Hokkaido Mathematical Journal Vol. 17 (1988), p. 355～371

Real hypersurfaces with cyclic-parallel Ricci tensor of a complex projective space

Jung-Hwan KWON* and Hisao NAKAGAWA
(Received August 10, 1987, Revised May 19, 1988)

Introduction.

The study of real hypersurfaces of a complex projective space P_nC was initiated by Takagi [11], who proved that all homogeneous hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. He showed also in [12, 13] that if a real hypersurface M of P_nC has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 and B. This result is recently generalized by Kimura [4], who proves that a real hypersurface M of P_nC has constant principal curvatures and $J\xi$ is principal if and only if M is locally congruent to one of the homogeneous hypersurfaces, where ξ denotes the unit normal and J is the complex structure of P_nC. In particular, real hypersurfaces of type A_1, A_2 and B of P_nC have been studied by several authors (cf. Cecil and Ryan [2], Kimura [5], Maeda [6] and Okumura [10]).

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been investigated from different points of view and there are some studies by Chen, Ludden and Montiel [3] and Montiel and Romero [9]. In particular, real hypersurfaces of H_nC, which are said *of type A*, similar to those of type A_1 and A_2 of P_nC were treated by Montiel and Romero [9].

Now, the Ricci tensor S is said to be *cyclic-parallel* if it satisfies

$$\mathfrak{S}\nabla S(X, Y, Z)=0$$

for any vector fields X, Y and Z, where $\mathfrak{S}$ and ∇ denote the cyclic sum and the Riemannian connection, respectively. It is noticed in § 4 that the Ricci tensors of real hypersurfaces of type A_1 or A_2 (resp. A) of P_nC (resp. H_nC) are cyclic-parallel. The purpose of this paper is to investigate this converse problem. Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$, whose Ricci tensor is cyclic-parallel. In § 3, it is verified that if $J\xi$ is principal, then all principal curvatures of M are constant and the number of distinct principal curvatures is at most 5. By means of this result and the classfication theorem due to Takagi [12] and Kimura [4], we can prove

THEOREM. *Let M be a real hypersurface of P_nC, whose Ricci tensor is cyclic-parallel. If $J\xi$ is principal, then M is locally congruent to one of homogeneous hypersurfaces of P_nC.*

In the last section, real hypersurfaces of P_nC whose Ricci tensors are cyclic-parallel are partially classified in the case where $J\xi$ is principal.

The authors would like to express their thanks to the referee for his valuable suggestions.

1. Preliminaries.

First of all, we recall a semi-Sasakian structure of a Riemannian manifold or a Lorentz manifold. Let $\bar{N}$ be a $(2n+1)$-dimensional semi-Riemannian manifold of index 0 or 1 with a semi-Riemannian metric tensor G. Let ϕ, $\bar{E}$ and $\bar{\omega}$ be a tensor field of type (1, 1), a vector field and a 1-form on $\bar{N}$, respectively, satisying the following properties:

$$\begin{cases} \bar{\omega}(U)=\varepsilon G(U, \bar{E}), \ \bar{\omega}(\bar{E})=1, \ G(\bar{E}, \bar{E})=\varepsilon, \\ \phi\bar{E}=0, \ \bar{\omega}\circ\phi=0, \ \phi^2=-1+\bar{\omega}\otimes\bar{E}, \\ G(\phi U, \phi V)=G(U, V)-\varepsilon\bar{\omega}(U)\bar{\omega}(V), \end{cases} \tag{1.1}$$

for any vetor fields U and V on $\bar{N}$, where I denotes the identity mapping and $\varepsilon=1$ or -1 according as $\bar{N}$ is Riemannian or Lorentz. In spite of the respective cases, the set $(\phi, \bar{E}, \bar{\omega}, G)$ is called an *almost contact metric structure* and $\bar{N}$ is called an *almost contact metric manifold.* If the almost contact metric atructure $(\phi, \bar{E}, \bar{\omega}, G)$ satisfies

$$\bar{D}_U\phi(V)=-G(U, V)\bar{E}+\varepsilon\bar{\omega}(U)V, \tag{1.2}$$

where $\bar{D}$ denotes the Levi-Civita connection of N, then it is called a *semi-Sasakian structure,* and $\bar{N}$ is called a *semi-Sasakian manifold.* As is easily seen, (1.1) and (1.2) imply

$$\bar{D}_U\bar{E}=\varepsilon\phi U, \ d\bar{\omega}(U, V)=G(\phi U, V), \ \bar{D}_U\phi(V)=\varepsilon\bar{R}'(U, E)V, \tag{1.3}$$

where $\bar{R}'$ denotes the Riemannian curvature tensor of $\bar{N}$, and hence $\bar{E}$ is the Killing vector field.

For a semi-Sasakian manifold $\bar{N}$ a plane section in the tangent space N_X at any point x of $\bar{N}$ is called a *ϕ-section* if it is spanned by a unit vector u orthogonal to $\bar{E}_X$ and ϕu. This section is non-degenerate in the case of the Lorentz manifold, because $\bar{E}$ is the time-like vector field. The sectional curvature of the ϕ-section is called a *ϕ-sectional curvature* and $\bar{N}$ is called a *semi-Sasakian space form if it has constant ϕ-sectional curvature.* Let $\bar{N}$ be a $(2n+1)$-dimensional semi-Sasakian space form of ϕ-sectional curvature c, which is denoted by $N_a^{2n+1}(c)$, where $a=0$ or 1 according as it is Riemannian or Lorentz. The Riemannian curvature tensor $\bar{R}'$ of $N_a^{2n+1}(c)$ is given by

$$\begin{aligned}\bar{R}'(U, V)W=[&(c+3\varepsilon)\{G(V, W)U-G(U, W)V\}\\&+(\varepsilon c-1)\omega(W)\{\omega(U)V-\omega(V)U\}\\&+(c-\varepsilon)\{(G(U, W)\omega(V)\\&-G(V, W)\omega(U)\}E+G(\phi V, W)\phi U\\&-G(\phi U, W)\phi V-2G(\phi U, V)\phi W]/4.\end{aligned} \tag{1.4}$$

In particular, if $c=\varepsilon$, then $N_a^{2n+1}(c)$ is of constant curvature c. For details, see cf. Takahashi [14] and Yano and Kon [15].

