Hokkaido Mathematical Journal Vol. 18 (1989), p. 187-194

Sharp function estimates for oscillatory singular integrals

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(Received January 14, 1988, Revised May 9, 1988)

1. Introduction. Given a real bilinear form $\langle Bx, y\rangle$, and a Calderón-Zygmund kernel $K(x)$, define the operator T by

$$(Tf)(x)=\text{p. v.}\int e^{i\langle Bx,y\rangle}K(x-y)f(y)\,dy.$$

Recently Phong and Stein [7] showed that T is bounded on $L^{p}(\boldsymbol{R}^{n})$, $1<p<\infty$. In the case $B=0$ (i. e., for Calderón-Zygmund singular integrals) it is known that T is weighted L^{p}-bounded for each weight in Muckenhoupt's A_{p} class $(1<p<\infty)$. It is now a standard way to deduce these weighted norm inequalities from the sharp function estimates of the following form

$$(Tf)^{\#}(x)\leqq C_{r}M_{r}f(x),\ x\in\boldsymbol{R}^{n},\ f\in C_{0}^{\infty}(\boldsymbol{R}^{n}).$$

Here $f^{\#}$ and $M_{r}f$ are the Fefferman-Stein sharp maximal function and the r-th Hardy-Littlewood maximal function respectively, i. e.,

$$f^{\#}(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_{Q}|f(y)-f_{Q}|\,dy,\ \text{and}$$
$$M_{r}f(x)=\sup_{x\in Q}\left(\frac{1}{|Q|}\int_{Q}|f(y)|^{r}dy\right)^{1/r},$$

where $f_{Q}=\frac{1}{|Q|}\int_{Q}f(y)\,dy$ and Q moves over all cubes with sides parallel to the coordinate axes.

The purpose of this note is to show that the modified sharp function estimates still hold for our oscillatory singular integrals, i. e.,

THEOREM 1. 1. *For any* $1<r<\infty$ *there exists a positive costant* C_{r} *such that*

$$(1.1)\qquad |Tf|^{\#}(x)\leqq C_{r}M_{r}f(x),\ x\in\boldsymbol{R}^{n},\ f\in C_{0}^{\infty}(\boldsymbol{R}^{n}).$$

As a consequence we get the same weighted norm inequalities as in the case of the usual Calderón-Zygmund singular integrals.

We should note the following two results. Firstly. in the one dimensional case one can show the above sharp function estimates by modifying the proof of Lemma 2. 2 in Chanillo-Kurtz-Sampson [2], but it seems to us that it is difficult to use their method in our case. Secondly, as for the weighted norm inequalties in the case $1<p<\infty$, our result generalizes a related one derived from their weighted weak (1, 1) estimates in Chanillo-Kurtz-Sampson [3].

By a Calderón-Zygmund kernel we mean a function $K(x)$ which is C^1 away from the origin, has mean value zero on each sphere centered at the origin and satisfies

$$(1.2)\qquad |K(x)|\leqq C|x|^{-n} \text{ and } |\nabla K(x)|\leqq C|x|^{-n-1}.$$

It is known in Ricci-Stein [8] and Chanillo-Christ [1] that T with $\langle Bx, y\rangle$ replaced by a real polynomial $P(x, y)$ is $L^p(\boldsymbol{R}^n)$-bounded and weak $(L^1(\boldsymbol{R}^n), L^1(\boldsymbol{R}^n))$-bounded, but for the present we cannot say anything about the sharp function estimates in this case.

Our proof of the Theorem is a refinement of the method in Phong-Stein [7].

We shall prepare, in the next section, some facts for proving Theorem 1. 1 and prove it in Section 3. In Section 4 we shall give several remarks and an extension of Theorem 1. 1 to the case of more general Calderón-Zygmund kernels.

We note that the letter C will always denote a positive constant which may vary in each occasion.

2. Preparation. We recall some facts and considerations in Phong-Stein [7] or Ricci-Stein [8].

LEMMA 2. 1. *Let T be as in the introduction. Then T is bounded on $L^p(\boldsymbol{R}^n)$, $1<p<\infty$.*

