1-\phi)f \in H_0^{m,r}(\Omega \cap B)$, $J^{m,r}f \in H_K^{m+1,r}(\mathbf{R}^n)^n$ and $(\nabla\phi)J^{m,r}f \in H_0^{m+1,r}(\Omega \cap B) \subseteq H_0^{m,r}(\Omega \cap B)$. Since $\operatorname{div} J^{m,r}f = f = 0$ on $\mathbf{R}^n \setminus \Omega$ we conclude easily that $\int_{\Omega \cap B} \operatorname{div}(1-\phi)J^{m,r}f dx = 0$. Thus we get $\operatorname{div} Rf = \operatorname{div}(1-\phi)J^{m,r}f + \operatorname{div} \phi J^{m,r}f = \operatorname{div} J^{m,r}f = f$.

From 2.4 it follows $R_{\Omega \cap B}^{m,r}(\operatorname{div}(1-\phi)J^{m,r}f) \in H_0^{m+1,r}(\Omega \cap B)^n \subseteq \widehat{H}_0^{m+1,r}(\Omega)^n$ and from 3.1 b) we conclude that $J^{m,r}f \in H_K^{m+1,r}(\mathbf{R}^n)^n$ and $\phi J^{m,r}f \in \widehat{H}_0^{m+1,r}(\Omega)^n$. Moreover we obtain

$$\begin{aligned} \|\nabla^{m+1} Rf\|_r &\leq c_1 (\|\nabla^m \operatorname{div}(1-\phi)J^{m,r}f\|_r + \|\nabla^{m+1}(\phi J^{m,r}f)\|_r) \\ &\leq c_2 (\|\nabla^m(1-\phi)f\|_r + \|\nabla^m(\nabla\phi)J^{m,r}f\|_r + \|\nabla^{m+1}(\phi J^{m,r}f)\|_r) \\ &\leq c_3 (\|f\|_{H^{m,r}(\Omega \cap B)} + \|J^{m,r}f\|_{H^{m+1,r}(\Omega \cap B)} \\ &\quad + \|\nabla^{m+1} J^{m,r}f\|_r + \|J^{m,r}f\|_{L^r(K)}) \\ &\leq c_4 (\|\nabla^m f\|_{L^r(\Omega)} + \|J^{m,r}f\|_{K,m,r}) \leq c_5 \|\nabla^m f\|_{L^r(\Omega)}. \end{aligned}$$

Here we used the estimates $\|f\|_{H^{m,r}(\Omega \cap B)} \leq c \|f\|_{\hat{H}_0^{m,r}(\Omega)}$ and $\|J^{m,r}f\|_{K,m+1,r} \leq c_1 \|f\|_{K,m,r} \leq c_2 \|f\|_{\hat{H}_0^{m,r}(\Omega)}$ which follow using the Poincaré inequality. This proves Theorem 3.3.

4. Applications to the gradient ∇

Using duality arguments the properties of the operator div lead immediately to results on the gradient ∇ . We give these results in Theorem 4.2 and Corollary 4.5. Bogovski [2, p.38] considered the case of the lowest regularity level $m=0$. Some of the estimates in 4.2 and 4.5 seem to be new and some of them are well known or recently proved by completely different methods (see [9, p.257] and [19]).

In what follows we use distributions in the sense of Schwartz [29]. Recall that $\langle T, v \rangle$ means the value of some distribution T at v . We need the following density property which is due to Bogovski [2, Lemma 4 and Lemma 8]. He gave a proof by a contradiction argument; we give here a simpler proof which is based on our Theorem 2.10.