Let $\bar{N}$ be a semi-Sasakian manifold with a structure $(\phi, \bar{E}, \bar{\omega}, G)$ and let N be a $2n$-dimensional semi-Riemannian hypersurface of $\bar{N}$ tangent to $\bar{E}$. By the same symbol G the induced semi-Riemannian metric of N is denoted. Each tangent space N_X at a point x of N is by definition a non-degenerate subspace of $\bar{N}_X$. Hence a property of a vector space furnished with a scalar product gives the direct sum docomposition $\bar{N}_X=N_X\oplus N_X^{\perp}$ and the normal space $N_X^{\perp}$ is non-degenerate. An endomorphism P' of the tangent bundle $T(N)$ and a 1-form F' with values is the normal bundle $N(N)$ are defined by

$$\phi X'=P'X'+F'X'.$$

Then P' is skew-symmetric, because ϕ is skew-symmetric, and the following relationships are given:

$$\begin{cases}G(F'X', \xi')+G(X', \phi\xi')=0,\\P'^2+\phi F'=-I+\omega'\otimes E', \quad F'P'=0,\\P'E'=F'E'=0\end{cases} \tag{1.5}$$

for any tangent vector X' and the unit normal ξ', where E' and ω' are the restriction of $\bar{E}$ and $\bar{\omega}$ to N, respectively. Let D be the Levi-Civita connection of N and let σ' and A' be the second fundamental form of N and the shape operator in the direction of the unit normal, respectively. The first equations of (1.3) and (1.5) and the Gauss equation give

$$\begin{cases}D_{X'}E'=\varepsilon P'X', \quad F'X'=\varepsilon\sigma'(X', E'), \quad A'E'=-\varepsilon\phi\xi',\\\sigma'(E', E')=0.\end{cases} \tag{1.6}$$

By means of the formulas of Gauss and Weingarten, we have

$$\begin{cases}D_{X'}P'(Y')=G(F'Y', \xi')A'X'+\phi\sigma'(X', Y')-G(X', Y')E'\\\qquad\qquad\qquad+\varepsilon\omega'(Y')X',\\D_{X'}P'(Y')=-\sigma'(X', P'Y').\end{cases} \tag{1.7}$$

Let R' and S' be the Riemannian curvature tensor and the Ricci tensor of N respectively. The Ricci tensor S' is given by

$$S'(X', Y')=\sum\varepsilon_j G(R'(E'_j, X')Y', E'_j)$$

relative to an orthonormal frame $\{E'_j\}$ such that $G(E'_i, E'_j)=\varepsilon_i\delta_{ij}$. In particular, if N is a semi-Sasakian space form of ϕ-sectional curvature c, then the Gauss equation of N is given by

$$\begin{aligned}
&R'(X', Y')Z'\\
&=[(c+3\varepsilon)\{G(Y', Z')X'-G(X', Z')Y'\}\\
&+(\varepsilon c-1)\omega'(Z')\{\omega'(X')Y'-\omega'(Y')X'\}\\
&+(c-\varepsilon)\{G(X', Z')\omega'(Y')-G(Y', Z')\omega'(X')\}E'\\
&+G(P'Y', Z')P'X'-G(P'X', Z')P'Y'-2G(P'X', Y')P'Z']/4\\
&+G(\sigma'(Y', Z'), \xi')A'X'-G(\sigma'(X', Z'), \xi')A'Y',
\end{aligned}$$

where $E_{2n}=E'$ and hence S' is given by

$$\begin{aligned}
(1.8)\qquad S'(X', Y')=&[(2n-1)(c+3\varepsilon)G(X', Y')\\
&-2(n-1)(\varepsilon c-1)\omega'(X')\omega'(Y')\\
&+(c-\varepsilon)\{3G(P'X', P'Y')-G(X', Y')\}]/4\\
&+\textstyle\sum_{j=1}^{2n-1}\{G(\sigma'(X', Y'), \sigma'(E'_j, E'_j))\\
&-G(\sigma'(X', E'_j), \sigma'(Y', E'_j))\}-\varepsilon G(F'X', F'Y').
\end{aligned}$$

Now, let $\bar{M}$ be a 2n-dimensional Kaehler manifold with an almost complex structure J and a Kaehler metric tensor g. Let M be a real hypersurface of $\bar{M}$ whose induced metric from that of $\bar{M}$ is denoted by the same symbol g. By the similar definition to that of the set of (P', F'), an endomorphism P of $T(M)$ and a 1-form F of $T(M)$ with values in $N(M)$ are defined by

$$JX=PX+FX.$$

Then P is skew-symmetric and moreover the following relationships between these operators are given:

$$\begin{aligned}
(1.9)\qquad &g(FX, \xi)+g(X, J\xi)=0,\\
&P^2=-I-JF,\quad FP=0.
\end{aligned}$$

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. A complex space form of constant holomorphic curvature $4c$ and of complex dimension n is denoted by $M_n(c)$. For the unit normal ξ to M in $\bar{M}$, the tangent vector $J\xi$ is denoted by $-E$. Then E is the unit vector field on M and a 1-form ω is defined by $F(X)=\omega(X)\xi$. As is well known, M admits an almost contact metric structure (P, E, ω, g). Let