Next following Phong-Stein [7], we define the modified sharp maximal function $f_E^{\#}$ by

$$(2.1)\qquad (f_E^{\#})(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)-f_Q^E|dy,$$

where $f_Q^E(x)=e_Q(x)\frac{1}{|Q|}\int_Q f(y)\bar{e}_Q(y)dy$, $e_Q(x)=\chi_Q(x)\exp(i\langle Bx, x_Q\rangle)$, and x_Q is the center of the cube Q. Then it holds (see [7, p. 128])

$$(2.2)\qquad |f|^{\#}(x)\leqq 2f_E^{\#}(x),\ x\in\boldsymbol{R}^n.$$

Hence to prove our theorem we have only to show

$$(2.3)\qquad (Tf)_E^{\#}(x)\leqq C_rM_rf(x),\ x\in\boldsymbol{R}^n,\ f\in C_0^\infty(\boldsymbol{R}^n).$$

On the other hand we have also with $\tau_h(f)(x)=f(x-h)$

(2.4) $$(\tau_{-h}(Tf))(x)=e^{i\langle Bh,h\rangle}e^{i\langle Bx,h\rangle}T(e^{i\langle Bh,\cdot\rangle}(\tau_{-h}f(\cdot)))(x).$$

Therefore we see easily that to prove our theorem it suffices to show

(2.5) $$\frac{1}{|Q|}\int_Q|(Tf)(y)-(Tf)_Q|dy\leqq C\,M_rf(x),$$

for any cube Q with center at the origin, for any $x\in Q$, and any $f\in C_0^\infty(\boldsymbol{R}^n)$. It is easily seen that $|Q|^{-1}\int_Q|f(y)-f_Q|dy\leqq 2(|Q'|/|Q|)|Q'|^{-1}\int_{Q'}|f(y)-f_{Q'}|dy$ for $Q\subset Q'$, and hence to show (2.5) it suffices to get

(2.6) $$\frac{1}{|Q|}\int_Q|(Tf)(y)-(Tf)_Q|dy\leqq C\,M_rf(0),$$

for any cube Q with center at the origin, and any $f\in C_0^\infty(\boldsymbol{R}^n)$.

3. Proof of the theorem. In the case rank $B=0$, i.e. $B=0$. it is well-known that the theorem holds (see Journé [6]), and hence we may assume rank $B\geqq 1$. Now let $1<r<\infty$ and $f\in C_0^\infty(\boldsymbol{R}^n)$. We shall show the estimate (2.6). We note that we may assume $1<r\leqq 2$. r' will denote the conjugate exponent of r, i.e. $1/r+1/r'=1$, and Q_γ will denote the cube with center at the origin and side length 2γ. Now fix a cube $Q=Q_\delta$ and decompose f as $f=f_1+f_2+f_3$, where $f_1=f$ in $Q_{2\delta}$ but $f_1=0$ otherwise; $f_2=f$ in $(Q_{2\delta})^c\cap Q_{1/\delta}$, $f_2=0$ otherwise; $f_3=f$ in $(Q_{2\delta})^c\cap(Q_{1/\delta})^c$, $f_3=0$ otherwise. Write $F=T(f)$, and $F_j=T(f_j)$, $j=1, 2, 3$. Then by Lemma 2.1

$$\int_Q|F_1|^rdx\leqq\int_{R^n}|F_1|^rdx\leqq C\int_{R^n}|f_1|^rdx=C\int_{Q_{2\delta}}|f|^rdx,$$

therefore

(3.1) $$\frac{1}{|Q|}\int_Q|F_1|dx\leqq\Big(\frac{1}{|Q|}\int_Q|F_1|^rdx\Big)^{1/r}\leqq C\Big(\frac{1}{|Q_{2\delta}|}\int_{Q_{2\delta}}|f|^rdx\Big)^{1/r}\leqq C\,M_rf(0).$$

Now

$$F_2(x)=T(f_2)(x)=\int e^{i\langle Bx,y\rangle}K(x-y)f_2(y)\,dy\,;$$

define the constant c_Q by $c_Q=\int K(-y)f_2(y)\,dy$. Then

$$F_2(x)-c_Q=\int(e^{i\langle Bx,y\rangle}K(x-y)-K(-y))f_2(y)\,dy.$$

However

$$e^{i\langle Bx,y\rangle}K(x-y)-K(-y)=(e^{i\langle Bx,y\rangle}-1)K(x-y)+\{K(x-y)-K(-y)\},$$

which is bounded by $C(\delta|y|^{-n+1}+\delta|y|^{-n-1})$, if $x\in Q_\delta$, and $y\in(Q_{2\delta})^c$. Hence if $x\in Q=Q_\delta$,