LEMMA 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be a bounded or an exterior Lipschitz domain. Then for all $m=1, 2, \dots$ and $1 < r < \infty$ the set $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega)^n : \operatorname{div} v = 0\}$ is a dense subspace of $\hat{H}_{0,\sigma}^{m,r}(\Omega) = \{v \in \hat{H}_0^{m,r}(\Omega)^n : \operatorname{div} v = 0\}$ with respect to the norm $\|\nabla^m v\|_r$.*

PROOF. Let $u \in \hat{H}_{0,\sigma}^{m,r}(\Omega)$ and let $u_j \in C_0^\infty(\Omega)^n$ be a sequence such that $\|\nabla^m(u - u_j)\|_r \rightarrow 0$ as $j \rightarrow \infty$. If Ω is bounded we get from 2.10, 2.4 c) that $R_\Omega^{m-1,r} \operatorname{div} u_j \in C_0^\infty(\Omega)^n$ and $v_j = u_j - R_\Omega^{m-1,r} \operatorname{div} u_j \in C_{0,\sigma}^\infty(\Omega)$ for $j=1, 2, \dots$. Using $\operatorname{div} u = 0$ it follows

$$\begin{aligned} \|\nabla^m(u - v_j)\|_r &\leq \|\nabla^m(u - u_j)\|_r + \|\nabla^m R_\Omega^{m-1,r} \operatorname{div} u_j\|_r \\ &\leq \|\nabla^m(u - u_j)\|_r + c_1 \|\nabla^{m-1} \operatorname{div}(u_j - u)\|_r \\ &\leq c_2 \|\nabla^m(u - u_j)\|_r \rightarrow 0 \end{aligned}$$

as $j \rightarrow \infty$.

Now we consider an exterior domain. Then the last argument does not work since $R_\Omega^{m-1,r} \operatorname{div} u_j$ need not have a compact support. However, using the construction of $R_{\Omega \cap B}^{m,r}$ for 3.3 and the properties of $R_{\Omega \cap B}^{m,r}$ (Theorem 2.4 c)) we see that $R_\Omega^{m-1,r} \operatorname{div} u_j \in C^\infty(\Omega)^n$ and that these functions vanish in a neighbourhood of $\partial\Omega$.

Let $G_j = \{x \in \mathbf{R}^n : j < |x| < 2j\}$, $B_j = \{x \in \mathbf{R}^n : |x| < 2j\}$ and let ϕ_j be defined as in the proof of 3.4. Then we get $G_j = \{jx : x \in G_1\}$. We put

$$v_j = u_j - \phi_j R_\Omega^{m-1,r} \operatorname{div} u_j + R_{G_j}^{m-1,r} (\nabla \phi_j) \cdot R_\Omega^{m-1,r} \operatorname{div} u_j.$$

Then it follows $v_j \in C_{0,\sigma}^\infty(\Omega)$ for sufficiently large j . Now we apply Theorem 2.10 to the domains G_j and obtain

$$\begin{aligned} \|\nabla^m R_{G_j}^{m-1,r}((\nabla\phi_j) \cdot R_\Omega^{m-1,r} \operatorname{div} u_j)\|_r &\leq c_1 \|\nabla^{m-1}(\nabla\phi_j) \cdot R_\Omega^{m-1,r} \operatorname{div} u_j\|_r, \\ \|\nabla^m(u - v_j)\|_r &\leq \|\nabla^m(u - u_j)\|_r + \|\nabla^m \phi_j R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)} \\ &\quad + c_2 \|\nabla^{m-1}(\nabla\phi_j) \cdot R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)}. \end{aligned}$$

Then we use $|\partial^\beta \phi_j| \leq c j^{-|\beta|}$ and the estimate

$$\|\partial^{\alpha-\beta} R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)} \leq c j^{|\beta|} \|\partial^\alpha R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)}$$

which follows from the Poincaré inequality since $R_\Omega^{m-1,r} \operatorname{div} u_j = 0$ near $\partial\Omega$. For $|\alpha| = m$ we obtain

$$\begin{aligned} \|\nabla^m \phi_j R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)} &\leq c_1 \sum_{\nu \leq m} j^{-\nu} \|\nabla^{m-\nu} R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)} \\ &\leq c_2 \|\nabla^m R_\Omega^{m-1,r} \operatorname{div} u_j\|_r \leq c_3 \|\nabla^{m-1} \operatorname{div} u_j\|_r = c_3 \|\nabla^{m-1} \operatorname{div}(u - u_j)\|_r \\ &\leq c_4 \|\nabla^m(u - u_j)\|_r. \end{aligned}$$

Similarly we get $\|\nabla^{m-1}(\nabla\phi_j) \cdot R_\Omega^{m-1,r} \operatorname{div} u_j\|_{L^r(B_j \cap \Omega)} \leq c \|\nabla^m(u - u_j)\|_r$. Thus it follows $\|\nabla^m(u - v_j)\|_r \leq c \|\nabla^m(u - u_j)\|_r \rightarrow 0$ as $j \rightarrow \infty$. This proves the lemma.