σ and A be a second fundamental form of M and a shape operator derived from ξ, respectively. The covariant derivative ∇P is defined by $\nabla_X P(Y) = \nabla_X(PY) - P\nabla_X Y$. Then it follows from the Gauss and the Weingarten formulas that it satisfies

$$(1.10)\quad \begin{cases} \nabla_X P(Y) = -g(AX, Y) + \omega(Y)AX, \\ \nabla_X E = PAX, \end{cases}$$

where ∇ denotes the Riemannian connection of M. By the Gauss equation, the Ricci tensor S of M is given by

$$(1.11)\quad \begin{aligned} S(X, Y) = c\{(2n+1)g(X, Y) - 3\omega(X)\omega(Y)\} \\ + hg(AX, Y) - g(AX, AY), \end{aligned}$$

where h denotes the trace of A, and by the Codazzi equation we have

$$(1.12)\quad \nabla_X A(Y) - \nabla_X A(X) = c\{\omega(X)PY - \omega(Y)PX + 2g(X, PY)E\}.$$

Frow now on, assume that the structure vector E is principal, that is, E is an eigenvector of A associated with an eigenvalue α. The equation (1. 10) implies that

$$(1.13)\quad \nabla_X A(E) = d\alpha(X)E + \alpha PAX - APAX,$$

from which is follows that

$$(1.14)\quad \begin{cases} 2APA = \alpha(AP + PA) + 2cP, \\ \beta(AP + PA) = 0, \ d\alpha = \beta\omega, \end{cases}$$

where $\beta = d\alpha(E)$. It implies that the principal curvature α is constant provided that $c > 0$. Suppose that $c < 0$. Consequently (1. 12), (1. 13) and (1. 14) give rise to

$$(1.15)\quad \begin{cases} \nabla_X A(E) = \alpha(PA - AP)X/2 - cPX + \beta\omega(X)E, \\ \nabla_E A(Y) = \alpha(PA - AP)Y/2 + \beta\omega(Y)E. \end{cases}$$

By combining these equations, the following relationship

$$(1.16)\quad dh(E) = \beta$$

is obtained. In fact, since the function h is the trace of the shape operator A, we have

$$\begin{aligned} dh(E) &= \sum\{g(\nabla_E A(E_j), E_j) + 2g(AE_j, \nabla_E E_j)\} \\ &= \sum\{\alpha g((PA - AP)E_j, E_j)/2 + \beta\omega(E_j)^2 + 2g(AE_j, \nabla_E E_j)\}, \end{aligned}$$

which is independent of the choice of the orthonormal frame $\{E_j\}$. Accordingly, without loss of generality, each E_j may be chosen as a principal vector.

For details stated in this section, see cf. Yano and Kon [15].

2. Hypersurfaces.

Let $\bar{N}$ be a $(2n+1)$-dimensional semi-Sasakian manifold equipped with the structure $(\phi, \bar{E}, \bar{\omega}, G)$. Assume that there is a fibration $\bar{\pi}: \bar{N}\to\bar{M}$, where $\bar{M}$ denotes the set of orbits of $\bar{E}$ and a real $2n$-dimensional Kaehler manifold. $\bar{N}$ is a principal circle bundle over $\bar{M}$ and $\bar{\omega}$ is a connection in this bundle, and we have the orthogonal decomposition $T_q(\bar{N})=T_{\bar{\pi}(q)}(\bar{M})+\mathrm{span}\{\bar{E}\}$. Let $*$ be the horizontal lift with respect to the connection $\bar{\omega}$. We donote the Kaehler structure of $\bar{M}$ by (J, g), where J is defined by $JX=d\bar{\pi}(\phi X^*)$. Then, by the construction we have

$$(2.1)\qquad (JX)^*=\phi X^*,\quad G(X^*, Y^*)=g(X, Y)$$

for any vector fields X and Y on $\bar{M}$. The following relation between the Riemannian connections $\bar{\nabla}$ of $\bar{M}$ and $\bar{D}$ on $\bar{N}$ is derived from the above properties:

$$(2.2)\qquad (\bar{\nabla}_X Y)^*=-\phi^2\bar{D}_{X^*}Y^*=\bar{D}_{X^*}Y^*-G(\phi X^*, Y^*)\bar{E},\quad \bar{D}_{X^*}E=\varepsilon\phi X^*.$$

Let N be a hypersurface tangent to $\bar{E}$ of $\bar{N}$. In the sequel, we assume that there is a fibration $\pi: N\to M$, where M is a real hypersurface of $\bar{M}$ such that the diagram

$$\begin{array}{ccc} & i' & \\ N & \rightarrow & \bar{N} \\ \pi\downarrow & & \downarrow\bar{\pi} \\ M & \rightarrow & \bar{M} \\ & i & \end{array}$$

is commutative and the immersion i' of N into $\bar{N}$ is a diffeomorphism of the fibres. This shows that we have the orthogonal decomposition $T_q(N)=(T_{\pi(q)}M)^*+\mathrm{span}\{E'_q\}$. Then the fibrations $\bar{\pi}: \bar{N}\to\bar{M}$ and $\pi: N\to M$ are both the Riemannian submersions in the sense of O'Neill. By G and g the induced semi-Riemannian tensors of N and M are denoted, respectively. Let D and ∇ be the Levi-Civita connections on N and M, and σ' and σ be the second fundamental forms of N and M, respectively. The associated shape operators are denoted by A' and A. The Gauss formulas for the immersions i' and i and (2.2) yield

$$(2.3)\qquad D_{X^*}Y^*=(\nabla_X Y)^*-G(\phi X^*, Y^*)E',\quad \sigma'(X^*, Y^*)=\sigma(X, Y)^*,$$

and by the Weingarten formulas for the immersions and (2.2) we have the following relations between the shape operators A' and A:

(2.4) $A'Y^*=(AY)^*+\varepsilon G(A'Y^*, E')E', \ D^{\perp}_{X^*}\xi^*=(\nabla^{\perp}_X\xi))^*,$

where $D^{\perp}$ and $\nabla^{\perp}$ are the covariant differentials with respect to the normal connections.

