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On the reciprocity formula for generalized Dedekind sums.

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T. Yoshida recently defined a function on a finite abelian group A;

$$\Delta_{A}[\lambda]:=\sum_{j=1}^{|A|-1}\left(\frac{j}{|A|}-\frac{1}{2}\right)\lambda^{j},$$

where λ is a linear character of A([2]). For a positive integer k and an integer h, the Dedekind sum $s(h, k)$ is defined by

$$s(h, k)=\sum_{j=1}^{k-1}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right),$$

$$((x))=x-[x]-\frac{1}{2}+\frac{1}{2}\delta(x),\ x\in\boldsymbol{R},$$

$$\delta(x)=\begin{cases}1 \text{ if } x \text{ is an integer,}\\ 0 \text{ otherwise.}\end{cases}$$

Yoshida has shown that if the order of λ is k, then

$$\langle\Delta_{A}[\lambda],\ \Delta_{A}[\lambda^{h}]\rangle_{A}=s(h, k),$$

where h is an integer and $\langle\ ,\ \rangle_{A}$ means a usual scalar product on the space of complex valued functions on A. Furthermore, by using the character theory, he proved the following well known formula;

$$s(h, k)+s(k, h)=\frac{k^{2}+h^{2}+1-3kh}{12kh},$$

where k and h are positive integers such that $(h, k)=1$. Now a generalized Dedekind sum is defined by

$$s(h, k, r)=\sum_{j=1}^{k-1}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj+r}{k}\right)\right),$$

where k is a positive integer, h is an integer and r is a real number([1]). Clearly $s(h, k, r)=s(h, k, -r)$. If r is an integer and k is the order of λ, then we have

$$\langle\Delta_{A}[\lambda],\ \Delta_{A}[\lambda^{h}]\cdot\lambda^{r}\rangle_{A}=s(h, k, r).$$

Here λ is as above. This formula will be proved in Corollary 4.

The purpose of this paper is to show the reciprocity formula for generalized Dedekind sums due to Knuth from a different angle approch.

THEOREM 1([1]). *Let k and h be positive integers such that $(k, h)=1$, and let r be a real number such that $-h<r<k$. For $z\in\boldsymbol{Z}$, put*

$$e(r, z)=\begin{cases}1 \text{ if } r=0 \text{ or} \\ \qquad r\neq 0,\ r\not\equiv 0 \pmod z, \\ 0 \text{ if } r\neq 0,\ r\equiv 0 \pmod z.\end{cases}$$

Then we have

$$s(h, k, r)+s(k, h, r)=\frac{k^2+h^2+1}{12kh}+\frac{[r]([r]+1-\delta(r))}{2kh} +\frac{1}{2}\left(\left[\frac{r}{k}\right]-\left[\frac{r}{h}\right]\right)-\frac{1}{4}(e(r, h)-e(r, k)+1).$$

If we assume that $k>h>0$ and $k>r\geqq 0$, then this formula is the same that as in [1].

Throughout this paper, let k and h be positive integers such that $(k, h)=1$. Let k' and h' be integers such that $kk'+hh'=1$. We start with the following lemma.

LEMMA 2. *Let c, d and r be integers such that $0\leq c<k$, $0\leq d<h$* and $0\leq r<k$.

(1) $$s(1, k, r)=\frac{(k-1)(k-2)}{12k}+\frac{r^2}{2k}-\frac{r}{2}+\frac{1}{4}\left(1-\delta\left(\frac{r}{k}\right)\right).$$

(2) $$s(hh', kh, hh'c+kk'd)=s(h, k, d-c)-\frac{1}{2}\left(\left(\frac{d-c}{k}\right)\right)\left(1-\delta\left(\frac{d}{h}\right)\right).$$

PROOF.

