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On Coxeter functors over tensor rings with duality conditions

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1. Introduction and preliminaries

Two pairs of functors play an outstanding role in the representation theory of finite-dimensional tensor algebras: The Coxeter functors C^{+} and C^{-} and the functors DTr and TrD, the importance of which has been discovered by Auslander and Reiten (see [5] resp. [1]; as usual Tr denotes the Auslander-Bridger transpose and D the duality with respect to the ground field). Brenner and Butler [4] and, independently, Gabriel [6] have proved the remarkable fact that there is an equivalence T of a very simple form such that C^{+} and $DTrT$ resp. C^{-} and $TDTr$ are isomorphic. The validity of a result of this sort for the larger class of artinian tensor rings with duality conditions has been conjectured by Auslander, Platzeck and Reiten [2] when they noticed that these rings possess a canonical selfduality D. In the present article this conjecture is confirmed. Actually we intend to show that Gabriel's proof can be adopted with certain modifications.

For the convenience of the reader we sum up some definitions and simple facts concerning tensor rings with duality conditions and their modules. Since it is more suitable for our purpose we prefer the equivalent language of modulations of quivers and their representations [5]. However, first let us agree upon some conventions. For modules M, N over some ring S we shall write (M, N) instead of $\mathrm{Hom}_{S}(M, N)$; furthermore we use the abbreviations $M^{*}=(M_{S}, S_{S})$ resp. ${}^{*}M=({}_{S}M, {}_{S}S)$ for a right resp. left S-module M. We shall place maps of left modules to the right of the argument and maps of right or bimodules to the left; accordingly the composition of maps is written. In the situation $M_{S}, {}_{S}N_{T}, P_{T}$ the canonical isomorphism $(M\otimes_{S}N_{T}, P_{T})\longrightarrow(M_{S}, (N_{T}, P_{T})_{S})$ is denoted by $f\longmapsto\hat{f}$.

In this paper we assume that Γ is a finite connected quiver without cycles and multiple arrows. The set of vertices resp. arrows of Γ is denoted by Γ_{0} resp. Γ_{1} and the domain resp. range of some arrow α by $d\alpha$ resp. $r\alpha$. Furthermore we assume that for each $x\in\Gamma_{0}$ we have a skew field F_x and for each arrow $\alpha: x \longrightarrow y$ a bimodule ${}_{F_x}M(\alpha)_{F_y}$ which is finite dimensional on either side, and a bimodule isomorphism ω_α : $M(\alpha)^* \longrightarrow {}^*M(\alpha)$. Now let $\mu = \alpha_k \cdots \alpha_1$ be some path of length $k \geq 1$ of Γ, starting in $x = d\alpha_1$ and ending in $y = r\alpha_k$. Then we have the bimodule ${}_{F_x}M(\mu)_{F_y} := M(\alpha_1) \otimes_{F_{r\alpha_1}} M(\alpha_2) \otimes \cdots \otimes_{F_{d\alpha_k}} M(\alpha_k)$ and inductively we define the bimodule isomorphism $M(\mu)^* \longrightarrow {}^*M(\mu)$ as follows. We suppose that $\mu = \tau\sigma$ for paths $\sigma: x \longrightarrow u$ and $\tau: u \longrightarrow y$ of smaller length for which ω_σ : $M(\sigma)^* \longrightarrow {}^*M(\sigma)$ and $\omega_\tau: M(\tau)^* \longrightarrow {}^*M(\tau)$ are already defined. Then ω_μ is to be the composition of isomorphisms

$$\begin{aligned} M(\mu)^* = (M(\sigma) \otimes_{F_u} M(\tau)_{F_y}, F_{y_{F_y}}) &\xrightarrow{\sim} (M(\sigma)_{F_u}, M(\tau)^*_{F_u}) \\ &\xrightarrow{(1,\, \omega_\tau)} (M(\sigma)_{F_u}, {}^*M(\tau)_{F_u}) \xrightarrow{\sim} ({}_{F_u}M(\tau), {}_{F_u}M(\sigma)^*) \\ &\xrightarrow{(1,\, \omega_\sigma)} ({}_{F_u}M(\tau), {}_{F_u}{}^*M(\sigma)) \xrightarrow{\sim} ({}_{F_x}M(\sigma) \otimes M(\tau), {}_{F_x}F_x) = {}^*M(\mu). \end{aligned}$$

Explicitly $(m_\sigma \otimes m_\tau)\omega_\mu(\varphi) := (m_\sigma)\omega_\sigma((m_\tau)\omega_\tau(\hat{\varphi}(-\sigma)))$ for $m_\sigma \in M(\sigma)$, $m_\tau \in M(\tau)$ and $\varphi \in M(\mu)^*$, where $(m_\tau)\omega_\tau(\hat{\varphi}(-\sigma))$ maps some $x_\sigma \in M(\sigma)$ to $(m_\tau)\omega_\tau(\hat{\varphi}(x_\sigma))$. For the empty path μ from x to x we put $M(\mu) := {}_{F_x}F_{xF_x}$ and for ω_μ we take the canonical isomorphism $F_x^* \longrightarrow {}^*F_x$.

A right representation V is given by right vector spaces $V(x)_{F_x}$, $x \in \Gamma_0$, and linear maps $V(\alpha)$: $V(d\alpha) \otimes_{F_{d\alpha}} M(\alpha)_{F_{r\alpha}} \longrightarrow V(r\alpha)_{F_{r\alpha}}$, $\alpha \in \Gamma_1$. Inductively we may define linear maps $V(\mu)$ for arbitrary paths μ. Again we assume that $\mu = \tau\sigma$ is decomposed as above and that $V(\sigma)$: $V(x) \otimes M(\sigma) \longrightarrow V(u)$ and $V(\tau)$: $V(u) \otimes M(\tau) \longrightarrow V(y)$ are already defined. Then we put $V(\mu) = V(\tau) \circ (V(\sigma) \otimes M(\tau))$. Similarly a left representation W of Γ consists of left vector spaces ${}_{F_x}W(X)$, $x \in \Gamma_0$, and homomorphisms $W(\alpha)$: ${}_{F_{d\alpha}}M(\alpha) \otimes_{F_{r\alpha}} W(r\alpha) \longrightarrow {}_{F_{d\alpha}}W(d\alpha)$, $\alpha \in \Gamma_1$. We shall denote the category of finite dimensional right resp. left representations by mod-Γ resp. Γ-mod. Recall that the tensor ring $R := \bigoplus_\mu M(\mu)$, where μ runs over the set of all paths of Γ, is a two-sided artinian ring and that mod-Γ resp. Γ-mod is equivalent to the category of finitely generated right resp. left modules over R. It may happen that some skew field G acts from the left on a right representation V, i.e. each $V(x)$ is a bimodule ${}_GV(x)_{F_x}$ and each $V(\alpha)$ a bimodule map. In this case we can define the tensor product $L \otimes_G V$ of V with a right vector space L by $(L \otimes V)(x) := L \otimes V(x)$ and $(L \otimes V)(\alpha): L \otimes V(\alpha)$. Similarly tensor products of left representations with left vector spaces may be defined.

