Pseudopotentials and symmetries of evolution equations

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§ 1. Introduction

In the development of the soliton theory the symmetries of differential equations played an important role. The Lie-Bäcklund transformation was defined and investigated by M. K. Ibragimov and R. L. Anderson [1], P. T. Olver [5], A. M. Vinogradov and I. S. Krasil'shchik [6] and others. It is known that the soliton equations have infinitely many Lie-Bäcklund transformations.

It is also known that some soliton equations (for example, the KdV equation) have a operator L called a recursion operator, which has the following properties: If ϕ is a Lie-Bäcklund transformation, then $L\phi$ is also a Lie-Bäcklund transformation. (See [5].) In general $L\phi$ can not be defined in the usual function space on the infinite-jet space, because L contains terms involving the integrals of the unknown function. Besides this, some observations show that it is useful to extend the notion of the Lie-Bäcklund transformation. (See [6], [7].) For these reasons, A. M. Vinogradov and I. S. Krasil'shchik defined a non-local symmetry of differential equations which is a natural generalization of Lie-Bäcklund transformation. In this paper we consider their symmetry in case of the evolution equations from a slightly different view point.

Let us consider the evolution equation

$$u_t = F(x, t, u, u_1, \dots, u_n) \quad (1.1)$$

where $u = u(x, t)$ and $u_i = \frac{\partial^i u}{\partial x^i}$. Considering u_i as an independent variable, we denote by R the infinite dimensional manifold having coordinates $(x, t, u, u_1, u_2, \dots)$. Let ∂ (resp. ∂_t) be the total derivative with respect to x (resp. t). We have $\partial\partial_t = \partial_t\partial$. Set $A = C^\infty(R)$. (In § 5 we consider only polynomial functions on R by some technical reasons.) The triplet $S = \{A; \partial, \partial_t\}$ is a geometric object corresponding to the differential equation (1.1). Then a Lie-Bäcklund transformation of (1.1) is a function ϕ contained in A satisfying

$$\partial_t \phi = D_\phi F, \quad (1.2)$$

where

$$D_\phi = \sum_{i=0}^{\infty} \partial^i \phi \partial / \partial u_i. \quad (1.3)$$

We can generalize these situation as follows. Introduce new variables $w_1, w_2, \dots, w_m$ and set $A_1 = C^\infty(R \times \{(w_1, w_2, \dots, w_m)\})$. Suppose that the derivations ∂ and ∂_t of A can be extended to derivations of A_1 which commute with each other. w_i 's are nothing but the pseudopotentials in the sense of [8]. Suppose that a function $\phi \in A_1$ satisfies (1.2). This is equivalent to the condition that the derivation D_ϕ from A to A_1 commutes with ∂_t . Then in § 4 we prove that there exists an extended differential algebra A_2 of A_1 such that D_ϕ can be extended to a derivation $A_1 \rightarrow A_2$ commuting with ∂, ∂_t . (See Theorem 1.) Using this theorem repeatedly, we know that there exists an extended differential algebra $S_\infty = \{A_\infty : \partial, \partial_t\}$ of S such that D_ϕ can be extended to a derivation of A_∞ commuting with ∂, ∂_t . This is the non-local symmetry in the sense of [7]. That is to say, Theorem 1 means that we can construct S_∞ and a non-local symmetry from the symmetry ϕ satisfying (1.2).

On the other hand it is well known that the soliton equations have symmetries called Bäcklund transformations. A Bäcklund transformation is defined by partial differential equations and is not obtained by integrating Lie-Bäcklund transformations. In § 3, we consider “the infinitesimal version” of Bäcklund transformations of the KdV, or the modified KdV equations and show that they are symmetries in the above sense. This suggests that the differential equations having many non-local symmetries are “good” equations such as soliton equations.

Finally in § 5 we construct S_∞ in case of the KdV equation. We note that we used the algebraic computation system REDUCE3 for the cumbersome calculations in this section.