(1) $$s(1, k, r)-s(1, k)=\sum_{j=1}^{k-1}\left(\left(\frac{j}{k}\right)\right)\left\{\left(\left(\frac{j+r}{k}\right)\right)-\left(\left(\frac{j}{k}\right)\right)\right\}$$
$$=\sum_{j=1}^{k-r-1}\left(\left(\frac{j}{k}\right)\right)\cdot\frac{r}{k}+\left(\left(\frac{k-r}{k}\right)\right)\left(\frac{r-k}{k}+\frac{1}{2}\right)+\sum_{j=k-r+1}^{k-1}\left(\left(\frac{j}{k}\right)\right)\left(\frac{r}{k}-1\right).$$

It is well known that, for a real number x and a positive integer n,

$$((-x))=-((x)),\quad ((x+n))=((x))$$

and $$((n\cdot x))=\sum_{i=0}^{n-1}\left(\left(\frac{i}{n}+x\right)\right).$$

Therefore the above sum is equal to $\sum_{j=1}^{r-1}\left(\left(\frac{j}{k}\right)\right)+\frac{1}{2}\left(\left(\frac{r}{k}\right)\right)$, and we have

$$\sum_{j=1}^{r-1}\left(\left(\frac{j}{k}\right)\right)+\frac{1}{2}\left(\left(\frac{r}{k}\right)\right)=\frac{r^2}{2k}-\frac{r}{2}+\frac{1}{4}\left(1-\delta\left(\frac{r}{k}\right)\right).$$

Since $s(1,k)=\dfrac{(k-1)(k-2)}{12k}$, we obtain the required equality.

(2) Since $kk'+hh'=1$, we have

$$\begin{aligned}s(hh',kh,hh'c+kk'd)&=\sum_{j=0}^{kh-1}\left(\left(\frac{j}{kh}\right)\right)\left(\left(\frac{hh'(j+c-d)+d}{kh}\right)\right)\\&=\sum_{\beta=0}^{k-1}\sum_{\alpha=0}^{h-1}\left(\left(\frac{\alpha k+\beta}{kh}\right)\right)\\&\qquad\left(\left(\frac{hh'(\alpha k+\beta+c-d)}{kh}+\frac{d}{kh}\right)\right)\\&=\sum_{\beta=0}^{k-1}\left\{\sum_{\alpha=0}^{h-1}\left(\left(\frac{\alpha}{h}+\frac{\beta}{kh}\right)\right)\right\}\\&\qquad\left(\left(\frac{hh'(\beta+c-d)}{kh}+\frac{d}{kh}\right)\right).\end{aligned}$$

Here we get $\sum_{\alpha=0}^{h-1}\left(\left(\frac{\alpha}{h}+\frac{\beta}{kh}\right)\right)=\left(\left(\frac{\beta}{k}\right)\right)$. Hence we have

$$\begin{aligned}s(hh',kh,hh'c+kk'd)&=\sum_{\beta=0}^{k-1}\left(\left(\frac{\beta}{k}\right)\right)\left(\left(\frac{h'(\beta+c-d)}{k}+\frac{d}{kh}\right)\right)\\&=\sum_{j=0}^{k-1}\left(\left(\frac{hj+d-c}{k}\right)\right)\left(\left(\frac{j}{k}+\frac{d}{kh}\right)\right).\end{aligned}$$

Now we can see

$$\left(\left(\frac{j}{k}+\frac{d}{kh}\right)\right)=\begin{cases}\left(\left(\frac{j}{k}\right)\right)+\frac{d}{kh} & \textit{if } j\neq 0,\\ \frac{d}{kh}-\frac{1}{2}\left(1-\delta\left(\frac{d}{h}\right)\right) & \textit{if } j=0.\end{cases}$$

Therefore we get the required formula, and the lemma was proved.

NOTE. The assertion (1) of Lemma 2 is a special case of the Knuth's formula which is in the proof of lemma 3 of the paper [1].

Next we state the properties of the function $\Delta_A[\lambda]$ without proofs.