$$|F_2(x)-c_Q|\leqq C\delta\left\{\int\frac{|f_2(y)|dy}{|y|^{n-1}}+\int\frac{|f_2(y)|dy}{|y|^{n+1}}\right\}$$
$$\leqq C\delta\left\{\int_{|y|\leqq C/\delta}\frac{|f(y)|dy}{|y|^{n-1}}+\int_{|y|\geqq\delta}\frac{|f(y)|dy}{|y|^{n+1}}\right\}$$
$$\leqq\delta\{C\delta^{-1}M_1f(0)+C\delta^{-1}M_1f(0)\}\leqq C\,M_1f(0).$$

because f_2 is supported in $(Q_{2\delta})^c\cap Q_{1/\delta}$, where $c^2=n$. The above estimates in terms of the Hardy-Littlewood maximal function are obtained by routine work, and we omit the proof. Thus

$$\frac{1}{|Q|}\int_Q|F_2(x)-c_Q|dx\leqq C\,M_1f(0)\leqq C\,M_rf(0). \tag{3.2}$$

Next,

$$F_3(x)=\int e^{i\langle Bx,y\rangle}\{K(x-y)-K(-y)\}f_3(y)\,dy$$
$$+\int e^{i\langle Bx,y\rangle}K(-y)f_3(y)\,dy=F_3^1(x)+F_3^2(x).$$

However $|K(x-y)-K(-y)|\leqq C|x|/|y|^{n+1}$ if $x\in Q$ and $y\in(Q_{2\delta})^c$, and therefore

$$|F_3^1(x)|\leqq C\delta\left\{\int_{|y|\geqq2\delta}\frac{|f(y)|dy}{|y|^{n+1}}\leqq C\,M_1f(0),\right.$$

which gives

$$\frac{1}{|Q|}\int_Q|F_3^1(x)|dx\leqq C\,M_1f(0)\leqq C\,M_rf(0). \tag{3.3}$$

Now note that there exist orthogonal matrices O_1 and O_2 such that O_2BO_1 is a diagonal matrix. Since rank $B\geqq1$, we may assume the first entry $\alpha_1\neq0$. Now write $x=(x_1,x')$, with $x_1\in\boldsymbol{R}^1$ and $x'\in\boldsymbol{R}^{n-1}$. Set $G(x)=F_3^2(O_1x)$, $\tilde{K}(y)=K(O_2^{-1}y)$, and $\tilde{f}_3(y)=f_3(O_2^{-1}y)$. Fix $c>0$ so that $|y'|>c/\delta$ if $|y|>1/\delta$ and $|y_1|\leqq c/\delta$. We decompose $\tilde{f}_3$ as $\tilde{f}_3=h_1+h_2$, where $h_1=\tilde{f}_3$ if $|y_1|\geqq c/\delta$, but $h_1=0$ otherwise. We have then

$$G(x)=\int e^{i(\alpha_1x_1y_1+\langle B'x',y'\rangle)}\tilde{K}(-y)\tilde{f}_3(y)\,dy$$
$$=\int e^{i\alpha_1x_1y_1}g(y_1,x')\,dy_1+\int e^{i(\alpha_1x_1y_1+\langle B'x',y'\rangle)}\tilde{K}(-y)h_2(y)\,dy$$
$$=G_1(x)+G_2(x),$$

where

$$g(y_1, x') = \int_{R^{n-1}} e^{i\langle B'x', y'\rangle} \tilde{K}(-y) h_1(y) \, dy'.$$

Now fix $0<a<1$ so that $ra>1$ and $r'n(1-a)>n-1$, i.e., $\frac{1}{r}<a<\frac{1}{r}+\left(1-\frac{1}{r}\right)\frac{1}{n}$. Then for $|y_1| \geqq c/\delta$ we have

$$\begin{aligned} |g(y_1, x')| &\leqq C\int_{R^{n-1}} \frac{|h_1(y)|dy'}{|y|^n} \\ &\leqq C\left(\int_{R^{n-1}} \frac{dy'}{|y|^{n(1-a)r'}}\right)^{1/r'} \left(\int_{R^{n-1}} \frac{|h_1(y)|^r dy'}{|y|^{nar}}\right)^{1/r}, \end{aligned}$$

and since $n(1-a)r'>n-1$, and $|y_1| \geqq c/\delta$,

$$\int_{R^{n-1}} \frac{dy'}{|y|^{n(1-a)r'}} \leqq C|y_1|^{-n(1-a)r'+n-1} \leqq C\delta^{n(1-a)r'-n+1}.$$