§ 2. Lie-Bäcklund transformations

Let us consider the evolution equation (1.1). We use the same notations as in § 1. Therefore R denotes the infinite dimensional manifold having coordinates $(x, t, u, u_1, u_2, \dots)$ and $A = C^\infty(R)$. Let ∂ (resp. ∂_t) be the total derivative with respect to x (resp. t), i. e.,

$$\partial = \partial / \partial x + \sum_{i=0}^{\infty} u_{i+1} \partial / \partial u_i$$

and

$$\partial_t = \partial / \partial t + \sum_{i=0}^{\infty} \partial^i F \partial / \partial u_i.$$

∂ and ∂_t are derivations of A which commute with each other.

A function $\phi \in A$ satisfying (1.2) is called a Lie-Bäcklund transformation (LBT) of (1.1). Note that (1.2) is equivalent to the condition $\partial_t D_\phi = D_\phi \partial_t$.

EXAMPLE. The KdV equation : $u_t = u_3 + 12uu_1$ (2.1)

Let $L = \partial^2 + 8u + 4u_1\partial^{-1}$. It is known that if ϕ is a LBT of (2.1), then $L\phi$ (if defined) is also a LBT. L is called a recursion operator. (See [5].) Obviously u_1 is a LBT, and so we have an infinite series $\phi_k = L^k u_1$ of LBT's. The first two of these are

$$\begin{aligned}\phi_1 &= u_3 + 12uu_1, \\ \phi_2 &= u_5 + 40u_1u_2 + 20uu_3 + 120u^2u_1.\end{aligned}\quad (2.2)$$

The evolution equations $u_t = \phi_k$ are nothing but the higher order KdV equations. Besides the above series there exist two LBT's:

$$\tilde{\phi}_1 = tu_1 + 1/12, \quad (2.3)$$

$$\tilde{\phi}_2 = tu_3 + xu_1/3 + 12tuu_1 + 2u/3. \quad (2.4)$$

(See [6].) Note that $L\tilde{\phi}_2$ is not defined in A .

§ 3. Symmetry

In this section, we consider a generalization of the Lie-Bäcklund transformation. Let $S = \{A; \partial, \partial_t\}$ be as in the previous sections. Let $w_1, w_2, \dots, w_m$ be new independent variables. Set $R_1 = R \times \{(w_1, w_2, \dots, w_m)\}$ and $A_1 = C^\infty(R_1)$. Suppose that the derivations ∂, ∂_t of A can be extended to derivations of A_1 such that $\partial\partial_t = \partial_t\partial$. For brevity, we use the same notations ∂, ∂_t for the corresponding derivations in A_1 . We call the differential algebra $S = \{A; \partial, \partial_t\}$ an extended differential algebra of S . w_i 's are called the pseudopotentials constituting S_1 . As before, for $\phi \in A_1$, D_ϕ denotes the derivation from A to A_1 defined by (1.3).

DEFINITION. ϕ is called a symmetry of (1.1) if and only if the following equation holds:

$$\partial_t \phi = D_\phi F. \quad (3.1)$$

Note that (3.1) is equivalent to the condition $\partial_t D_\phi = D_\phi \partial_t$.

EXAMPLE 1. The modified KdV equation:

$$u_t = u_3 + 6u^2u_1. \quad (3.2)$$

Define a potential w by

$$\begin{cases} w_x = u \\ w_t = u_2 + 2u^3. \end{cases}$$

Let $\tilde{w}$ be the solution of the following differential equation :

$$\begin{cases} (\tilde{w} - w)_x = k \sin(\tilde{w} + w) \end{cases} \quad (3.3 \text{ a})$$

$$\begin{cases} (\tilde{w} - w)_t = k\{(\tilde{w}_{xx} + w_{xx})\cos(\tilde{w} + w) \\ + (\tilde{w}_x^2 + w_x^2)\sin(\tilde{w} + w)\}, \end{cases} \quad (3.3 \text{ b})$$