LEMMA 3([2]). *Let A be a finite abelian group and B a subgroup of A. Let λ be a linear character of A.*

(1) *If $\lambda^k=1_A$, then we have* $\Delta_A[\lambda]=\sum_{j=1}^{k-1}\left(\frac{j}{k}-\frac{1}{2}\right)\lambda^j$.

(2) $\Delta_A[\lambda]_{|B}=\Delta_B[\lambda_{|B}]$, *here $_{|B}$ denotes the restriction to B.*

(3) *Let* $a \in A$. *Then* $\Delta_A[\lambda](a^{-1}) = -\Delta_A[\lambda](a)$, *and*

$$\Delta_A[\lambda] = \begin{cases} 0 & \text{if } \lambda(a)=1, \\ \frac{1}{2}\cdot\frac{\lambda(a)+1}{\lambda(a)-1} & \text{if } \lambda(a)\neq 1. \end{cases}$$

COROLLARY 4. *Under the notation of Lemma* 3, *if the order of* λ *is* k, *then*

$$\langle \Delta_A[\lambda],\ \Delta_A[\lambda^h]\cdot\lambda^r\rangle_A = s(h, k, r),$$

where h *and* r *are integers.*

PROOF. By (1) of Lemma 3, we have $\Delta_A[\lambda] = \sum_{j=1}^{k-1}\left(\frac{j}{k}-\frac{1}{2}\right)\lambda^j$. Therefore we have

$$\begin{aligned}\langle \Delta_A[\lambda],\ \Delta_A[\lambda^h]\cdot\lambda^r\rangle_A &= \left\langle \sum_{j=1}^{k-1}\left(\frac{j}{k}-\frac{1}{2}\right)\lambda^j,\ \sum_{i=1}^{k-1}\left(\frac{i}{k}-\frac{1}{2}\right)\lambda^{hi+r}\right\rangle_A \\ &= \sum_{\substack{i=1 \\ hi+r\not\equiv 0 (mod\ k)}}^{k-1} \left(\frac{hi+r}{k} - \left[\frac{hi+r}{k}\right] - \frac{1}{2}\right)\left(\frac{i}{k}-\frac{1}{2}\right).\end{aligned}$$

This proves the corollary.

Hereafter, let M and N be cyclic groups of order k and h, and let λ and μ be generators of the linear characters of M and N, respectively. Put $A = M\times N$. In the paper [2] Yoshida defined the function θ on A as

$$\begin{aligned}\theta = {} & \Delta_A[\lambda\times 1_N]\cdot\Delta_A[1_M\times\mu] \\ & -\Delta_A[\lambda\times\mu](\Delta_A[\lambda\times 1_N]+\Delta_A[1_M\times\mu]),\end{aligned}$$

and proved the preceding reciprocity formula for Dedekind sums by considering the product $\langle\theta, 1_A\rangle_A$

PROPOSITION 5. *Let* r *be an integer such that* $-h<r<k$. *Then we have*

$$\begin{aligned} s(h, k, r)+s(k, h, r) = {} & \frac{k^2+h^2+1}{12kh} + \frac{r^2}{2kh} \\ & + \frac{1}{2}\left(\left[\frac{r}{k}\right]-\left[\frac{r}{h}\right]\right) - \frac{1}{4}(e(r, h)-e(r, k)+1).\end{aligned}$$

PROOF. Under the above notation, let c and d be integers such that $0\leq c<k$ and $0\leq d<h$. We consider the product $\langle\theta, \lambda^c\times\mu^d\rangle_A$ to show the equality. Let m and n be generators of M and N respectively. Then we have the equality;

$$\theta(m^i\times n^j)=-\frac{1}{4}\ \textit{for}\ 0<i<k\ \textit{and}\ 0<j<h,$$

which is proved by (3) of Lemma 3(see [2]). Put $\omega_k=\lambda(m)$ and $\omega_h=\mu(n)$. Since $\theta(1_A)=0$, we have