On the other hand, for the orthogonal operators (P', F') and (P, F) of the immersions i' and i respectively, (2.1) means that

(2.5) $(PX)^*=P'X^*, \ (FX)^*=F'X^*, \ (J\xi)^*=\phi\xi^*,$

and by (1.6) and (2.4) it turns out that

(2.6) $A'Y^*=(AY)^*+G(F'Y^*, \xi^*)E', \ A'E'=-\varepsilon\phi\xi^*.$

For the relationship between covariant derivatives of the second fundamental form σ' of N and σ of M, it follows from (1.3), (1.6), (2.3) and (2.4) that we have

(2.7) $$\begin{cases} D_{X^*}\sigma'(Y^*, Z^*)=\{\nabla_X\sigma(Y, Z)+\varepsilon g(PX, Y)FZ+\varepsilon g(PX, Z)FY\}^*, \\ D_{X^*}\sigma'(Y^*, E^*)=D_{E'}\sigma'(X^*, Y^*)=-\varepsilon\{\sigma(X, PY)+\sigma(PX, Y)\}^*, \\ D_{X^*}\sigma'(E', E')=D_{E'}\sigma'(E', X^*)=-2F'P'X^*. \end{cases}$$

By means of (1.1) and (2.2), a straightforward calculation gives rise to

(2.8) $$(\bar{R}(X, Y)Z)^*=\bar{R}'(X^*, Y^*)Z^*+\varepsilon\{G(Z^*, \phi Y^*)\phi X^* \\ -G(Z^*, \phi X^*)\phi Y^*-2G(Y^*, \phi X^*)\phi Z^*\}$$

and by choosing the orthonormal frame field in which E′ is included, it turns out that

(2.9) $\bar{S}(X, Y)=\bar{S}'(X^*, Y^*)+2\varepsilon g(X, Y).$

Then, by making use of (2.3), (2.6), (2.7) and (2.8) it follows from the Gauss equations of N and M that we have

(2.10) $$(R(X, Y)Z)^*=R'(X^*, Y^*)Z^*+\varepsilon\{g(PY, Z)PX \\ -g(PX, Z)PY-2g(PX, Y)PZ\}^* \\ +\{-G(F'X^*, \xi^*)G(A'Y^*, Z^*)+G(F'Y^*, \xi^*)G(A'X^*, Z^*)\}E^*$$

and hence it turns out that

(2.11) $S(X, Y)=S'(X^*, Y^*)+2\varepsilon g(PX, PY).$

In particular, if $\bar{N}$ is a semi-Sasakian space form of ϕ-holomorphic curvature c, then we have by (1.8)

(2.12) $S'(X^*, Y^*)=[(2n-1)(c+3\varepsilon)g(X, Y)$

$$\begin{aligned}
&+(c-\varepsilon)\{3g(PX, PY)-g(X, Y)\}]/4\\
&+\textstyle\sum_{j=1}^{2n-1}\{g(\sigma(X, Y),\ \sigma(E_j, E_j))\\
&-g(\sigma(X, E_j),\ \sigma(Y, E_j))\}\\
&-\varepsilon g(FX, FY),
\end{aligned}$$

$$S'(X^*, E^*)=\varepsilon\textstyle\sum_{j=1}^{2n-1}\{g(FX, \sigma(E_j, E_j))-g(\sigma X, E_j),\ FE_j)\},$$

$$S'(E', E')=(2n-1)c-\textstyle\sum_{j=1}^{2n-1}g(FE_j, FE_j).$$

Finally, the following property between the covariant derivatives of Ricci tensors S' and S is given. The proof is omitted, because it is only the straightforward calculation in which many formulas mentioned above are used.

LEMMA 2.1. *Let $\bar{N}$ be a semi-Sasakian space form of constant ϕ-sectional curvature c and N be semi-Riemannian hypersurface tangent to the structure vector $\bar{E}$. Assume that there exist fibrations $\bar{\pi}: \bar{N}\to\bar{M}$ and $\pi: N\to M$, where M is a hypersurface of a Kaehler manifold $\bar{M}$. If the one is compatible with the other, then we have*

$$\begin{aligned}
(2.13)\qquad D_{X^*}S'(Y^*, Z^*)=&\nabla_X S(Y, Z)+g(PX, Y)S'(E', Z^*)\\
&+g(PX, Z)S'(E', Y^*)\\
&-2\varepsilon\{g(\sigma(Y, PZ),\ FX)+g(\sigma(Z, PY),\ FX)\}\\
&+2\varepsilon\{g(\sigma(X, Y),\ FPZ)+g(\sigma(X, Z),\ FPY)\}
\end{aligned}$$

for any vector fields X, Y and Z tangent to M.

REMARK. Lemma 2.1 holds in the case where N and M are semi-Riemannian submanifolds of N and M, respectively.

3. Cyclic-parallel Ricci tensors.

This section is devoted to the investigation about the principal curvatures of a real hypersurface of a complex space form whose Ricci tensor is cyclic-parallel. The Ricci tensor S of the semi-Riemannian manifold is said to be *cyclic-parallel,* if it satisfies $\mathfrak{S}\nabla S=0$, where $\mathfrak{S}$ denotes the cyclic sum, that is, it satisfies

$$(3.1)\qquad \mathfrak{S}\nabla S(X, Y, Z)=\nabla_X S(Y, Z)+\nabla_Y S(Z, X)+\nabla_Z S(X, Y)=0$$

for any tangent vector fields X, Y and Z, which is equivalent to $\nabla S(X, X, X)=0$. For this condition, refer to Besse [1].