where k is an arbitrary constant. Then, if u is a solution of (3.2), $\tilde{u} = \tilde{w}_x$ is also a solution of (3.2). The transformation $u \longrightarrow \tilde{u}$ is called a Bäcklund transformation of (3.2). Note that $\tilde{w}$ depends on k and the transformation $u \longrightarrow \tilde{u}$ is a 1-parameter family of transformations. Although it is not a transformation in the strict sense, we can consider its infinitesimal version as follows. Assuming that $\tilde{w} = w$ when $k=0$, we set $W = \frac{\partial \tilde{w}}{\partial k} \Big|_{k=0}$. Then differentiating (3.3) with respect to k and setting $k=0$, we get

$$\begin{cases} W_x = \sin 2w, \end{cases} \quad (3.4 \text{ a})$$

$$\begin{cases} W_t = 2w_{xx} \cos 2w + 2w_x^2 \sin 2w. \end{cases} \quad (3.4 \text{ b})$$

By the definition of W we can say that $\phi = \sin 2w$ is “the infinitesimal transformation” of (3.3). The significance of (3.4 b) will be clear in the next section. The infinitesimal transformation ϕ can be interpreted as follows. Let $R_1 = R \times \{w\}$ and $A_1 = C^\infty(R_1)$. Extend the derivations ∂ and ∂_t of A to the derivations of A_1 by

$$\begin{cases} \partial w = u \end{cases} \quad (3.5 \text{ a})$$

$$\begin{cases} \partial_t w = u_2 + 2u^3. \end{cases} \quad (3.5 \text{ b})$$

It is easy to see that $\partial \partial_t = \partial_t \partial$ holds in A_1 . Now define the derivation $D_\phi : A \longrightarrow A_1$ by (1.3). Then it is clear that $D_\phi \partial = \partial D_\phi$, and moreover we can show by direct calculations that $D_\phi \partial_t = \partial_t D_\phi$. That is to say, ϕ is a symmetry in the sense of the above definition.

EXAMPLE 2. The KdV equation

In this case the Bäcklund transformation can be stated as follows. (See [8].) Define a pseudopotential y by

$$\begin{cases} y_x = 2u + y^2 - k \end{cases} \quad (3.6 \text{ a})$$

$$\begin{cases} y_t = 4(u+k)(2u+y^2-k) + 2u_2 + 4u_1y. \end{cases} \quad (3.6 \text{ b})$$

Then, if u is a solution of (2.1), $\tilde{u} = -u - y^2 + k$ is also a solution of (2.1). Although the following arguments are quite formal and heuristic, we can consider the infinitesimal version as follows. First of all, to declare the

dependence of y (resp. $\tilde{u}$) on k let us write $y(k)$ (resp. $\tilde{u}(k)$) instead of y (resp. $\tilde{u}$). We denote by $T(k)$ the transformation $u \longrightarrow \tilde{u}(k)$. Note that $T(0)$ is not the identity transformation, so that $\left. \frac{\partial T(k)}{\partial k} \right|_{k=0}$ is not an infinitesimal transformation. Therefore, fixing k_0 , we set $T(k_0)^{-1}(\tilde{u}(k)) = U(k)$. We have

$$\tilde{u}(k) = -U(k) - Y(k)^2 + k_0,$$

where $Y(k)$ is the pseudopotential defined by

$$\begin{cases} Y(k)_x = 2U(k) + Y(k)^2 - k_0 \\ Y(k)_t = 4(U(k) + k_0)(2U(k) + Y(k)^2 - k_0) \\ \quad + 2U(k)_{xx} + 4U(k)_x Y(k). \end{cases} \quad (3.7 \text{ a})$$

$$(3.7 \text{ b})$$

Hence we have

$$\begin{aligned} U(k) &= -\tilde{u}(k) - Y(k)^2 + k_0 \\ &= u + y(k)^2 - k - Y(k)^2 + k_0. \end{aligned} \quad (3.8)$$

Set $\phi = \left. \frac{\partial U(k)}{\partial k} \right|_{k=k_0}$, $w = y(k_0)$, $w' = \left. \frac{\partial y(k)}{\partial k} \right|_{k=k_0}$ and $v = \left. \frac{\partial Y(k)}{\partial k} \right|_{k=k_0}$.