Next we recall the definition of the indecomposable projective and injective representations. For each vertex x the indecomposable projective right representation P_x is given by $P_x(s) := \bigoplus_{x \xrightarrow{\mu} s} M(\mu)$, μ running over the set of paths from x to s, and for an arrow $\alpha: s \longrightarrow t$ by the map $P_x(\alpha)$: $P_x(s)\otimes M(\alpha) \longrightarrow P_x(t)$, $P_x((m_\mu)\otimes m_\alpha):=(m_\mu\otimes m_\alpha)$. Let $\beta: x \longrightarrow y$ be an arrow of Γ. Then we may form $M(\beta)\otimes_{F_y}P_y$ and it is quickly checked that $P_\beta: M(\beta)\otimes P_y \longrightarrow P_x$ with $P_\beta(s)(m_\beta\otimes(m_\nu)):=(m_\beta\otimes m_\nu)$ for $m_\beta\in M(\beta)$ and $(m_\nu)\in P_y(s)$ defines a morphism. The indecomposable injective right representations I_x, $x\in\Gamma_0$, are defined by $I_x(s):=\bigoplus_{s\xrightarrow{\mu}x} M(\mu)^*$ and $I_x(\alpha): I_x(s)\otimes M(\alpha) \longrightarrow I_x(t)$, with $I_x(\alpha)(\varphi_\mu\otimes m_\alpha)((m_\rho)):=\sum_{\rho\alpha=\mu}\varphi_\mu(m_\alpha\otimes m_\rho)$ for $\varphi_\mu\in M(\mu)^*$, $m_\alpha\in M(\alpha)$ and $(m_\rho)\in \bigoplus_{t\xrightarrow{\rho}x} M(\rho)$, in short $I_x(\alpha)(\varphi_\mu\otimes m_\alpha)=\sum_{\rho\alpha=\mu}\varphi_\mu(m_\alpha\otimes -\rho)$. For each arrow $\beta: x\longrightarrow y$ we have the morphism $I_\beta: M(\beta)\otimes_{F_y}I_y \longrightarrow I_x$ given by $I_\beta(m_\beta\otimes\psi_\rho)((m_\mu)):=\sum_{\rho=\beta\mu}(m_\beta)\omega_\beta(\psi_\rho(m_\mu\otimes-\beta))$ for $m_\beta\in M(\beta)$, some path $\rho: s\longrightarrow y$, $\psi_\rho\in M(\rho)^*$ and $(m_\mu)\in\bigoplus_{s\xrightarrow{\mu}x} M(\mu)$, in shorter notation $I_\beta(s)(m_\beta\otimes\psi_\rho)=\sum_{\rho=\beta\mu}(m_\beta)\omega_\beta(\widehat{\psi}_\rho(-\mu))$. In an analogous way the indecomposable projective and injective left representations ${}_xP$ and ${}_xI$ are introduced.

Auslander, Reiten and Platzeck [2] have noticed that there exists a duality mod-$\Gamma \underset{D}{\overset{D}{\rightleftarrows}} \Gamma$-mod. It works as follows. If we start with a left representation W then DW is defined by $DW(x):={}^*W(x)$ and for an arrow $\alpha: x\longrightarrow y$ the linear map $DW(\alpha): DW(x)\otimes M(\alpha)\longrightarrow DW(y)$ is the composition

$$\begin{aligned}{}^*W(x)\otimes_{F_x}M(\alpha) &\xrightarrow{{}^*W(\alpha)\otimes M(\alpha)} {}^*(M(\alpha)\otimes_{F_y}W(y))\otimes_{F_x}M(\alpha)\\ &\xrightarrow{\sim}({}_{F_x}M(\alpha)\otimes_{F_y}W(y),\ {}_{F_x}M(\alpha))\xrightarrow{\sim}({}_{F_y}W(y),\ {}_{F_y}({}_{F_x}M(\alpha),\ {}_{F_x}M(\alpha)))\\ &\longrightarrow({}_{F_y}W(y),\ {}_{F_y}F_y)={}^*W(y);\end{aligned}$$

the last map is induced by the bimodule homomorphism $({}_{F_x}M(\alpha), {}_{F_x}M(\alpha)) \longrightarrow F_y$, $f\longmapsto\sum_{j=1}^{k}\omega_\alpha^{-1}(\psi_{x,j})((m_{x,j})f)$, where $(m_{x,j},\psi_{x,j})_{1\le j\le k}$ is a dual basis of ${}_{F_x}M(\alpha)$. Explicitly $(w)(DW(\alpha)(\psi\otimes m_\alpha)):=\omega_\alpha^{-1}((-\alpha\otimes w)W(\alpha)\circ\psi)m_\alpha)$ for $\psi\in{}^*W(x)$, $m_\alpha\in M(\alpha)$ and $w\in W(y)$. Similarly the dual DV of a right representation V is given by $DV(x):=V(x)^*$, $x\in\Gamma_0$, and $DV(\alpha)$: $M(\alpha)\otimes DV(y)\longrightarrow DV(x)$ is determined by the formula $((m_\alpha\otimes\psi)DV(\alpha))(v):=(m_\alpha)\omega_\alpha(\psi\circ V(\alpha)(v\otimes -))$. It is easy to see that the duality D is weakly symmetric, i.e. $D(P_x/J(P_x))\cong{}_xP/J({}_xP)$ where $J(P_x)$ resp. $J({}_xP)$ denotes the unique maximal subrepresentation of P_x resp. ${}_xP$.