Let M be a real hypersurface of $M_n(c)(c\neq 0)$ whose Ricci tensor is cyclic-parallel. Then M admits an almost contact metric structure (P, E, ω, g). Assume that the structure vector field E is principal. The principal curvature is denoted by α. Then it follows from some formulas given in § 1 that (3.1) is reduced to

$$(3.2)\quad \begin{aligned} & h\{g(\nabla_X A(Y),Z)+g(\nabla_Y A(Z),X)+g(\nabla_Z A(X),Y\} \\ & +\{Xhg(AY,Z)+Yhg(AZ,X)+Zhg(AX,Y)\} \\ & -\{g(AX,\nabla_Y A(Z)+\nabla_Z A(Y)) \\ & +g(AY,\nabla_Z A(X)+\nabla_X A(Z)) \\ & +g(AZ,\nabla_X A(Y)+\nabla_Y (X)\} \\ & -3c\{\omega(X)g(BY,Z)+\omega(Y)g(BZ,X)+\omega(Z)g(BX,Y)\}=0, \end{aligned}$$

where B denotes the operator of $T(M)$ defined by $PA-AP$.

First of all, the constancy of the principal curvature α is proved. In the case of P_nC, the fact is true without the condition that S is cyclic-parallel.

LEMMA 3.1. *Let M be a real hypersurface of $M_n(c)$, $(c\neq 0)$, whose Ricci tensor is cyclic-parallel. If E is principal, then the corresponding principal curvature α is constant.*

PROOF. Putting $Z=E$ in (3.2) and taking account of (1.15) and (1.16), we have

$$(3.3)\quad \begin{aligned} & (3\alpha h-8c-2\alpha^2)B-2\alpha(PA^2-A^2P)+2\alpha(dh\otimes E+\omega\otimes \mathrm{grad}\ h) \\ & +2\beta A+6\beta(h-2\alpha)\omega\otimes E=0, \end{aligned}$$

where $\beta=d\alpha(E)$. If this operator acts on E, then it turns out that

$$(3.4)\quad \alpha dh=\beta(4\alpha-3h)\omega,$$

from which together with (1.16) it follows that

$$(3.5)\quad \beta(\alpha-h)=0.$$

Let U be the set consisting of points of M at which the function β is not zero. Suppose that U is not empty. Then we have

$$(3.6)\quad PA+AP=0,\ \alpha=h$$

by means of (1.14) and (3.5). Accordingly the following equation is derived from (3.3):

$$(\alpha^2-8c)PA-\alpha\beta\omega\otimes E+\beta A=0.$$

For a principal vector X on U orthogonal to E with a principal curvature λ, we have

$$(\alpha^2-8c)\lambda PX+\beta\lambda X=0$$

Since X and PX are mutually orthogonal, it means that $\lambda=0$ on U. This together with (3.6) implies that

$$AX=0,\ APX=0,$$

which show that the shape operator A and the structure tensor P commute each other on U. The same argument as those of Okumura [10] ($c>0$) and Montiel and Romero [9] ($c<0$) proves that α is constant on U. By (1.15) it turns out that $\beta=0$, which is a contradiction. Consequently U is empty and therefore $\beta=0$ on M. q. e. d.

Since α is constant, (3.3) and (3, 4) give

$$\alpha dh=0,\ (3\alpha h-8c-2\alpha^2)B-2\alpha(PA^2-A^2P)=0. \tag{3.7}$$

By making use of this equation, the following theorem is proved. By means of the congruence theorem due to Kimura [4], the main theorem mentioned in the introduction is a direct consequence of the following result.

THEOREM 3.2. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, whose Ricci tensor is cyclic-parallel. If the structure vector E is principal, then all principal curvatures of M are constant and the number of distinct principal curvatures are at most* 5.

PROOF. Let X be a principal vector orthogonal to E with a principal curvature λ. Then it follows from (1.14) that

$$(2\lambda-\alpha)APX=(\lambda\alpha+2c)PX.$$

Let V be the set consisting of points at which the function $2\lambda-\alpha$ is non-zero. In the case of $c>0$, V is entirely equal to M. Suppose that V is not empty. Then PX is also principal on the open set V and its corresponding principal curvature μ is given by

$$\mu=(\alpha\lambda+2c)/(2\lambda-\alpha).$$

Consequently, as the relationship between principal curvatures λ and μ, (3.7) is reduced to

$$(\lambda-\mu)\{\alpha(\lambda+\mu)-k\}=0,\ k=(3\alpha h-8c-2\alpha^2)/2, \tag{3.8}$$

which is the quartic equation of variable λ whose coefficients are not necessarily constant.

Suppose that $\alpha=0$. Then (3.8) is regarded as $c(\lambda-\mu)=0$ and hence $\lambda=\mu$, which implies that $\lambda^2=c>0$, because of the definition of μ. It means that λ is constant on V and hence the continuity of λ shows that V coincides with M.

On the other hand, suppose that $\alpha\neq 0$. It is seen that the function h is constant by (3.7) and hence (3.8) is the quartic equation of λ whose coefficients are constant. It means that λ is constant on V and hence on M, and the number d of the distinct principal curvatures is at most 5.