Differentiating (3.8) with respect to k and then setting $k = k_0$, we get

$$\phi = 2ww' - 1 - 2wv, \quad (3.9)$$

where we assumed that $U(k_0) = u$ and $Y(k_0) = y(k_0) = w$. Since k_0 is arbitrary, let us write k instead of k_0 . Note that $w' = \frac{\partial w}{\partial k}$ and v are the pseudopotentials defined by

$$\begin{cases} w'_x = 2ww' - 1 \\ w'_t = 4(2u + w^2 - k) + 4(u + k)(2ww' - 1) + 4u_1 w' \end{cases} \quad (3.10 \text{ a})$$

$$(3.10 \text{ b})$$

and

$$\begin{cases} v_x = 2(2ww' - 1 - wv) \\ v_t = 4(-u_1 v + 2u_1 w' - 2uvw + 4uw'w \\ \quad - 2kvw + 4kw'w + w^2 - 3k). \end{cases} \quad (3.11 \text{ a})$$

$$(3.11 \text{ b})$$

As in the case of the modified KdV equation, let us extend the differential algebra $S = \{A; \partial, \partial_t\}$ as follows. Set $R_1 = R \times \{(w, w', v)\}$ and $A_1 = C^\infty(R_1)$. Extend the derivations ∂, ∂_t to the derivations of A_1 by using the right hand sides of the equations (3.6), (3.10) and (3.11). For example, $\partial w' = 2ww' - 1$ and $\partial v = 2(2ww' - 1 - wv)$. Then it is easy to see that $\partial \partial_t = \partial_t \partial$. Moreover we can prove by direct calculations that the derivation $D_\psi: A \longrightarrow A_1$ defined by (1.3) commutes with ∂_t .

§ 4. Extension of derivation

Let R and S be as in the previous sections. Let $S_1 = \{A_1; \partial, \partial_t\}$ be an extended differential algebra of S and $\phi \in A_1$ be a symmetry. Let $w_1, w_2, \dots, w_m$ be the pseudopotentials constituting S_1 . Let $z_1, z_2, \dots, z_l$ be new independent variables. Set $R_2 = R_1 \times \{(z_1, z_2, \dots, z_l)\}$ and $A_2 = C^\infty(R_2)$. Suppose that the derivations ∂, ∂_t of A_1 can be extended to commutative derivations of A_2 . Then we say that $S_2 = \{A_2; \partial, \partial_t\}$ is an extended differential algebra of S_1 . We can prove

THEOREM 1. *There exists an extended differential algebra $S_2 = \{A_2; \partial, \partial_t\}$ of S_1 such that D_ϕ can be extended to a derivation from A_1 to A_2 commuting with ∂, ∂_t .*

PROOF. Set $\partial w_\alpha = f_\alpha$ and $\partial_t w_\alpha = g_\alpha$, where $f_\alpha, g_\alpha \in A_1$. We have $\partial_t f_\alpha = \partial g_\alpha$. Suppose that there exists S_2 . Set $D_\phi w_\alpha = z_\alpha$. Then the commutativity of D_ϕ with ∂ and ∂_t implies

$$\begin{aligned} \partial z_\alpha &= D_\phi f_\alpha \\ &= \sum \frac{\partial f_\alpha}{\partial u_i} \partial^i \phi + \sum \frac{\partial f_\alpha}{\partial w_\beta} z_\beta, \end{aligned} \quad (4.1 a)$$

$$\begin{aligned} \partial_t z_\alpha &= D_\phi g_\alpha \\ &= \sum \frac{\partial g_\alpha}{\partial u_i} \partial^i \phi + \sum \frac{\partial g_\alpha}{\partial w_\beta} z_\beta, \end{aligned} \quad (4.1 b)$$