At last we define a left representation V^t for each right representation V as follows. Let $V^t(x) := (V, P_x)$ be the set of morphisms from V to P_x, and for an arrow $\alpha : x \longrightarrow y$ let $V^t(\alpha) : M(\alpha) \otimes V^t(y) \longrightarrow V^t(x)$ be given by $(m_\alpha \otimes h) V^t(\alpha) := m_\alpha \otimes h$ for $m_\alpha \in M(\alpha)$ and $h \in V^t(y)$, where $(m_\alpha \otimes h)(s)(v) := m_\alpha \otimes h(s)(v)$ for $s \in \Gamma_0$, $v \in V(s)$. In addition, for a morphism $f : U \longrightarrow V$ the morphism $f^t : V^t \longrightarrow U^t$ is simply given by composition, i. e. $(h) f^t(x) := h \circ f$. It is obvious that the functor $V \longmapsto V^t$ is right exact. Furthermore, note that in case $V_1 \xrightarrow{f} V_0 \longrightarrow V \longrightarrow 0$ is a projective resolution of V such that the kernel of f is contained in the radical of V_1, then the cokernel of $f^t : V_0^t \longrightarrow V_1^t$ is the Auslander-Bridger transpose $Tr\ V$ of V.

LEMMA 1: *For each $x \in \Gamma_0$ and each finite dimensional right F_x-vector space L we have a functorial isomorphism*

$$j = j_{L,x} : L \otimes_{F_x} I_x \longrightarrow D((L \otimes P_x)^t)$$

where

$$j(s) : L \otimes_{F_x} I_x(s) \longrightarrow {}^*(L \otimes P_x, P_s)$$

is given by the formula

$$(h) j(s)(1 \otimes \varphi_\mu) := (h(x)(l \otimes 1)) \omega_\mu(\varphi_\mu)$$

for $l \in L$, some path $\mu : s \longrightarrow x$, $\varphi_\mu \in M(\mu)^*$ and $h \in (L \otimes P_x, P_s)$.

PROOF: We confine ourselves to show that j is a morphism. We have to realize that for each arrow $\alpha : s \longrightarrow t$ the diagram

$$\begin{array}{ccc} L \otimes I_x(s) \otimes M(\alpha) & \xrightarrow{j(s) \otimes M(\alpha)} & {}^*(L \otimes P_x, P_s) \otimes M(\alpha) \\ {\scriptstyle L \otimes I_x(\alpha)} \downarrow & & \downarrow {\scriptstyle \gamma} \\ L \otimes I_x(t) & \xrightarrow[j(t)]{} & {}^*(L \otimes P_x, P_t) \end{array}$$

commutes with $\gamma = D((L \otimes P_x)^t)(\alpha)$. We take $l \in L$, a path $\mu : s \longrightarrow x$, $\varphi_\mu \in M(\mu)^*$, $m_\alpha \in M(\alpha)$, $g \in (L \otimes P_x, P_t)$ and put $g(x)(l \otimes 1) := (x_\sigma) \in P_t(x) = \bigoplus_{t \xrightarrow{\sigma} x} M(\sigma)$. Then

$$\begin{aligned} (g)(\gamma(j(s) \otimes 1)(l \otimes \varphi_\mu \otimes m_\alpha)) &= (g)\gamma(j(s)(l \otimes \varphi_\mu) \otimes m_\alpha) \\ &= \omega_\alpha^{-1}((-\alpha \otimes g)(L \otimes P_x)^t(\alpha) \circ j(s)(l \otimes \varphi_\mu))(m_\alpha) \\ &= \omega_\alpha^{-1}((-\alpha \otimes g) \circ j(s)(l \otimes \varphi_\mu))(m_\alpha) \\ &= \omega_\alpha^{-1}((-\alpha \otimes g(x)(l \otimes 1)) \omega_\mu(\varphi_\mu))(m_\alpha) \\ &= \sum_{\sigma\alpha = \mu} \omega_\alpha^{-1}((-\alpha \otimes x_\sigma) \omega_\mu(\varphi_\mu))(m_\alpha). \end{aligned}$$

On the other hand

$$(g)(j(t)(1\otimes I_x(\alpha))(l\otimes\varphi_\mu\otimes m_\alpha))=(g)(j(t)(l\otimes I_x(\alpha)(\varphi_\mu\otimes m_\alpha)))$$
$$=\sum_{\sigma\alpha=\mu}(g)j(t)(l\otimes\varphi_\mu(m_\alpha\otimes-\sigma))$$
$$=\textstyle\sum_{\sigma\alpha=\mu}g(x)(l\otimes 1)\omega_\sigma(\varphi_\mu(m_\alpha\otimes-\sigma))$$
$$=\sum_{\sigma\alpha=\mu}(x_\sigma)\omega_\sigma(\varphi_\mu(m_\alpha\otimes-\sigma)).$$

Since we have

$$x_\sigma\omega_\mu(\varphi_\mu)=\omega_\alpha((x_\sigma)\omega_\sigma(\hat{\varphi}_\mu(-\alpha)))$$

for $\mu=\sigma\alpha$, we obtain

$$\omega_\alpha^{-1}((-\alpha\otimes x_\sigma)\omega_\mu(\varphi_\mu))(m_\alpha)=((x_\sigma)\omega_\sigma(\hat{\varphi}_\mu(-\alpha)))(m_\alpha)$$
$$=(x_\sigma)\omega_\sigma(\varphi_\mu(m_\alpha\otimes-\sigma)),$$

hence the two sums coincide.