THEOREM 4.2 *Let $m=0, 1, 2, \dots$, $1 < r < \infty$, and $n \geq 2$.*

a) *Suppose $\Omega \subseteq \mathbf{R}^n$ is a bounded Lipschitz domain and $T \in H^{-m-1,r}(\Omega)^n$ is a distribution such that $\langle T, v \rangle = 0$ for all $v \in C_0^\infty(\Omega)^n$ with $\operatorname{div} v = 0$. Then there exists a $p \in H^{-m,r}(\Omega)$ determined up to a constant C such that $T = \nabla p$ and*

$$(4.3) \quad \inf_c \|p + C\|_{H^{-m,r}(\Omega)} \leq c \|\nabla p\|_{H^{-m-1,r}(\Omega)^n}$$

where $c = c(\Omega, m, r) > 0$ is a constant. If p is an arbitrary distribution on Ω with $\nabla p \in H^{-m-1,r}(\Omega)^n$, then it holds $p \in H^{-m,r}(\Omega)$ and the estimate (4.3).

b) *Suppose $\Omega \subseteq \mathbf{R}^n$ is an exterior Lipschitz domain and $T \in \widehat{H}^{-m-1,r}(\Omega)^n$ is a distribution such that $\langle T, v \rangle = 0$ for all $v \in C_0^\infty(\Omega)^n$ with $\operatorname{div} v = 0$. Then there exists a unique $p \in \widehat{H}^{-m,r}(\Omega)$ such that $T = \nabla p$ and*

$$(4.4) \quad \|p\|_{\widehat{H}^{-m,r}(\Omega)} \leq c \|\nabla p\|_{\widehat{H}^{-m-1,r}(\Omega)^n}$$

where $c = c(\Omega, m, r) > 0$ is a constant. If p is an arbitrary distribution on Ω such that $\nabla p \in \widehat{H}^{-m-1,r}(\Omega)^n$, then there is a constant $C(p)$ with $p + C(p) \in H^{-m,r}(\Omega)$ and (4.4) is true with $p + C(p)$ instead of p .

COROLLARY 4.5. *Suppose $1 < r < \infty$.*

a) Let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be a bounded Lipschitz domain and $m \in \mathbf{N}_0$. Then

$$\inf_Q \|p + Q\|_r \leq c \|\nabla^{m+1} p\|_r$$

for all $p \in H^{m+1,r}(\Omega)$ where $c = c(\Omega, m, r) > 0$ is a constant and \inf is taken over all polynomials Q of degree $d(Q) \leq m$

b) Let $\Omega \subseteq \mathbf{R}^n$ ($n \geq 2$) be an exterior Lipschitz domain. Then

$$\|p\|_{L^r(\Omega)} \leq c \|\nabla p\|_{\hat{H}^{-1,r}(\Omega)^n}$$

for all $p \in L^r(\Omega)$. For every distribution p such that $\nabla p \in \hat{H}^{-1,r}(\Omega)^n$, there exists a constant $C(p)$ with $p + C(p) \in L^r(\Omega)$ and this estimate is true with $p + C(p)$ instead of p .

c) Let Ω be as in b), $r > n(n-1)^{-1}$, and γ with $\frac{1}{n} + \frac{1}{r} = \frac{1}{\gamma}$. Then

$$\|p\|_{L^r(\Omega)} \leq c \|\nabla p\|_{L^r(\Omega)^n}$$

for all $p \in L^r(\Omega)$ with $\nabla p \in L^r(\Omega)^n$. For every distribution p such that $\nabla p \in L^r(\Omega)^n$ there exists a constant $C(p)$ with $p + C(p) \in L^r(\Omega)$ and this estimate is true with $p + C(p)$ instead of p ; here $c = c(\Omega, r) > 0$ does not depend on p .