By the Hausdorff-Young theorem

$$\begin{aligned} &\left(\int |G_1(x_1, x')|^{r'} dx_1\right)^{1/r'} \leqq C\left(\int |g(y_1, x')|^r dy_1\right)^{1/r} \\ &\leqq C\delta^{n(1-a)-(n-1)/r'} \left(\int_{R^n} \frac{|h_1(y)|^r dy}{|y|^{nar}}\right)^{1/r} \\ &\leqq C\delta^{n(1-a)-(n-1)/r'} \left(\int_{|y|>1/\delta|} \frac{|f_1(y)|^r dy}{|y|^{nar}}\right)^{1/r} \\ &\leqq C\delta^{n(1-a)-(n-1)/r'} \delta^{na-n/r} M_r f(0) = C\delta^{1/r'} M_r f(0), \end{aligned}$$

because $h_1(y)=0$ for $|y| \leqq 1/\delta$. Hence an extra integration in x' gives

$$\int_{O_1^{-1}(Q)} |G_1(x)|^{r'} dx \leqq C\delta^n (M_r f(0))^{r'}. \tag{3.4}$$

Next

$$\begin{aligned} |G_2(x)| &\leqq C\int_{R^n} \frac{|h_2(y)|}{|y|^n} dy \leqq C\int_{|y_1|<c/\delta, |y'|>c/\delta} \frac{|h_2(y)|}{|y|^n} dy \\ &\leqq C\left(\int_{|y_1|<c/\delta} \int_{|y'|>c/\delta} \frac{dy' dy_1}{|y|^{nr'(1-a)}}\right)^{1/r'} \left(\int_{|y| \geqq 1/\delta} \frac{|f(y)|^r}{|y|^{nra}} dy\right)^{1/r} \\ &= C\, I_1 \times I_2. \end{aligned}$$

Then

$$\begin{aligned} (I_1)^{r'} &\leqq C\int_0^{c/\delta} \left(\int_{c/\delta}^{\infty} \frac{t^{n-2} dt}{(y_1+t)^{nr'(1-a)}}\right) dy_1 \\ &\leqq C\int_0^{c/\delta} \left(\int_{c/\delta}^{\infty} \frac{dt}{(y_1+t)^{nr'(1-a)-n+2}}\right) dy_1 \end{aligned}$$

$$\leqq C\int_0^{c/\delta}\frac{dy_1}{(y_1+c/\delta)^{nr'(1-a)-n+1}}\leqq C\delta^{nr'(1-a)-n}.$$

$$I_2\leqq C\delta^{na-n/r}M_rf(0).$$

Hence

$$|G_2(x)|\leqq C\ M_rf(0),$$

and so

$$(3.5)\qquad \left(\int_{O_1^{-1}(Q)}|G_2(x)|^{r'}dx\right)^{1/r'}\leqq C\delta^{n/r'}M_rf(0).$$

By (3. 4) and (3. 5) we get

$$(3.6)\qquad \frac{1}{|Q|}\int_Q|F_3^2(x)|dx\leqq\left(\frac{1}{|Q|}\int_Q|F_3^2(x)|^{r'}dx\right)^{1/r'}$$

$$=\left(\frac{1}{|Q|}\int_{O_1^{-1}(Q)}|G(x)|^{r'}dx\right)^{1/r'}\leqq C\ M_rf(0).$$

Combining (3. 1), (3. 2), (3. 3), and (3. 6) we get

$$\frac{1}{|Q|}\int_Q|F(x)-c_Q|dx\leqq C\ M_rf(0),$$

which implies

$$(3.7)\qquad \frac{1}{|Q|}\int_Q|F(x)-F_Q|dx\leqq 2C\ M_rf(0).$$

This proves the desired estimate (2. 6) and completes the proof of the theorem.

4. Additional results. With a minor change of the proof of Theorem 1. 1 we have the following

THEOREM 4. 1. *Let K be a Calderón-Zygmund kernel and $\langle Bx, y\rangle$ be a real bilinear form. Define the operator S by*

$$(Sf)(x)=\text{p. v.}\int e^{i\langle B(x-y),\,x-y\rangle}K(x-y)f(y)\,dy.$$

Then for any $1<r<\infty$ there exists a positive constant C_r such that

$$(4.1)\qquad (Sf)^{\#}(x)\leqq C_rM_rf(x),\ x\in\boldsymbol{R}^n,\ f\in C_0^\infty(\boldsymbol{R}^n).$$

SKETCH OF THE PROOF. We may assume B is symmetric and rank $B\geqq 1$. Since S is translation-invariant, we have only to get the estimate (2. 6) for Sf, in this case, too. Since

$$Sf(x)=e^{i\langle Bx,x\rangle}\text{p. v.}\int e^{-2i\langle Bx,y\rangle}K(x-y)e^{i\langle By,y\rangle}f(y)\,dy,$$

