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On positive solutions of quasi-linear elliptic equations

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ABSTRACT. In this note we prove the existence of positive solutions of the Dirichlet problem for a quasi-linear elliptic equation. Our boundary data belongs to L^2 and a corresponding solution is in a weighted Sobolev space.

1. Introduction.

Let $Q\subset R_n$ be a bounded domain with the boundary ∂Q of class C^2. In Q we consider the Dirichlet problem

(1) $Lu=-\sum_{i,j=1}^{n}D_i(a_{ij}(x, u)D_ju)+a_0(x)u=f(x, u)$ in Q,

(2) $u(x)=\phi(x)$ on ∂Q,

where ϕ is a non-negative function in $L^2(\partial Q)$.

Throughout this paper we make the following assumptions

(A) There is a positive constant γ such that

$$\gamma^{-1}|\xi|^2\leq\sum_{i,j=1}^{n}a_{ij}(x, u)\ \xi_i\xi_j\leq\gamma|\xi|^2$$

for all $\xi\in R_n$ and $(x, u)\in Q\times R$; $a_{ij}(x, u)=a_{ji}(x, u)$ $(i, j=1, \dots, n)$ for all $(x, u)\in Q\times R$. Moreover, we assume that $a_{ij}(\cdot, \cdot)\in C(\bar{Q}\times R)$ $(i, j=1, \dots, n)$ and for each $u\in R$, $a_{ij}(\cdot, u)\in C^1(\bar{Q})$ $(i, j=1, \dots, n)$ and that there exist functions $A_{ij}\in C^1(\bar{Q})$ such that

$$\lim_{|u|\to\infty} a_{ij}(x, u)=A_{ij}(x) \text{ and } \lim_{|u|\to\infty} D_x a_{ij}(x, u)=D_x A_{ij}(x)\ (i, j=1, \dots, n)$$

uniformly on $\bar{Q}$. Finally, the coefficient $a_0(x)$ is non-negative and belongs to $L^\infty(Q)$.

(B) The nonlinearity $f: Q\times R\to R$ satisfies the Carathéodory conditions, i. e.

(i) for each $u\in R$, the function $x\to f(x, u)$ is measurable in Q,

(ii) for each $x\in Q(a. e.)$, the function $u\to f(x, u)$ is continuous on R.

Further assumptions on f will be formulated later on.

In this note we use the notion of a generalized (weak) solution of (1) involving the Sobolev spaces $W_{loc}^{1,2}(Q)$, $W^{1,2}(Q)$ and $\mathring{W}^{1,2}(Q)$ (for the definitions of these spaces see [10]).

A function u is said to be a generalized (weak) solution of (1) if $u\in W_{\mathrm{loc}}^{1,2}(Q)$ and satisfies

(3) $$\int_{Q}\sum_{i,j=1}^{n}[a_{ij}(x, u)D_{i}u\ D_{j}v+a_{0}(x)u\ v]\ dx=\int_{Q}f(x, u)v\ dx$$

for each $v\in W^{1,2}(Q)$ with compact support, provided $f(\cdot, u(\cdot))\in L_{\mathrm{loc}}^{2}(Q)$.

There is an extensive literature on positive solutions for semi-linear elliptic equations (see survey articles [1] and [8]). Most of these results are concerned with solutions with zero or smooth boundary data for semi-linear elliptic equations. Terefore solutions belong to the usual Sobolev space $W^{1,2}(Q)$ or to the Hölder space $C^{2,\alpha}(\bar{Q})$, depending on the regularity of coefficients. The results of this paper are related to those of [7], where some existence theorems of positive solutions in $C^{2,\alpha}(\bar{Q})$ for quasi-linear elliptic equations were obtained.

In this paper we assume that $\phi\in L^{2}(\partial Q)$ and consequently we cannot expect to find a solution in the Sobolev space $W^{1,2}(Q)$. On the other hand, the boundary condition (2) requires a proper formulation due to the fact that not every function in $L^{2}(\partial Q)$ is a trace of an element from $W^{1,2}(Q)$.

To describe our approach to the problem (1), (2) we need some terminology. It follows from the regularity of the boundary ∂Q that there exists a number $\delta_{0}>0$ such that, for $\delta\in(0, \delta_{0})$, the domain $Q_{\delta}=Q\cap\{x ; \min_{y\in\partial Q}|x-y|>\delta\}$ with the bonudary ∂Q_{δ} possesses the property that to each $x_{0}\in\partial Q$ there exists a unique point $x_{\delta}(x_{0})\in\partial Q_{\delta}$ such that $x_{\delta}(x_{0})=x_{0}-\delta\nu(x_{0})$, where $\nu(x_{0})$ is the outward normal to ∂Q at x_{0}. The above relation gives a one-to-one mapping, of class C^{1}, ∂Q onto ∂Q_{δ}.