Bogovski [2, p. 38] proved a partial assertion of 4.2 a) and b) in the special case $m=0$. The estimate in 4.5 a) is well known and proved in [20] by another method. The estimate 4.5 b) has been proved in [6] for the special case $r=2$. 4.5 c) has been proved recently in [9, p. 257] and in [19] by completely different methods.

From 4.5 c) we get for exterior domains the estimate

$$\|p\|_{L^r(\Omega)} \leq c \|\nabla p\|_{L^r(\Omega)^n}, \quad 1 < \gamma < n, \quad \frac{1}{\gamma} = \frac{1}{n} + \frac{1}{r},$$

for all (restrictions to Ω of) $p \in C_0^\infty(\mathbf{R}^n)$. Usually this estimate is known only for all $p \in C_0^\infty(\Omega)$ [7].

PROOF OF 4.2 AND 4.5. Let r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$.

a) Let Ω be as in 4.2 a). From 2.4 we get in particular that the operator $\operatorname{div} : H_0^{m+1,r'}(\Omega)^n \rightarrow H_0^{m,r'}(\Omega)$ has a closed range $R(\operatorname{div})$. From the closed range theorem [28] we conclude that the transposed operator $-\nabla : H^{-m,r}(\Omega) \rightarrow H^{-m-1,r}(\Omega)^n$ has also a closed range $R(-\nabla)$. The null space $N(-\nabla) = \{v \in H^{-m,r}(\Omega) : \nabla v = 0\}$ is the space of constants. It follows the validity of (4.3) for all $p \in H^{-m,r}(\Omega)$. Let T be given as in 4.2 a). Using the density property in 4.1 we see that $\langle T, v \rangle = 0$ holds even for all

$v \in N(\operatorname{div}) = \{v \in H_0^{m+1,r'}(\Omega)^n : \operatorname{div} v = 0\}$. Then the closed range theorem yields that $T \in R(-\nabla)$ and $T = \nabla p$ with some $p \in H^{-m,r}(\Omega)$ and that (4.3) is true. If p is a distribution with $\nabla p \in H^{-m-1,r}(\Omega)^n$, then we get $\langle \nabla p, v \rangle = -\langle p, \operatorname{div} v \rangle = 0$ for all $v \in C_0^\infty(\Omega)^n$ with $\operatorname{div} v = 0$. Thus we obtain some $p_1 \in H^{-m,r}(\Omega)$ with $\nabla p = \nabla p_1$. It follows $p_1 = p + C$ with a constant C and therefore $p \in H^{-m,r}(\Omega)$. This proves 4.2 a). The estimate 4.5 a) follows from (4.3). To see this we consider first the case $m=0$, and using the elementary inequality $\|\nabla p\|_{H^{-1,r}(\Omega)^n} \leq c \|\nabla p\|_r$ for bounded domains, we obtain from (4.3) the estimate $\inf_c \|p + C\|_r \leq c \|\nabla p\|_r$. Then we get 4.5 a) when we apply this estimate repeatedly.

b) Let Ω be as in 4.2 b). In this case the argument is similar as for 4.2 a). We have only to replace the spaces $H_0^{m+1,r'}(\Omega)^n$, $H_0^{m,r'}(\Omega)$, and $H^{-m-1,r}(\Omega)^n$ by $\hat{H}_0^{m+1,r'}(\Omega)^n$, $\hat{H}_0^{m,r'}(\Omega)$, and $\hat{H}^{-m-1,r}(\Omega)^n$, respectively. Then we have to apply Theorem 3.3 instead of Theorem 2.4, and we have to use that $N(-\nabla) = \{0\}$ for exterior domains. Then the assertion 4.2 b) follows as above. 4.5 b) is the special case $m=0$ in 4.2 b). 4.5 c) follows from 4.5 b) when we use the embedding $L^r(\Omega)^n \subseteq \hat{H}^{-1,r}(\Omega)^n$ which is a consequence of Sobolev's embedding theorem [7, p. 24].

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