$$\begin{aligned}\langle\theta,\lambda^c\times\mu^d\rangle_A&=(k<\theta_{|M}\cdot\lambda^c\rangle_M+h\langle\theta_{|N}\cdot\mu^d\rangle_N\\&\quad+\sum_{i=1}^{k-1}\sum_{j=1}^{h-1}\theta(m^i\times n^j)\cdot\lambda(m^{ci})\cdot\mu(n^{dj}))/kh\\&=(k\langle\Delta_M[\lambda],\ \Delta_M[\lambda]\cdot\lambda^c\rangle_M+h\langle\Delta_N[\mu],\ \Delta_N[\mu]\cdot\mu^d\rangle_N\\&\quad-\frac{1}{4}\cdot\sum_{i=1}^{k-1}\sum_{j=1}^{h-1}\omega_k^{ci}\cdot\omega_h^{dj})/kh\end{aligned}$$

by (2) and (3) of Lemma 3. Here we get

$$\sum_{i=1}^{k-1}\sum_{j=1}^{h-1}\omega_k^{ci}\cdot\omega_h^{dj}=\left(k\delta\left(\frac{c}{k}\right)-1\right)\left(h\delta\left(\frac{d}{h}\right)-1\right).$$

Hence $\langle\theta,\lambda^c\times\mu^d\rangle_A$ is equal to

$$\frac{s(1,k,r)}{h}+\frac{s(1,h,r)}{k}-\frac{1}{4kh}\left\{\left(k\delta\left(\frac{c}{k}\right)-1\right)\left(h\delta\left(\frac{d}{h}\right)-1\right)\right\},$$

by Corollary 4. Therefore, by (1) of Lemma 2, we have

$$(*)\qquad\langle\theta,\lambda^c\times\mu^d\rangle_A=\frac{k^2+h^2+1}{12kh}+\frac{c^2+d^2}{2kh}-\frac{1}{2}\left(\frac{c}{h}+\frac{d}{k}\right)-\frac{1}{4}\delta\left(\frac{cd}{kh}\right).$$

On the other hand,

$$\begin{aligned}\langle\theta,\lambda^c\times\mu^d\rangle_A&=s(hh',kh,hh'c+kk'd)+s(kk',kh,hh'c+kk'd)\\&\quad+\langle\Delta_A[\lambda\times1_N]\cdot\Delta_A[1_M\times\mu],\ \lambda^c\times\mu^d\rangle_A\\&=s(h,k,d-c)+s(k,h,c-d)\\&\quad-\frac{1}{2}\left(\left(\frac{d-c}{k}\right)\right)\left(1-\delta\left(\frac{d}{h}\right)\right)-\frac{1}{2}\left(\left(\frac{c-d}{h}\right)\right)\left(1-\delta\left(\frac{c}{k}\right)\right)\\&\quad+\left(\frac{c}{k}-\frac{1}{2}+\frac{1}{2}\delta\left(\frac{c}{k}\right)\right)\left(\frac{d}{h}-\frac{1}{2}+\frac{1}{2}\delta\left(\frac{d}{h}\right)\right)\end{aligned}$$

by (2) Lemma 2 and Corollary 4. Here we get

$$\begin{aligned}\left(\left(\frac{d-c}{k}\right)\right)\left(1-\delta\left(\frac{d}{h}\right)\right)&=\left(\left(\frac{d-c}{k}\right)\right)+\left(\left(\frac{c}{k}\right)\right)\delta\left(\frac{d}{h}\right)\\&=\frac{d-c}{k}-\left[\frac{d-c}{k}\right]-\frac{1}{2}+\frac{1}{2}\delta\left(\frac{d-c}{k}\right)\\&\quad+\left(\left(\frac{c}{k}\right)\right)\delta\left(\frac{d}{h}\right).\end{aligned}$$