S is also bounded on $L^p(\boldsymbol{R}^n)$, $1<p<\infty$. We decompose f as in the proof of Theorem 1.1 and replace $\exp(i\langle Bx,y\rangle)$ by $\exp(i\langle B(x-y),x-y\rangle)$. Then the proof there works also for f_1 and f_2 with c_Q replaced by $c_Q=\int\exp(i\langle By,y\rangle)K(-y)f_2(y)\,dy$. The contribution of F_3^1 is the same as before. As for F_3^2, we have only to work with $-2B$ and $\exp(i\langle By,y\rangle)f_3(y)$ in place of B and $f_3(y)$, respectively. This completes the proof.

REMARK 4.2. If B is symmetric, then Theorem 4.1 implies Theorem 1.1.

REMARK 4.3. In the conclusion of Theorem 1.1, one cannot replace $|Tf|^{\#}(x)$ by $(Tf)^{\#}(x)$. In fact, consider the following one-dimensional case. Let $Tf(x)=\text{p. v.}\int e^{2ixy}(x-y)^{-1}f(y)\,dy$ and set $G_A(x)=T(\chi_{(0,A)}(y)\exp(-iy^2))(x)$, $A>0$. Then, for $A+\frac{1}{A}<x<A+\frac{10}{A}$, we have by easy calculation

$$G_A(x)=e^{ix^2}\int_0^A e^{-i(x-y)^2}\frac{dy}{x-y}=\frac{1}{2}e^{ix^2}\int_{(x-A)^2}^{x^2}e^{-is}\frac{ds}{s}$$
$$=\frac{1}{2}e^{ix^2}\int_{1/A^2}^{1}\frac{ds}{s}+g_A(x),$$

where $g_A(x)$ is a uniformly bounded function as $A\to+\infty$. Now, if $\left(A+\frac{5}{A}+b\right)^2=\left(A+\frac{5}{A}\right)^2+c$, then b is asymptotically equal to $\frac{c}{2A}$, as $A\to+\infty$. Hence, if $\left(A+\frac{5}{A}\right)^2=2k\pi$ (k: a large integer), then we get easily

$$(\text{Re } G_A)^{\#}\left(A+\frac{5}{A}\right)>C\log A,$$

which implies that for every $r>1$ there exists no $C_r>0$ such that

$$(Tf)^{\#}(x)\leqq C_rM_rf(x),\quad x\in\boldsymbol{R},\ f\in C_0^\infty(\boldsymbol{R}).$$

Finally we mention that Theorem 1.1 can be extended to the case of more general singular integrals, that is, we have

THEOREM 4.4. *Let V be an $L^2(\boldsymbol{R}^n)$-bounded linear operator whose distributional kernel $K(x,y)$ satisfies $|K(x,y)|\leqq C|x-y|^{-n}$ and $|\nabla K(x,y)|\leqq$*

$C|x-y|^{-n-1}$. *Let* $\langle Bx, y\rangle$ *be a real bilinear form and consider the oscillatory singular integral operator* T *defined by*

$$Tf(x)=V(e^{i\langle Bx,\cdot\rangle}f(\cdot))(x),\ f\in C_0^\infty(\boldsymbol{R}^n).$$

Then the conclusion of Theorem 1.1 *holds.*

SKETCH OF THE PROOF. It is known that T is bounded on $L^r(\boldsymbol{R}^n)$, $1<r<\infty$, see [8, p. 192]. It is also true that

$$(\tau_{-h}T\tau_h)(f)(x)=e^{i\langle B(x+h),h\rangle}(\tau_{-h}V\tau_h)(e^{i\langle Bx,\cdot\rangle}f(\cdot)e^{i\langle Bh,\cdot\rangle})(x).$$

Since the distributional kernel of $\tau_{-h}V\tau_h$ is $K(x+h, y+h)$, we see from the proof of Theorem 1.1 that

$$(T_hf)_E^\#(0)\leqq C_rM_rf(0),$$

where $T_hf=(\tau_{-h}V\tau_h)(e^{i\langle Bx,\cdot\rangle}f(\cdot))$ and the constant C_r is independent of $h\in\boldsymbol{R}^n$. Hence we obtain easily

$$(Tf)_E^\#(h)\leqq C_rM_r(\tau_{-h}f(\cdot)e^{i\langle Bh,\cdot\rangle})(0)=C_rM_rf(h),\ h\in\boldsymbol{R}^n.$$

This completes the proof.

Note also that a similar result holds in Theorem 4.1 for the above kernels.

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