Next, the case where V is emtpy is considered. Then we have $2\lambda=\alpha$ on M and hence $\alpha\lambda+2c=0$, $\lambda^2=-c>0$ on M. Accordingly $\lambda\neq\alpha$ and $\alpha\neq 0$, and hence h is constant on M. Suppose that there exist a point x and a principal vector u at x orthogonal to E_x with a principal curvature τ such that $\tau\neq\alpha/2$. Then $P_x u$ becomes a principal vector with a principal curvature

$$(\alpha\tau+2c)/(2\tau-\alpha)\neq\alpha/2$$

and from (3. 3) it follows that

$$(2\tau-\alpha)(3h-2\tau-\alpha)=0.$$

Accordingly, $\tau=(3h-\alpha)/2$ and it is different from α. In fact, suppose that $\tau=\alpha$ and its multiplicity is equal to p. Then we have $h=\alpha$, which yields $(2n-1+p)\alpha=0$, a contradiction. This shows that there exist distinct constant principal curvatures α, $\alpha/2$ and $(3h-\alpha)/2$. q. e. d.

REMARK 1. In a complex projective space Kimura [4] proved that if all principal curvatures are constant and if E is principal, then $d\leqq 5$.

REMARK 2. In a complex hyperbolic space, Montiel and Romero [10] gave an example of a real hpersurface whose distinct principal curvatures are α and $\alpha/2$ with multiplicites 1 and $2n-2$. It is stated in the next section.

REMARK 3. Under the condition $\nabla(\mathfrak{S}\nabla S)=0$, the same conclusion as that in this section is obtained.

4. Examples.

In this section, some standard examples of real hypersurfaces of $M_n(c)$ $(c\neq 0)$ whose Ricci tensors are cyclic-parallel are given. In the complex Euclidean space $\boldsymbol{C}^{n+1}$ equipped with the Hermitian form F, the Euclidean metric of $\boldsymbol{C}^{n+1}$ which is identified with $\boldsymbol{R}^{2n+2}$ is given by Re F. For the unit sphere $S^{2n+1}=\{z\in\boldsymbol{C}^{n+1}: F(z, z)=1\}$ the tangent space T_zS^{2n+1} at each point z can be identified with $\{w\in\boldsymbol{C}^{n+1}: \text{Re } F(z, w)=0\}$. Let T'_z be the orthogonal complement of the vector iz in T_zS^{2n+1}. When the sphere S^{2n+1} is considered as a principal fibre bundle over $P_n\boldsymbol{C}$ with the structure group S^1 and the projection π, there is a connection such that T'_z is the horizontal subspace at z which is invariant under the S^1-action. The Fubini-Study metric g of constant holomorphic sectional curvature 4 is given by $g_P(X, Y)=\text{Re } F_Z(X^*, Y^*)$ for any tangent vectors X and Y in $T_P(P_n\boldsymbol{C})$, where

z is any point of S^{2n+1} with $\pi(z)=p$ and, X^* and Y^* are the vectors in T'_z such that $d\pi X^*=X$ and $d\pi Y^*=Y$. On the other hand, the complex structure $J:w\rightarrow iw$ in T'_z is compatible with the action of S^1 and induces the almost complex structure J on P_nC such that $d\pi\circ i=J\circ d\pi$. Then P_nC is a complex projective space with constant holomorphic curvature 4.

Now, for any positive number r a hypersurface $N_0(2n, r)$ of S^{2n+1} is defined by

$$N_0(2n, r)=\{(z_1, \ldots, z_{n+1})\in S^{2n+1}\subset C^{n+1}: \textstyle\sum_{j=1}^{n}|z_j|^2=r|z_{n+1}|^2\}.$$

For an integer $m(2\leqq m\leqq n-1)$ and a positive number s, a hypersurface $N(2n, m, s)$ of S^{2n+1} is defined by

$$N(2n, m, s)=\{(z_1, \ldots, z_{n+1})\in S^{2n+1}\subset C^{n+1}: \textstyle\sum_{j=1}^{m}|z_j|^2=s\sum_{j=m+1}^{n+1}|z_j|^2\}.$$

Then it is seen that $N_0(2n, r)$ and $N(2n, m, s)$ are both isoparametric hypersurfaces of S^{2n+1} which have two distinct constant principal curvatures [12, 13], and the second fundamental forms are parallel.

For a real hypersurface M of P_nC it is known that we can construct a real hypersurface N of S^{2n+1} which is a principal S^1-bundle over M with totally geodesic fibres and the projection π. Moreover, the projection is compatible with the Hopf fibration $\bar{\pi}: S^{2n+1}\rightarrow P_nC$, that is, the diagram

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2n+1} \\ \pi\downarrow & & \downarrow\bar{\pi} \\ M & \xrightarrow[i]{} & P_nC \end{array}$$

is commutative (i' and i being the respective immersions). Since the second fundamental forms of the immersions i' of the examples mentioned above are parallel, so are the Ricci tensors. It follows from this result together with Lemma 2.1 that $M_0(2n-1, r)=\pi(N_0(2n, r))$ and $M(2n-1, m, s)=\pi(N(2n, m, s))(n\geqq 3)$ are examples of real hypersurfaces of P_nC whose Ricci tensors are cyclic-parallel, because the shape operator and the induced structure tensor P commute with each other.

REMARK 1. It is known [11] that $M_0(2n-1, r)$ and $M(2n-1, m, s)$ are both compact connected real hypersurfaces of P_nC with constant two or three distinct principal curvatures respectively, which are said to be *of type* A_1 and A_2 respectively.

REMARK 2. It is shown in [2] and [10] that $M_0(2n-1, r)$ and $M(2n-1, m, s)$, $s=(m-1)/(n-m)$, are pseudo-Einstein. From this property that the Ricci tensor is cyclic-parallel can be checked by the direct calculation.