Let us denote by X_α (resp. Y_α) the right hand side of (4.1 a) (resp. (4.1 b)). We now prove that (4.1) is integrable. First note that

$$\partial_t = \sum \partial^i F \partial / \partial u_i + \sum g_\alpha \partial / \partial w_\alpha$$

in A_1 and hence we have

$$\begin{aligned} [\partial_t, \partial / \partial u_i] &= - \sum \frac{\partial}{\partial u_i} (\partial^j F) \partial / \partial u_j - \sum \frac{\partial g_\alpha}{\partial u_i} \partial / \partial w_\alpha, \\ [\partial_t, \partial / \partial w_\alpha] &= - \sum \frac{\partial g_\beta}{\partial w_\alpha} \partial / \partial w_\beta. \end{aligned}$$

Using the above equations, we have

$$\begin{aligned} \partial_t X_\alpha &= \left\{ \left(\partial_t \frac{\partial f_\alpha}{\partial u_i} \right) \partial^i \phi + \frac{\partial f_\alpha}{\partial u_i} \partial_t \partial^i \phi \right\} \\ &\quad + \left(\partial_t \frac{\partial f_\alpha}{\partial w_\beta} \right) z_\beta + \frac{\partial f_\alpha}{\partial w_\beta} \left(\frac{\partial g_\beta}{\partial u_i} \partial^i \phi + \frac{\partial g_\beta}{\partial w_\gamma} z_\gamma \right) \\ &= \left\{ \frac{\partial}{\partial u_i} (\partial_t f_\alpha) - \frac{\partial}{\partial u_i} (\partial^j F) \frac{\partial f_\alpha}{\partial u_j} - \frac{\partial g_\beta}{\partial u_i} \frac{\partial f_\alpha}{\partial w_\beta} \right\} \partial^i \phi \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial f_\alpha}{\partial u_i} \partial_t \partial^i \phi + \left\{ \frac{\partial}{\partial w_\beta} (\partial_t f_\alpha) - \frac{\partial g_\gamma}{\partial w_\beta} \frac{\partial f_\alpha}{\partial w_\gamma} \right\} z_\beta \\
& + \frac{\partial f_\alpha}{\partial w_\beta} \left(\frac{\partial g_\beta}{\partial u_i} \partial^i \phi + \frac{\partial g_\beta}{\partial w_\gamma} z_\gamma \right) \\
& = \left\{ \frac{\partial}{\partial u_i} (\partial_t f_\alpha) - \frac{\partial}{\partial u_i} (\partial^j F) \frac{\partial f_\alpha}{\partial u_j} \right\} \partial^i \phi \\
& + \frac{\partial f_\alpha}{\partial u_i} \partial_t \partial^i \phi + \frac{\partial}{\partial w_\beta} (\partial_t f_\alpha) z_\beta,
\end{aligned} \tag{4.2}$$

where, using the Einstein rule, we omitted the summation sign. Similarly we have

$$\partial Y_\alpha = \frac{\partial}{\partial u_i} (\partial g_\alpha) \partial^i \phi + \frac{\partial}{\partial w_\beta} (\partial g_\alpha) z_\beta, \tag{4.3}$$

where we used the equalities :

$$\begin{aligned}
[\partial, \partial / \partial u_i] &= -\partial / \partial u_{i-1} - \frac{\partial f_\alpha}{\partial u_i} \partial / \partial w_\alpha, \\
[\partial, \partial / \partial w_\alpha] &= -\frac{\partial f_\beta}{\partial w_\alpha} \partial / \partial w_\beta.
\end{aligned}$$