Therefore we have

$$\langle\theta, \lambda^c\times\mu^d\rangle_A = s(h, k, d-c)+s(k, h, c-d) + \frac{cd}{kh}-\frac{1}{2}\left(\frac{d}{k}+\frac{c}{h}-\left[\frac{d-c}{k}\right]-\left[\frac{c-d}{h}\right]\right) - \frac{1}{4}\left(\delta\left(\frac{d-c}{k}\right)+\delta\left(\frac{c-d}{h}\right)-3+\delta\left(\frac{cd}{kh}\right)\right).$$

Now by (*), we obtain the equality;

$$s(h, k, d-c)+s(k, h, c-d) = \frac{k^2+h^2+1}{12kh}+\frac{(c-d)^2}{2kh}-\frac{1}{2}\left(\left[\frac{d-c}{k}\right]+\left[\frac{c-d}{h}\right]\right) + \frac{1}{4}\left(\delta\left(\frac{d-c}{k}\right)+\delta\left(\frac{c-d}{h}\right)-3\right).$$

Put $r=c-d$. Then $-h<r<k$, and we have

$$\left[\frac{d-c}{k}\right]=\left[\frac{-r}{k}\right]=-\left[\frac{r}{k}\right]-1+\delta\left(\frac{r}{k}\right).$$

Furthermore $\delta\left(\frac{r}{h}\right)-\delta\left(\frac{r}{k}\right)=e(r, k)-e(r, h)$, this proves the proposition.

PROOF OF THEOREM 1. Let r be a real number such that $-h<r<k$. Put $r=l+x$ where l is an integer such that $-h<l<k$ and x is a real number such that $-1<x<1$. Furthermore we assume that both l and x are positive or negative unless $l=0$. Then we have the following lemma. See also the proofs of Lemmas 1 and 3 in [1].

LEMMA 6. *We have*

$$s(h, k, r)+s(k, h, r)=s(h, k, l)+s(k, h, l) + \left\{\frac{|l|}{2kh}-\frac{1}{4}\left|\delta\left(\frac{l}{k}\right)-\delta\left(\frac{l}{h}\right)\right|\right\}(1-\delta(r)).$$

PROOF. First we have

$$s(h, k, r)=\sum_{j=1}^{k-1}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj+l+x}{k}\right)\right) = \sum_{j=0}^{k-1}\left(\left(\frac{h'(j-l)}{k}\right)\right)\left(\left(\frac{j+x}{k}\right)\right) = s(h, k, l)+\frac{x}{2|x|}\left(\left(\frac{h'l}{k}\right)\right)(1-\delta(r)).$$

For $s(k, h, r)$ we have the similar form. Furthermore we have

$$\left(\left(\frac{h'l}{k}\right)\right)+\left(\left(\frac{k'l}{h}\right)\right)=\frac{l}{kh}+\frac{1}{2}\left(\delta\left(\frac{l}{k}\right)-\delta\left(\frac{l}{h}\right)\right).$$

Therefore we get the required formula from our assumption, and the lemma was proved.

By Proposition 5 and Lemma 6, we have

$$s(h,k,r)+s(k,h,r)=\frac{k^2+h^2+1}{12kh}$$
$$+\frac{[r]([r]+1-\delta(r))}{2kh}+\frac{1}{2}\left(\left[\frac{r}{k}\right]-\left[\frac{r}{h}\right]\right)+\frac{1}{2}(1-e(l,k))(1-\delta(r))$$
$$-\frac{1}{4}\left\{e(l,h)-e(l,k)+1+\left|\delta\left(\frac{l}{k}\right)-\delta\left(\frac{l}{h}\right)\right|(1-\delta(r))\right\}.$$

Finally we can see

$$e(l,h)-e(l,k)+\left|\delta\left(\frac{l}{k}\right)-\delta\left(\frac{l}{h}\right)\right|(1-\delta(r))$$
$$=e(r,h)-e(r,k)+2(1-e(l,k))(1-\delta(r)),$$

and this prove the theorem.

References

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