Now, some examples of real hypersurfaces of $H_n\boldsymbol{C}$ are considered. In $\boldsymbol{C}^{n+1}$ with the standard basis, a Hermitian form F is defined by

$$F(z, w)=-z_0\bar{w}_0+\sum_{k=1}^{n} z_k\bar{w}_k,$$

where $z=(z_0, \ldots, z_n)$ and $w=(w_0, \ldots, w_n)$ are in $\boldsymbol{C}^{n+1}$. The Minkowski space $(\boldsymbol{C}^{n+1}, F)$ is simply denoted by $\boldsymbol{C}_1^{n+1}$. The scalar product given by Re $F(z, w)$ is a semi-Riemannian metric of index 2 in $\boldsymbol{C}_1^{n+1}$. Let H_1^{2n+1} be a real hypersurface of $\boldsymbol{C}_1^{n+1}$ defined by

$$H_1^{2n+1}=\{z\in \boldsymbol{C}_1^{n+1} : F(z, z)=-1\},$$

and let G be a semi-Riemannian metric of H_1^{2n+1} induced from the complex Lorentz metric Re F of $\boldsymbol{C}_1^{n+1}$. Then (H_1^{2n+1}, G) is the Lorentz manifold of constant curvature -1, which is called an *anti-de Sitter space*. For any point z of H_1^{2n+1} the tangent space $T_zH_1^{2n+1}$ can be identified with $\{w\in \boldsymbol{C}_1^{n+1} :$ Re $F(z, w)=0\}$. Moreover, similar to the case of the complex projective space, it is known in [9] that H_1^{2n+1} *is a principal* S^1-bundle over a complex hyperbolic space $H_n\boldsymbol{C}$ with the projection $\pi : H_1^{2n+1}\rightarrow H_n\boldsymbol{C}$, which is a semi-Riemannian submersion with the fundamental tensor J and time-like totally geodesic fibres.

Now, for given integers p and q with $p+q=n-1$ and $r\in\boldsymbol{R}$ with $0<r<1$, a Lorentz hypersurface $N_{p,q}(r)$ of H_1^{2n+1} is defined by

$$N_{p,q}(r)=\{(z_0, \ldots, z_n)\in H_1^{2n+1} : r(-|z_0|^2+\textstyle\sum_{j=1}^{p}|z_j|^2)=-\sum_{j=p+1}^{n}|z_j|^2\}$$

and a Lorentz hypersurface N_n of H_1^{2n+1} is given by

$$N_n=\{(z_0, \ldots, z_n)\in H_1^{2n+1} : |z_0-z_1|^2=1\}.$$

Then it is seen from [8] that $N_{p,q}(r)$ is isometric to $H_1^{2p+1}(1/(\mathrm{r}-1))\times S^{2q+1}(r/(1-r))$ and the second fundamental forms of $N_{p,q}(r)$ and N_n are both parallel, and hence so are the Ricci tensors.

Since $N_{p,q}(r)$ and N_n are S^1-invariant, $M_{p,q}(r)=\pi(N_{p,q}(r))$ and $M_n=\pi(N_n)$ are real hypersurfaces of $H_n\boldsymbol{C}$. Then $\pi : N_{p,q}(r)\rightarrow M_{p,q}(r)$ and $\pi : N_n\rightarrow M_n$ are semi-Riemannian submersions which are compatible with the S^1- *fibration* $\pi : H_1^{2n+1}\rightarrow H_n\boldsymbol{C}$. By means of Lemma 2. 1 it follows that $M_{p,q}(r)$ and M_n are examples of real hypersurfaces of $H_n\boldsymbol{C}$ whose Ricci tensors are cyclic-parallel, because the shape operator and the structure tensor commute with each other.

Real hypersurfaces of H_nC are due to Montiel [8] and Montiel and Romero [9].

REMARK 3. It is seen that $M_{p,q}(r)$ and M_n are complete connected real hypersurfaces of H_nC with constant two or three distinct principal curvatures, which are said to be *of type A*.

5. Classifications in P_nC.

In this section the complete connected real hypersurface of P_nC whose Ricci tensor is cyclic-parallel is considered. Let M be such a hypersurface of P_nC and assume that the structure vector E is principal. Then it is already seen that all principal curvatures are constant and the number d of distinct principal curvatures is at most 5. Let $\lambda_a(a=0,\dots,4)$ be distinct principal curvatures with multiplicities m_a, respectively, defined by

$$\begin{aligned} &\lambda_0=\alpha, \\ (5.1)\qquad &\lambda_1,\ \lambda_2: \text{ the roots of } x^2-\alpha x-1=0, \\ &\lambda_3,\ \lambda_4: \text{ the roots } x^2-kx/\alpha+k/2+1=0, \end{aligned}$$

where $k=(3\alpha h-8-2\alpha^2)/2(\alpha\neq 0)$. Accordingly, by means of a theorem due to Kimural [4], M is congruent to an open set of a homogeneous real hypersurface of P_nC.

In connection with the examples given in the previous section, we next investigate whether or not the homogeneous real hypersurfaces of P_nC which are not of type A_1 or A_2 satisfy the condition $\mathfrak{S}\nabla S=0$. In order to answer this purpose, the sufficient condition for the cyclic-parallelism of the Ricci tensor is first considered.

LEMMA 5.1. *Let M be a real hypersurface of P_nC. If the structure vector is principal, then the Ricci tensor is cyclic-parallel if and only if*

$$\mathfrak{S}\nabla S|E^\perp=0,\ \mathfrak{S}\nabla S(X,\ Y,\ E)=0$$

for any vector fields X and Y, where $E^\perp$ denotes the orthogonal complement of E.

PROOF. It suffices to prove only the " if " part. Since the operator $\mathfrak{S}\nabla S$ is trilinear, we have

$$\mathfrak{S}\nabla S(X,\ Y,\ Z+fE)=\mathfrak{S}\nabla S(X,\ Y,\ Z) \text{ for any function } f,$$

from which together with the assumption $\mathfrak{S}\nabla S|E^\perp=0$ it follows that

$$\mathfrak{S}\nabla S(X,\ Y,\ Z)=0 \text{ for any vector fields } X \text{ and } Y \text{ of } E^\perp.$$

By repeating the similar argument to the above one, the conclusion is given.
q. e. d.