By (4.2), (4.3) and $\partial_t f_\alpha = \partial g_\alpha$ we obtain

$$\begin{aligned}
\partial_t X_\alpha - \partial Y_\alpha &= \frac{\partial f_\alpha}{\partial u_i} \left\{ \partial^i \partial_t \phi - \frac{\partial}{\partial u_j} (\partial^j F) \partial^i \phi \right\} \\
&= \frac{\partial f_\alpha}{\partial u_i} (\partial^i \partial_t \phi - D_\phi \partial^i F) \\
&= \frac{\partial f_\alpha}{\partial u_i} \partial^i (\partial_t \phi - D_\phi F) \\
&= 0.
\end{aligned}$$

Therefore, if the system (4.1) for z_α has no solution in A_1 , we can define new pseudopotentials z_α by (4.1). This completes the proof. q. e. d.

Using the above theorem repeatedly, we know the following: There exists an extended differential algebra $S_\infty = \{A_\infty; \partial, \partial_t\}$ of S such that the derivation D_ϕ can be extended to a derivation of A_∞ commuting with ∂, ∂_t . (The pseudopotentials constituting S_∞ may be infinite.) This is nothing but the non-local symmetry in the sense of [7].

REMARK 1. Note that in general z_α is not unique when it can be integrated in A_1 .

REMARK 2. Let us consider the symmetry of the modified KdV equation. We use the same notations as in § 3, so that A_1 denotes $C^\infty(R \times$

$\{w\}$), where w is defined by (3.5). Set $D_\phi w = W$, where $\phi = \sin 2w$. Then it is shown that (3.4) is nothing but the equation (4.1) which must be satisfied by W .

§ 5. Symmetry of the KdV equation

In this section we consider only polynomial functions by some technical reasons. Therefore A denotes the space $\mathbf{R}[x, t, u, u_1, \dots]$ of polynomial functions on $R = \{(x, t, u, u_1, \dots)\}$.

Let us consider the symmetry ϕ defined by (3.9) of the KdV equation. Since $w = w(k)$ depends on k , set

$$w_i = \frac{1}{i!} \left. \frac{\partial^i w}{\partial k^i} \right|_{k=0}.$$

Therefore $w(k) = \sum_{i=0}^{\infty} w_i k^i$ holds formally. Similarly set $v = \sum_{i=0}^{\infty} v_i k^i$. w_i and v_i are pseudopotentials. The actions of ∂ and ∂_t to w_i (resp. v_i) are determined by the formal expansions of (3.6) (resp. (3.11)) with respect to k . For example we have

$$\partial w_i = \sum_{r=0}^i w_r w_{i-r} \quad (5.1)$$

for $i \geq 2$ and

$$\partial v_i = 4 \sum_{r=0}^i (r+1) w_{i-r} w_{r+1} - 2 \sum_{r=0}^i w_r v_{i-r} \quad (5.2)$$

for $i \geq 1$. We set $R_\infty = R \times \{(w_0, w_1, \dots; v_0, v_1, \dots)\}$ and denote by A_∞ the space of polynomial functions on R_∞ . Then $S_\infty = \{A_\infty; \partial, \partial_t\}$ is an extended differential algebra of S .

REMARK. For $f \in A_\infty$ let us define the degree of f in such a way that

$$\begin{aligned} \deg x &= 1, \deg t = 3, \deg u_i = i + 2 \\ \deg w_i &= 1 - 2i \text{ and } \deg v_i = -1 - 2i. \end{aligned}$$

If we define $\deg k = 2$, then $w(k)$ (resp. $v(k)$) is homogeneous of degree 1 (resp. -1). From this it is shown that if $\deg f = r$ for $f \in A_\infty$, then $\deg(\partial f) = r + 1$ and $\deg(\partial_t f) = r + 3$.

First we prove

LEMMA. For $f \in A_\infty$, suppose

$$\partial f = c w_0 f,$$

where c is a constant. Then f depends only on t . In particular if $c \neq 0$, then

$f=0$.