Let V be a right representation and $\alpha: x\longrightarrow y$ an arrow of Γ. Then it is easy to check that the morphisms

$$V(x)\otimes P_\alpha: V(x)\otimes_{P_x}M(\alpha)\otimes_{F_y}P_y\longrightarrow V(x)\otimes_{F_x}P_x$$

and

$$V(\alpha)\otimes P_y: V(x)\otimes_{F_x}M(\alpha)\otimes_{F_y}P_y\longrightarrow V(y)\otimes_{F_y}P_y$$

induce commutative diagrams

$$\begin{array}{ccc} V(x)\otimes M(\alpha)\otimes I_y & \xrightarrow{V(x)\otimes I_\alpha} & V(x)\otimes I_x \\ \downarrow j & & \downarrow j \\ D((V(x)\otimes M(\alpha)\otimes P_y)^t) & \xrightarrow[D((V(x)\otimes P_\alpha)^t)]{} & D((V(x)\otimes P_x)^t) \end{array}$$

and

$$\begin{array}{ccc} V(x)\otimes M(\alpha)\otimes I_y & \xrightarrow{V(\alpha)\otimes I_y} & V(x)\otimes I_y \\ \downarrow j & & \downarrow j \\ D((V(x)\otimes M(\alpha)\otimes P_y)^t) & \xrightarrow[D((V(\alpha)\otimes P_y)^t)]{} & D((V(y)\otimes P_y)^t). \end{array}$$

2. Proof of the main result

The first step in the proof of the equivalence $C^+\cong DTrT$ for a certain equivalence $T:\text{mod-}\Gamma\longrightarrow\Gamma\text{-mod}$ consists in a precise description of the Auslander-Reiten translate $DTrV$ for a right representation V. We begin with the construction of a projective resolution of V. For each vertex x we have a morphism $\zeta_x : V(x)\otimes_{F_x}P_x \longrightarrow V$ given by

$$\zeta_x(s)(v\otimes(m_\mu)) := \sum_{x \xrightarrow{\mu} s} V(\mu)(v\otimes m_\mu) \text{ for } v\in V(x)$$

and $(m_\mu)\in P_x(s)$. Bundling up the ζ_x we arrive at the morphism $\zeta=(\zeta_x)$: $\bigoplus_{x\in\Gamma_0} V(x)\otimes P_x \longrightarrow V$. In order to describe the kernel of ζ, we define morphisms

$$\partial, \varepsilon : \bigoplus_{\alpha\in\Gamma_1} V(d\alpha)\otimes M(\alpha)\otimes P_{r\alpha} \longrightarrow \bigoplus_{x\in\Gamma_0} V(x)\otimes P_x$$

by fixing their components $\partial_{x,\alpha}, \varepsilon_{x,\alpha} : V(d\alpha)\otimes M(\alpha)\otimes P_{r\alpha} \longrightarrow V(x)\otimes P_x$ as follows: $\varepsilon_{r\alpha,\alpha}=V(\alpha)\otimes P_{r\alpha}$, $\varepsilon_{x,\alpha}=0$ for $x\neq r\alpha$, $\partial_{d\alpha,\alpha}=V(d\alpha)\otimes P_\alpha$ and $\partial_{x,\alpha}=0$ for $x\neq d\alpha$. It is easy to see that $\zeta\partial=\zeta\varepsilon$.

PROPOSITION 2: *The sequence*

$$0 \longrightarrow \bigoplus_\alpha V(d\alpha)\otimes M(\alpha)\otimes P_{r\alpha} \xrightarrow{\partial-\varepsilon} \bigoplus_x V(x)\otimes P_x \xrightarrow{\zeta} V \longrightarrow 0$$

constitutes a projective resolution of V.

PROOF: For each $s\in\Gamma_0$ we consider the F_s-linear maps

$$\tau(s) : V(s) \longrightarrow V(s)\otimes P_s(s) \text{ and}$$
$$\sigma(s) : V(x)\otimes P_x(s) \longrightarrow \bigoplus_\alpha V(d\alpha)\otimes M(\alpha)\otimes P_{r\alpha}(s)$$

defined in the following way. For $v\in V(s)$ we put $\tau(s)(v) := v\otimes 1$; for $w\in V(x)$, some path $\mu=\alpha_k\cdots\alpha_1 : x\longrightarrow s$ and $m_i\in M(\alpha_i)$ we put

$$\sigma(s)(w\otimes m_1\otimes\cdots\otimes m_k) := \sum_{i=1}^{k} V(\alpha_{i-1}\cdots\alpha_1)(w\otimes m_1\otimes\cdots\otimes m_{i-1})\otimes m_i\otimes\cdots\otimes m_k.$$

Simple calculations prove the equations $\sigma(s)(\partial(s)-\varepsilon(s))=1$, $(\partial(s)-\varepsilon(s))\sigma(s)+\tau(s)\zeta(s)=1$, and $\zeta(s)\tau(s)=1$, hence the sequence is exact. Obviously the first and the second terms are projective.

Applying the functor $D((-)^t)$ to the above sequence and taking into account the remark concluding section 1, we arrive at the exact sequence

$$0 \longrightarrow DTrV \longrightarrow \bigoplus_\alpha V(d\alpha)\otimes M(\alpha)\otimes I_{r\alpha} \xrightarrow{\theta-\eta} \bigoplus_x V(x)\otimes I_x$$

where $\eta_{r\alpha,\alpha}=V(\alpha)\otimes I_{r\alpha}$, $\eta_{x,\alpha}=0$ for $x\neq r\alpha$, $\theta_{d\alpha,\alpha}=V(d\alpha)\otimes I_\alpha$ and $\theta_{x,\alpha}=0$ for $x\neq d\alpha$.