By taking account of (3. 2) and (3. 3), it is easily seen that the second condition is equivalent to the equation (3. 7), because h is constant.

LEMMA 5. 2. *Let M be a real hypersurface of P_nC. If E is principal, then it satisfies the condition $\mathfrak{S}\nabla S(X, Y, E)=0$ for any vector fields X and Y if and only if the function h is constant and*

$$(5.2) \qquad \alpha(PA^2-A^2P)-k(PA-AP)=0.$$

where $k=(3\alpha h-2\alpha^2-8)/2$.

Let M be a homogeneous hypersurface of type B, C, D or E of P_nC. Then it has always principal curvatures λ_3 and λ_4 with the same multiplicities m_3 and m_4, which satisfy $\lambda_3\lambda_4=-1$ (cf. [12], Table). Accordingly, in order for M to satisfy the condition $\mathfrak{S}\nabla S(X, Y, E)=0$, the principal curvatures λ_3 and λ_4 must satisfy the relation (5. 2), in other words, they ought to be the roots of the second equation of (5. 1). This means that $h=2\alpha/3$ is a necessary and sufficient condition. Since h is given by $h=\alpha+m_1(\lambda_1+\lambda_2)+m_3(\lambda_3+\lambda_4)$, we have

$$h=(1+m_1)\alpha-4m_3/\alpha,$$

because of $\lambda_1+\lambda_2=\alpha$ and $\lambda_3+\lambda_4=-4/\alpha$. It yields that $h=2\alpha/3$ is equivalent to

$$\alpha^2=12(n-1),\ 24/(3n-8),\ 48/13 \text{ or } 72/25,$$

according as the homogenous hypersurface M of type B, C, D or E.

We next consider the condition $\mathfrak{S}\nabla S|E^{\perp}=0$. Suppose that the number d of distinct principal curvatures of a real hypersurface of P_nC is at most three, say α, λ and μ. Since any vector fields $X_a(a=1, 2, 3)$ orthogonal to E have the direct sum decomposition $X_a=X_{a1}+X_{a2}$ such that $AX_{a1}=\lambda X_{a1}$ and $AX_{a2}=\mu X_{a2}$, we have $g(\nabla_{X1}A(X_2), X_3)=\sum g(\nabla_{Xab}A(X_{cd}), X_{ef})$. Since $g(\nabla_X A(Y), Z)$ is symmetric with respect to X, Y and Z orthogonal to E because of (1. 12), we may consider without loss of generality that, in each term $g(\nabla_X A(Y), Z)$ of the right hand side of the above equation, Y and Z are both principal vectors corresponding to the principal curvature λ. Consequently, since the shape operetor is self-adjoint, we get

$$g(\nabla_X A(Y), Z)=g(\nabla_X(AY)-A\nabla_X Y, Z)=0$$

from which it follows that $g(\nabla_{X1}A(X_2), X_3)=0$ for any vector fields X_a orthogonal to E, and hence the condition

$$\mathfrak{S}\nabla S|E^{\perp}=0$$

is satisfied. It yields that the homogeneous real hypersurface of type B satisfies the above condition. For a real number $t(0<t<1)$ we denote by $N(2n, t)$ a hypersurface of S^{2n+1} defined by $|\sum_{j=1}^{n+1} z_j^2|^2=t$ and $\sum_{j=1}^{n+1} |z_j|^2=1$ for $(z_1, \dots, z_{n+1})\in \boldsymbol{C}^{n+1}$. Then it is seen by Takagi [13] that the hypersurface has constant principal curvatures $\lambda_a(a=1,\dots, 4)$ with multiplicities 1, 1, $n-1$ and $n-1$, and that $t=\sin^2 2\theta$. Thus, for the projection π of the Hopf fibration of S^{2n+1} onto $P_n\boldsymbol{C}$, $M(2n-1, t)=\pi(N(2n, t))$ is a compact real hypersurface of type B and, since $t=1/(3n-2)$ is equivalent to $\alpha^2=12(n-1)$, the Ricci tensor of $M(2n-1, 1/(3n-2))$ is cyclic-parallel, because of $\alpha=2\cot 2\theta$. By means of Lemmas 3. 1, 5. 1 and 5. 2 we can prove the following

THEOREM 5. 3. *$M_0(2n-1, r)$, $M(2n-1, m, s)$ and $M(2n-1, 1/(3n-2))$ are complete and connected real hypersurfaces of $P_n\boldsymbol{C}$ whose Ricci tensor is cyclic-parallel and whose structure vector is principal.*

REMARK 1. Let X, Y and Z be principal vectors orthogonal to E associated with principal curvatures λ, μ and σ, respectively. Then the following equation is derived from (3. 8):

$$\mathfrak{S}\nabla S(X, Y, Z)=\{3h-2(\lambda+\mu+\sigma)\}g(\nabla_X A(Y), Z).$$

For the homogenous real hypersurface of type C, D or E of $P_n\boldsymbol{C}$ whose value of α^2 is given by $24/(3n-8)$, $48/13$ or $72/25$, the above relationship means that $\mathfrak{S}\nabla S|E^{\perp}=0$ if and only if $g(\nabla_X A(Y), Z)=0$ for any vector fields orthogonal to E.

REMARK 2. Let M be a complete and connected real hypersurface of $H_n\boldsymbol{C}$. Montiel and Romero [9] proved that M is congruent to $M_{p,q}(r)$ or M_n provided that $B=PA-AP=0$. Accordingly it seems to be interesting whether or not Theorem 5. 3 holds in the case of $H_n\boldsymbol{C}$.

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Taegu Univ.

Univ. of Tsukuba