PROOF. Since $\partial u_i = u_{i+1}$, it is easy to see that f is independent of u_i 's. Similarly f does not depend on w_0 . Next suppose that f is a polynomial of $x, t, w_1, \dots, w_n$ and set

$$f = \sum_{r=0}^N a_r w_n^r,$$

where a_r is a polynomial independent of w_n . We will prove the lemma for the above f by induction on n . Assume that

$$\partial f = c w_0 f. \quad (5.3)$$

Since $\partial w_n = 2w_0 w_n + (\text{the terms independent of } w_n)$ from (5.1), we obtain, by comparing the coefficients of w_n^N in (5.3), the following equation ;

$$\partial a_N + 2N a_N w_0 = c w_0 a_N.$$

Therefore the induction assumption implies $a_N = 0$ if $c \neq 2N$. When $c = 2N$, a_N is a polynomial of t . In this case, let us consider the coefficient of w_n^{N-1} in (5.3). Using $c = 2N$, we have

$$\partial a_{N-1} - 2a_{N-1} w_0 + N a_N (\partial w_n - 2w_0 w_n) = 0. \quad (5.4)$$

Set $a_{N-1} = \sum b_\alpha w^\alpha$, where b_α is independent of w_i and $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, $w^\alpha = w_1^{\alpha_1} \dots w_{n-1}^{\alpha_{n-1}}$. Considering the terms containing w_0 in (5.4), we obtain

$$\sum 2|\alpha| b_\alpha w^\alpha - 2 \sum b_\alpha w^\alpha = 0,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_{n-1}$. This implies $|\alpha| = 1$, that is, a_{N-1} is linear with respect to w_i . Then it is easy to see that (5.4) implies $a_N = a_{N-1} = 0$. Therefore we know that the lemma holds for f independent of v_i 's.

Finally let us consider the general case. Let $f = f(x, t; w_1, w_2, \dots; v_0, \dots, v_n)$. Then similar induction arguments on n implies that the lemma holds for general f . q. e. d.

In particular we have

PROPOSITION. $\ker \partial \cap \ker \partial_t = \mathbf{R}$.

Let us return to the symmetry $\psi = \psi(k)$. Set $\psi(k) = \sum_{i=0}^{\infty} \psi_i k^i$. Each ψ_i is a symmetry. We want to extend the derivation D_{ψ_i} to a derivation $A_\infty \longrightarrow A_\infty$. First of all we will show the uniqueness of the extension. Set $D_{\psi_i} w_0 = \alpha_0$. Then α_0 satisfies

$$\partial \alpha_0 = D_{\psi_i} \partial w_0$$

$$\begin{aligned}
&= D_{\phi_i}(2u + w_0^2) \\
&= 2\phi_i + 2w_0\alpha_0.
\end{aligned}$$

Hence if α'_0 satisfies the same equation, we have

$$\partial(\alpha_0 - \alpha'_0) = 2w_0(\alpha_0 - \alpha'_0).$$

Then the lemma implies $\alpha_0 = \alpha'_0$. Suppose that the extension $D_{\phi_i}: \mathbf{R}[x, t; u, u_1, \dots; w_0, \dots, w_j] \longrightarrow A_\infty$ is unique. Set $D_{\phi_i}w_r = \alpha_r$ for $0 \leq r \leq j+1$. We have, from (5.1),

$$\partial\alpha_{j+1} = 2w_0\alpha_{j+1} + \beta,$$

where β is a polynomial of w_r 's and $\alpha_0, \dots, \alpha_j$. Therefore we again apply the lemma to prove that α_{j+1} is unique. By similar arguments we can show that the image $D_{\phi_i}v_r$ of v_r is also unique.