Now let Γ^1 denote an isomorphic copy of Γ, the isomorphism being denoted by $x \longmapsto x^1$ on the vertices and by $\alpha \longmapsto \alpha^1$ on the arrows. Furthermore, let $\overrightarrow{\Gamma\Gamma}$ denote the quiver consisting of the disjoint union of Γ and Γ^1 together with a new arrow $\alpha^1 : y^1 \longrightarrow x$ for each arrow $\alpha : x \longrightarrow y$ of Γ. We put $M(\alpha^1) := M(\alpha)$ and $M(\alpha') := {}^*M(\alpha)$. A right representation W of $\overrightarrow{\Gamma\Gamma}$ is called bound, if for all $x \in \Gamma_0$ and $w \in W(x^1)$ the equation

$$\sum_{\overset{\alpha}{\longrightarrow} x} W(\alpha\alpha')(w \otimes \varepsilon_{r\alpha}) + \sum_{x \overset{\beta}{\longrightarrow}} W(\beta'\beta^1)(w \otimes \varepsilon_{d\beta}) = 0$$

holds. In these sums, α and β run over the sets of arrows of Γ ending and starting in x, respectively, $\varepsilon_{r\alpha} := \sum_j \psi_{d\alpha,j} \otimes m_{d\alpha,j} \in {}^*M(\alpha) \otimes M(\alpha)$, where $(m_{d\alpha,j}, \psi_{d\alpha,j})_j$ is some dual basis of ${}_{F_{d\alpha}}M(\alpha)$, and $\varepsilon_{d\beta} := \sum_k m_{r\beta,k} \otimes \omega_\beta(\psi_{r\beta,k}) \in M(\beta) \otimes {}^*M(\beta)$, where $(m_{r\beta,k}, \psi_{r\beta,k})_k$ is a dual basis of $M(\beta)_{F_{r\beta}}$. (The elements $\varepsilon_{r\alpha}$ and $\varepsilon_{d\beta}$ are independent of the choice of the dual bases, furthermore, $\xi \cdot \varepsilon_{r\alpha} = \varepsilon_{r\alpha} \cdot \xi$ for all $\xi \in F_{r\alpha}$ and $\eta \cdot \varepsilon_{d\beta} = \varepsilon_{d\beta} \cdot \eta$ for all $\eta \in F_{d\beta}$.) We denote by mod-$\widetilde{\Gamma\Gamma}$ the category of bound right representations of $\overrightarrow{\Gamma\Gamma}$ and by $\rho :$ mod-$\widetilde{\Gamma\Gamma} \longrightarrow$ mod-Γ resp. $\rho' :$ mod-$\widetilde{\Gamma\Gamma} \longrightarrow$ mod-Γ^1 the restriction functors.

THEOREM 3: 1) *ρ admits a right adjoint functor* $\rho^* : \textit{mod-}\Gamma \longrightarrow \textit{mod-}\widetilde{\Gamma\Gamma}$.

2) *The composition of functors* $C^+ : \textit{mod-}\Gamma \overset{\rho^*}{\longrightarrow} \textit{mod-}\widetilde{\Gamma\Gamma} \overset{\rho'}{\longrightarrow} \textit{mod-}\Gamma^1 \overset{\sim}{\longrightarrow}$ *mod-*Γ *is isomorphic to DTrT, where* $T : \textit{mod-}\Gamma \longrightarrow \textit{mod-}\Gamma$ *is defined by* $(TV)(x) := V(x)$ *for* $x \in \Gamma_0$ *and* $(TV)(\alpha) := -V(\alpha)$ *for* $\alpha \in \Gamma_1$.

In the next section we shall show that the functor C^+ coincides with the Coxeter functor defined by Dlab and Ringel [5].

PROOF: According to the remark following Prop. 2, $DTrTV$ is the kernel in the exact sequence

$$(*) \qquad 0 \longrightarrow DTrTV \longrightarrow \bigoplus_\alpha V(d\alpha) \otimes M(\alpha) \otimes I_{r\alpha} \overset{\theta+\eta}{\longrightarrow} \bigoplus_x V(x) \otimes I_x.$$

In order to indicate a right adjoint ρ^* of ρ it is useful to interpret the middle and the end term of sequence $(*)$ as spaces of linear maps. We consider the right representations $F_\alpha(V)$, $\alpha \in \Gamma_1$, and $G_x(V)$, $x \in \Gamma_0$, depending functorially on V, which are defined as follows. For $s \in \Gamma_0$ we put

$$F_\alpha(V)(s) := \Big(\bigoplus_{s \overset{\nu}{\longrightarrow} r\alpha} M(\nu^1) \otimes M(\alpha')_{F_{d\alpha}}, V(d\alpha)_{F_{d\alpha}} \Big),$$

$$G_x(V)(s):=(\bigoplus_{s\xrightarrow{\rho}x} M(\rho^1)_{Fx},\ V(x)_{Fx}),$$

and for some arrow $\beta: s \longrightarrow t$ we let $F_\alpha(V)(\beta): F_\alpha(V)(s)\otimes M(\beta)\longrightarrow F_\alpha(V)(t)$ be defined by $F_\alpha(V)(\beta)(\phi\otimes m_\beta):=\phi(m_\beta\otimes -)$ and $G_x(V)(\beta): G_x(V)(s)\otimes M(\beta)\longrightarrow G_x(V)(t)$ by $G_x(V)(\beta)(\Psi\otimes m_\beta):=\Psi(m_\beta\otimes -)$. There are functorial isomorphisms

$$p_\alpha: V(d\alpha)\otimes M(\alpha)\otimes I_{r\alpha}\longrightarrow F_\alpha(V) \quad \text{and}$$
$$q_x: V(x)\otimes I_x\longrightarrow G_x(V),$$

given by $p_\alpha(s)(v\otimes m_\alpha\otimes\psi)(z\otimes\varphi_{\alpha'}):=v\cdot(m_\alpha\cdot\psi(z))\varphi_{\alpha'}$ for $v\in V(d\alpha)$, $m_\alpha\in M(\alpha)$, $\psi\in I_{r\alpha}(s)$, $z\in \bigoplus_{s\xrightarrow{\nu}r\alpha} M(\nu^1)$, $\varphi_{\alpha'}\in M(\alpha')$, and by $q_x(w\otimes\psi)(z):=w\cdot\psi(z)$ for $w\in V(x)$, $\psi\in I_x(s)$ and $z\in\bigoplus_{s\xrightarrow{\rho}x} M(\rho^1)$. At last, if we define $\theta'_{x,\alpha}$, $\eta'_{x,\alpha}: F_\alpha(V)\longrightarrow G_x(V)$ by the formulae $\theta'_{d\alpha,\alpha}(s)(\phi_\alpha):=\phi_\alpha(-\otimes\varepsilon_{d\alpha})$, $\theta'_{x,\alpha}(s):=0$ for $x\neq d\alpha$, $\eta'_{r\alpha,\alpha}(s)(\phi_\alpha): V(\alpha)(\phi_\alpha\otimes M(\alpha))(-\otimes\varepsilon_{r\alpha})$ and $\eta'_{x,\alpha}(s):=0$ for $x\neq r\alpha$, then the diagrams

$$\begin{array}{ccc}
V(d\alpha)\otimes M(\alpha)\otimes I_{r\alpha} & \xrightarrow{\theta_{d\alpha,\alpha}} & V(d\alpha)\otimes I_{d\alpha} \\
{\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle q_{d\alpha}} \\
F_\alpha(V) & \xrightarrow[\theta'_{d\alpha,\alpha}]{} & G_{d\alpha}(V)
\end{array}$$

and

$$\begin{array}{ccc}
V(d\alpha)\otimes M(\alpha)\otimes I_{r\alpha} & \xrightarrow{\eta_{r\alpha,\alpha}} & V(r\alpha)\otimes I_{r\alpha} \\
{\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle q_{r\alpha}} \\
F_\alpha(V) & \xrightarrow[\eta'_{r\alpha,\alpha}]{} & G_{r\alpha}(V)
\end{array}$$

commute and we have the exact sequence

$$0\longrightarrow DTrTV\longrightarrow\bigoplus_\alpha F_\alpha(V)\xrightarrow{\theta'+\eta'}\bigoplus_x G_x(V).$$

Now $\rho^*: \text{mod-}\Gamma\longrightarrow\text{mod-}\widetilde{\Gamma\Gamma}$ is constructed as follows. For $V\in\text{mod-}\Gamma$ and $s\in\Gamma_0$ let $\rho^* V(s):=V(s)$ and $\rho^* V(s^1):=\operatorname{Ker}(\theta'(s)+\eta'(s))$. Note that $(\phi_\alpha)\in\bigoplus_\alpha F_\alpha(V)(s)$ belongs to $\rho^* V(s^1)$ if and only if the equations

$$(**)\qquad \sum_{\xrightarrow{\alpha}x} V(\alpha)((\phi_\alpha\otimes M(\alpha))(z\otimes\varepsilon_{r\alpha}))+\sum_{x\xrightarrow{\beta}}\phi_\beta(z\otimes\varepsilon_{d\beta})=0$$

hold for all $x \in \Gamma_0$ and $z \in \bigoplus_{s \xrightarrow{\rho} x} M(\rho^1)$. For some arrow $\beta : s \longrightarrow t$ of Γ we require that $\rho^* V(\beta) = V(\beta)$, that $\rho^* V(\beta^1) : \rho^* V(s^1) \otimes M(\beta) \longrightarrow \rho^* V(t^1)$ is given by $\rho^* V(\beta^1)(\phi \otimes m_\beta) := \phi(m_\beta \otimes -)$, and $\rho^* V(\beta') : \rho^* V(t^1) \otimes M(\beta') \longrightarrow \rho^* V(s) = V(s)$ by $\rho^* V(\beta')(\phi \otimes \varphi_{\beta'}) = \phi_\beta(\varphi_{\beta'})$, ϕ_β being the β-th component of $\phi \in \rho^* V(t^1)$. To show that $\rho^* V$ is bound, let $x \in \Gamma_0$ and $\phi \in \rho^* V(x^1)$; then

$$\sum_{\xrightarrow{\alpha} x} \rho^* V(\alpha\alpha')(\phi \otimes \varepsilon_{r\alpha}) + \sum_{x \xrightarrow{\beta}} \rho^* V(\beta'\beta^1)(\phi \otimes \varepsilon_{d\beta})$$
$$= \sum_{\xrightarrow{\alpha} x} \rho^* V(\alpha)(\rho^* V(\alpha') \otimes M(\alpha))(\phi \otimes \varepsilon_{r\alpha})$$
$$+ \sum_{x \xrightarrow{\beta}} \rho^* V(\beta')(\rho^* V(\beta^1) \otimes M(\beta'))(\phi \otimes \varepsilon_{d\beta})$$
$$= \sum_{\xrightarrow{\alpha} x} V(\alpha)(\phi_\alpha \otimes M(\alpha))(\varepsilon_{r\alpha}) + \sum_{x \xrightarrow{\beta}} \phi_\beta(\varepsilon_{d\beta}) = 0,$$

taking into account equation (**). It remains to show that there is an isomorphism

$$\text{mod-}\widetilde{\Gamma\Gamma}\ (X, \rho^* V) \xrightarrow{\sim} \text{mod-}\Gamma(\rho X, V)$$

functorial in $X \in \text{mod-}\widetilde{\Gamma\Gamma}$ and $V \in \text{mod-}\Gamma$. We start with a family of linear maps $f(s) : X(s) \longrightarrow \rho^* V(s) = V(s)$ and $f(s^1) : X(s^1) \longrightarrow \rho^* V(s^1)$, $s \in \Gamma_0$; to indicate that $f(s^1)(z)$ is a linear map for $z \in X(s^1)$ we write $f(s^1)(z) := f(s^1)(z, -)$. This family constitutes a morphism $X \longrightarrow \rho^* V$ if and only if for each arrow $\beta : s \longrightarrow t$ in Γ the following equations hold:

i) $V(\beta)(f(s)(z) \otimes m_\beta) = f(t)(X(\beta)(z \otimes m_\beta))$, where $m_\beta \in M(\beta)$ and $z \in X(s)$;

ii) $f(s^1)(z, m_\beta \otimes -) = f(t^1)(X(\beta^1)(z \otimes m_\beta), -)$ for $m_\beta \in M(\beta)$ and $z \in X(s^1)$;

iii) $f(s)(X(\beta')(z \otimes \varphi_{\beta'})) = f(t^1)(z, \varphi_{\beta'})$ for $\varphi_{\beta'} \in M(\beta')$ and $z \in X(s^1)$.

It is easily verified that ii) and iii) are equivalent to the single equation

iv) $f(u^1)(z, w) = f(t)(X(\beta'\mu^1)(z \otimes w))$ for all paths $\mu : u \longrightarrow t$ in Γ, $z \in X(u^1)$ and $w \in M(\mu^1) \otimes M(\beta')$.