Next suppose that the 1-parameter family of derivations $D_{\psi(k)}$ can be extended to a derivation $A_\infty \longrightarrow A_\infty$ and set

$$D_{\psi(k)} = \sum k^i D_i.$$

Since $[\partial, D_i] = [\partial_t, D_i] = 0$ and

$$D_{\psi(k)}u = \psi(k) = \sum \phi_i k^i,$$

we have $D_i = D_{\phi_i}$.

Under these preparations we set

$$\begin{aligned}
\xi &= (p - q)\{v(p) - w'(p)\}\tilde{w}^2 + \tilde{w} + v(p) - w'(p), \\
\eta &= 2\frac{\partial\tilde{w}}{\partial q} - \frac{\tilde{w} - w'(p)}{p - q} + \tilde{w}(w'(p) - v(p))(4w'(q) - 2v(q) - \tilde{w}) \\
&\quad - \frac{v(p) - v(q)}{p - q},
\end{aligned}$$

where p, q are parameters and

$$\tilde{w} = \frac{w(p) - w(q)}{p - q}.$$

Note that ξ and η have no singularity at $p = q$. Hence we can set $\xi = \sum \xi_{ij} p^i q^j$ and $\eta = \sum \eta_{ij} p^i q^j$. Let us define the derivation D_{ϕ_i} in A_∞ by $D_{\psi(p)}w(q) = \xi$ and $D_{\psi(p)}v(q) = \eta$. In other word, we set $D_{\phi_i}w_j = \xi_{ij}$ and $D_{\phi_i}v_j = \eta_{ij}$. (Note that $D_{\psi(p)} = \sum p^i D_{\phi_i}$.) Then we can prove by direct calculations that ξ satisfies the following equations;

$$\begin{cases} \partial\xi = D_{\psi(p)}\partial w(q), & (5.5 \text{ a}) \\ \partial_t\xi = D_{\psi(p)}\partial_t w(q), & (5.5 \text{ b}) \end{cases}$$

η also satisfies similar equations to (5.5). These means that D_{ϕ_i} commutes with ∂ and ∂_t in A_∞ .

For symmetries ϕ, ϕ' let us define the Poisson bracket $\{\phi, \phi'\}$ by

$$\begin{aligned}\{\phi, \phi'\} &= [D_\phi, D_{\phi'}](u) \\ &= D_\phi \phi' - D_{\phi'} \phi.\end{aligned}$$

Then we can prove by direct calculations that $\{\phi(p), \phi(q)\} = 0$.

To resume, we have

THEOREM 2. *D_{ϕ_i} can be uniquely extended to a derivation from A_∞ to A_∞ . Moreover $\{\phi_i, \phi_j\} = 0$ holds for any i, j .*

Finally we give a remark on the KdV hierarchy. Let $\phi_k = L^k u_1$, where L is the recursion operator. The totality of the evolution equations $u_t = \phi_k$ is called the KdV hierarchy. In particular consider the 5-th order KdV equation

$$u_t = \phi_2. \quad (5.6)$$

Define a pseudopotential y by

$$\begin{cases} y_x = 2u + y^2 - k \\ y_t = 4(u_2 + 4ku + 4k^2 + 6u^2)y^2 + 4(u_3 + 12uu_1 + 4ku_1)y \\ \quad + 2(u_4 + 16uu_2 + 2ku_2 + 12u_1^2 + 24u^3 \\ \quad + 4ku^2 + 8k^2u - 8k^3). \end{cases} \quad (5.7a)$$

$$(5.7b)$$

It is shown that the transformation $u \longrightarrow \tilde{u} = -u - y^2 + k$ is also a Bäcklund transformation of (5.6). Note that the right hand of (5.7a) is identical with the right hand of (3.6a). This suggest that the transformation $u \longrightarrow \tilde{u}$ represents a symmetry of the KdV hierarchy.

It is well known that the symmetry of the KdV hierarchy is a certain Kac-Moody Lie algebra (for example, see [3]). The relation between the Kac-Moody Lie algebra and the symmetry in this paper seems to be an interesting problem.

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