Equation i) expresses the fact that $(f(s))_{s \in \Gamma_0}$ is a morphism $\rho X \longrightarrow V$. Hence the restriction to Γ of a morphism $f : X \longrightarrow \rho^* V$ yields a morphism $\rho X \longrightarrow V$, which by iv) uniquely determines f. Conversely, given a morphism $\rho X \longrightarrow V$, equation iv) enables us to expand it to a morphism $X \longrightarrow \rho^* V$ whose restriction to Γ coincides with the original one. Hence ρ^* is a right adjoint of ρ; according to the construction of ρ^* we have $C^+ \cong DTrT$.

3. Description of C^+ by reflection functors

Now we show that the Coxeter functor C^+ as defined in the preceding section is isomorphic to the composition of reflection functors. In the latter form it has originally been introduced by Bernstein, Gelfand and Ponomarev for quiver algebras [3] and some years later by Dlab and Ringel for modulated quivers [5]. We assume $\Gamma_0=\{1, 2, \cdots, n\}$ and that the orientation of Γ is admissible for sinks, i. e. that $i>j$ for each arrow $i \longrightarrow j$. By K_i, $0\leq i\leq n$, we denote the full subquiver of $\overrightarrow{\Gamma\Gamma}$ containing the edges $1, \cdots, n, 1^1, \cdots, i^1$; note that $K_0=\Gamma$, $K_n=\overrightarrow{\Gamma\Gamma}$ and that i^1 is a source of K_i. Furthermore, we denote by mod-$\widetilde{K}_i$ the full subcategory of right representations W of K_i which satisfy the equations

$$\sum_{\xrightarrow{\alpha} x} W(\alpha\alpha')(z\otimes\varepsilon_{r\alpha})+\sum_{x\xrightarrow{\beta}} W(\beta'\beta^1)(z\otimes\varepsilon_{d\beta})=0$$

for all $x\in\{1, \cdots, i\}$ and $z\in W(i^1)$; again, α and β run over the set of arrows of Γ ending and starting in x, respectively.

PROPOSITION 4: *The restriction functor* ρ_i : *mod-*$\widetilde{K}_i\longrightarrow$ *mod-*$\widetilde{K}_{i-1}$ *possesses a right adjoint* $\rho^{(i)}$: *mod-*$\widetilde{K}_{i-1}\longrightarrow$ *mod-*$\widetilde{K}_i$.

PROOF: Let $V\in$ mod-$\widetilde{K}_{i-1}$. We require that $\rho^{(i)}V(x):=V(x)$ for the edges x of K_{i-1} and that $\rho^{(i)}V(i^1)$ is the kernel in the exact sequence

$$0\longrightarrow \rho^{(i)}V(i^1)\overset{\tau}{\hookrightarrow}\bigoplus_{\xrightarrow{\lambda} i} V(d\lambda)\otimes M(\lambda)\xrightarrow{(V(\lambda))} V(i)$$

where λ runs over the arrows of K_{i-1} ending in i. The λ-th projection of the direct sum is denoted by pr_λ. Next we have to define $\rho^{(i)}V$ on the arrows of K_i. For an arrow γ of K_{i-1} we put $\rho^{(i)}V(\gamma):=V(\gamma)$. For some arrow $\alpha: j\longrightarrow i$ in Γ we let $\rho^{(i)}V(\alpha'): \rho^{(i)}V(i^1)\otimes M(\alpha')\longrightarrow \rho^{(i)}V(j)$ be the composition $\rho^{(i)}V(\alpha')=\pi_\alpha\circ((pr_\alpha\circ\tau)\otimes M(\alpha'))$ where π_α : $V(d\alpha)\otimes M(\alpha)\otimes M(\alpha')\longrightarrow V(d\alpha)$ is given by $\pi_\alpha(v\otimes m_\alpha\otimes\varphi_{\alpha'}):=v.\ (m_\alpha)\varphi_{\alpha'}$. A straightforward calculation shows that $(\rho^{(i)}V(\alpha')\otimes M(\alpha))(z\otimes\varepsilon_{r\alpha})=pr_\alpha(z)$ for all $z\in\rho^{(i)}V(i^1)$. On the other hand, for the arrows $\beta: i\longrightarrow j$ in Γ we let

$$\rho^{(i)}V(\beta^1): \rho^{(i)}V(i^1)\otimes M(\beta^1)\longrightarrow \rho^{(i)}V(j^1)$$

be the composition $\rho^{(i)}V(\beta^1)=\pi_{\beta'}\circ((pr_{\beta'}\circ\tau)\otimes M(\beta^1))$ where $\pi_{\beta'}$: $V(d\beta')\otimes M(\beta')\otimes M(\beta)\longrightarrow V(d\beta')$ is defined by $\pi_{\beta'}(v\otimes\varphi_{\beta'}\otimes m_\beta):=v\cdot\omega_\beta^{-1}(\varphi_{\beta'})(m_\beta)$. In this case we have $(\rho^{(i)}V(\beta^1)\otimes M(\beta'))(z\otimes\varepsilon_{d\beta})=pr_{\beta'}(z)$ for all $z\in\rho^{(i)}V(i^1)$. In consequence

$$\sum_{\xrightarrow{\alpha} i} \rho^{(i)} V(\alpha\alpha')(z\otimes\varepsilon_{r\alpha}) + \sum_{i\xrightarrow{\beta}} \rho^{(i)} V(\beta'\beta^1)(z\otimes\varepsilon_{d\beta})$$
$$= \sum_{\xrightarrow{\alpha} i} \rho^{(i)} V(\alpha)(\rho^{(i)} V(\alpha')\otimes M(\alpha))(z\otimes\varepsilon_{r\alpha})$$
$$+ \sum_{i\xrightarrow{\beta}} \rho^{(i)} V(\beta^1)(\rho^{(i)} V(\beta^1)\otimes M(\beta'))(z\otimes\varepsilon_{d\beta})$$
$$= \sum_{\xrightarrow{\alpha} i} V(\alpha)pr_\alpha(z) + \sum_{i\xrightarrow{\beta}} V(\beta')pr_{\beta'}(z) = 0 \text{ for all } z\in\rho^{(i)} V(i^1),$$

i.e. the representation $\rho^{(i)} V$ belongs to mod-$\tilde{K}_i$. It is obvious how to define $\rho^{(i)}$ on morphisms.

To show that $\rho^{(i)}$ is a right adjoint of ρ_i, we check that the restriction map

$$\text{mod-}\tilde{K}_i(X, \rho^{(i)} V) \longrightarrow \text{mod-}\tilde{K}_{i-1}(\rho_i X, V)$$

is an isomorphism. To show that it is injective, let $f: X\longrightarrow \rho^{(i)} V$ be a morphism whose restriction to K_{i-1} is zero. For each arrow $\alpha: j\longrightarrow i$ in Γ we have $\rho^{(i)} V(\alpha')(f(i^1)\otimes M(\alpha')) = f(j)X(\alpha') = 0$, hence $pr_\alpha f(i^1)(z) = (\rho^{(i)} V(\alpha')\otimes M(\alpha))(f(i^1)(z)\otimes\varepsilon_{r\alpha}) = 0$ for all $z\in\rho^{(i)} V(i^1)$. Similarly we have $\rho^{(i)} V(\beta^1)(f(i^1)\otimes M(\beta)) = 0$ and $pr_{\beta'} f(i^1)(z) = 0$ for the arrows $\beta: i\longrightarrow j$ in Γ. Consequently $f(i^1)(z) = 0$ for all $z\in\rho^{(i)} V(i^1)$. To show surjectivity we start with a morphism $g: \rho_i X\longrightarrow V$. We consider the linear map φ: $X(i^1)\longrightarrow \bigoplus_{\xrightarrow{\lambda} i} X(d\lambda)\otimes M(\lambda)$ such that $pr_\alpha\varphi(z) = (X(\alpha')\otimes M(\alpha))(z\otimes\varepsilon_{r\alpha})$ for each arrow $\alpha: j\longrightarrow i$ and $pr_{\beta'}\varphi(z) = (X(\beta')\otimes M(\beta'))(z\otimes\varepsilon_{d\beta})$ for the arrows $\beta: i\longrightarrow j$. The equation

$$0 = \sum_{\xrightarrow{\alpha} i} X(\alpha\alpha')(z\otimes\varepsilon_{r\alpha}) + \sum_{i\xrightarrow{\beta}} X(\beta'\beta^1)(z\otimes\varepsilon_{d\beta})$$
$$= \sum_{\xrightarrow{\alpha} i} X(\alpha)(X(\alpha')\otimes M(\alpha))(z\otimes\varepsilon_{r\alpha})$$
$$+ \sum_{i\xrightarrow{\beta}} X(\beta')(X(\beta^1)\otimes M(\beta'))(z\otimes\varepsilon_{d\beta})$$

implies that the composition of maps

$$X(i^1) \xrightarrow{\varphi} \bigoplus_{\xrightarrow{\lambda} i} X(d\lambda)\otimes M(\lambda) \xrightarrow{(X(\lambda))} X(i)$$

is zero, hence there exists a unique linear map $f(i^1)$ making the diagram

$$\begin{array}{ccc} X(i^1) & \xrightarrow{\quad\varphi\quad} & \bigoplus_{\xrightarrow{\lambda} i} X(d\lambda)\otimes M(\lambda) \\ {\scriptstyle f(i^1)}\downarrow & & \downarrow{\scriptstyle \oplus g(d\lambda)\otimes M(\lambda)} \\ \rho^{(i)}V(i^1) & \xhookrightarrow{\quad\tau\quad} & \bigoplus_{\xrightarrow{\lambda} i} V(d\lambda)\otimes M(\lambda) \end{array}$$

commutative. It is easy to see that $f(i^1)$ and the maps $g(x)$, $x\in K_{i-1}$, constitute a morphism $X\longrightarrow \rho^* V$ whose restriction to K_{i-1} is g.

As a consequence, the composition of functors $\rho^{(n)}\cdots\rho^{(1)}$ is a right adjoint of $\rho=\rho_1\cdots\rho_n$, hence isomorphic to the functor ρ^* constructed in section 2. Now let $\Gamma_0:=\Gamma$, let Γ_i be the full subquiver of K_i containing the edges $K_i\backslash\{1,\cdots,i\}$ for $1\leq i\leq n$, $j_i:\text{mod-}\Gamma_i\longrightarrow\text{mod-}\tilde{K}_i$ the trivial extension, $p_i:\text{mod-}\tilde{K}_i\longrightarrow\text{mod-}\Gamma_i$ the restriction functor and $\sigma_i:=p_i\rho^{(i)}j_{i-1}:\text{mod-}\Gamma_{i-1}\longrightarrow\text{mod-}\Gamma_i$; note that $\Gamma_0=K_0=\Gamma$, $\Gamma_n=\Gamma^1$, j_0 is the identity and $p_n=\rho'$. Now it is obvious that $\sigma_1,\cdots,\sigma_n$ are the well-known reflection functors and that $\rho'\rho^{(n)}\cdots\rho^{(1)}=\sigma_n\cdots\sigma_1$, hence $C^+=\rho'\rho^*\cong\sigma_n\cdots\sigma_1$.

We conclude with an example (compare [7], p. 292). Let F be a skew field endowed with a derivation $F\longrightarrow F$, $\xi\longmapsto\xi'$, and ${}_FM:={}_FF\times{}_FF$ with canonical basis $x=(1,0)$ and $y=(0,1)$. By use of the derivation M can also be made a right vector space by $x\xi:=\xi x$ and $y\xi:=\xi y+\xi' x$. It is easy to see that x, y is a basis of this new vector space M_F as well, that ${}_FM_F$ is a bimodule and the map $M^*\longrightarrow{}^*M$, $\varphi\longmapsto\tilde{\varphi}$, with $\tilde{\varphi}(x):=\varphi(x)$, $\tilde{\varphi}(y):=\varphi(x)'+\varphi(y)$ is a bimodule isomorphism. If we take for Γ the quiver $2\xrightarrow{\alpha}1$ and put $F_1=F_2=F$, $M(\alpha):=M$, then the assumptions of section 1 are fulfilled and theorem 3 applies. In the present special case T is isomorphic to the identity functor, hence we even have $C^+\cong DTr$.

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