According to Lemma 14. 16 in [10] (p. 355), the distance function $r(x)=\mathrm{dist}\ (x, \partial Q)$ belongs to $C^{2}(\bar{Q}-Q_{\delta_{0}})$ if δ_{0} is sufficiently small. We denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x)=r(x)$ for $x\in\bar{Q}-Q_{\delta_{0}}$, $\rho\in C_{2}(\bar{Q})$, $\rho(x)\geq\frac{3\delta_{0}}{4}$ in $Q_{\delta_{0}}$, $\gamma_{1}^{-1}\ r(x)\leq\rho(x)\leq\gamma_{1}\ r(x)$ in Q for some positive constant γ_{1}.

Guided by the results of [3], [4] and [5], we adopt the following approach to the Dirichlet problem (1), (2).

Let $\phi\in L^{2}(\partial Q)$. A weak solution $u\in W_{\mathrm{loc}}^{1,2}(Q)$ of (1) is a solution of the Dirichlet problem with the boundary conditin (2) if

$$\lim_{\delta\to 0}\int_{\partial Q}[u(x_{\delta}(x))-\phi(x)]^{2}dS_{x}=0.$$

It follows from [4], that if the problem (1), (2) admits a solution u such that $f(\cdot, u(\cdot))\in L^{2}(Q)$, then $u\in\tilde{W}^{1,2}$ where $\tilde{W}^{1,2}(Q)$ is a weighted Sobolev space defined by

$$\tilde{W}^{1,2}(Q)=\{u\,;\ u\in W^{1,2}_{\mathrm{loc}}(Q)\ \text{and}\int_Q|Du(x)|^2r(x)\,dx+\int_Q u(x)^2dx<\infty\}$$

and equipped with the norm

$$\|u\|^2_{\tilde{W}^{1,2}}=\int_Q|Du(x)|^2r(x)\,dx+\int_Q u(x)^2\,dx.$$

To proceed further we set for every $v\in L^2(Q)$

$$L^v_u=-\sum_{i,j=1}^n D_i(a_{ij}(x,\ v(x))D_ju)+a_0(x)u$$

and consider the eigenvalue problem in $\overset{\circ}{W}{}^{1,2}(Q)$

(EVP) $L^v_u=\lambda m(x)u$ in Q,
$u(x)=0$ on ∂Q,

where $m\in L^\infty(Q)$ and $m(x)>0$ on some subset of Q of positive measure. By virtue of Theorem 1.13 in [8] the first positive eigenvalue $\lambda_1(m, v)$ is simple and the corresponding eigenfunction can be taken positive on Q. Set

$$\mathscr{H}_1(m)=\inf_{v\in L^2(Q)}\lambda_1\ (m,\ v)$$

Combining the argument of the proof of Proposition 1.11 in [8] with the variational characterization of eigenvalues (Proposition 1.10 in [8]), it is easy to check that $\mathscr{H}_1(m)>0$. Let $\bar{\mu}(m)$ be the first eigenvalue associated with the eigenvalue problem in $\overset{\circ}{W}{}^{1,2}(Q)$

$$(\mathrm{EVP})_1\quad\begin{cases}-\sum_{i,j=1}^n D_i(A_{ij}(x)D_ju)+a_0(x)u=\lambda m(x)u\ \text{in}\ Q,\\ u(x)=0\ \text{on}\ \partial Q.\end{cases}$$

It is obvious that $\mathscr{H}_1(m)\leq\bar{\mu}(m)$. One can give examples of quasilinear elliptic operators for which cases $\bar{\mu}(m)=\mathscr{H}_1(m)$ and $\mathscr{H}_1(m)<\bar{\mu}(m)$ occur (for more details see [6]).

2. Main result.

To establish our main theorem we need some modification of results contained in papers [5] and [6] for the Dirichlet problem

(4) $Lu=\mu m(x)u+h(x)$ in Q.

(5) $u(x)=\phi(x)$ on ∂Q,

where $h\in L^2(Q)$ and $\mu\geq 0$ is a parameter.

LEMMA, 1. *Let* $0<\mu\leq\mathscr{H}_1(m)$ *and* $\mathscr{H}_1(m)<\bar{\mu}(m)$. *Then for each* $\phi\in C^1(\partial Q)$ *there exists at least one solution* $u\in W^{1,2}(Q)$ *of the problem* (4), (5), *which is non-negative if* $\phi\geq 0$ *on* ∂Q *and* $h\geq 0$ *on* Q.

PROOF. If $\mu<\mathscr{H}_1(m)$ the result is an immediate consequence of the Schauder fixed point theorem. To show that this continues to hold for $\mu=\mathscr{H}_1(m)$, we consider for each integer $k>1$ the Dirichlet problem for the equation

$$\text{(4k)}\quad Lu=(1-\frac{1}{k})\ \mathscr{H}_1(m)\ m(x)u+h(x) \text{ in } Q$$

with the boundary condition (5). Since ϕ can be extended to an element $\Phi\in C^1(\bar{Q})$ by means of the transformation $u-\Phi$, the problem (4k), (5) can be reduced to the Dirichlet problem in $\overset{\circ}{W}{}^{1,2}(Q)$. By the previous case, for each k there exists a solution $u_k\in\overset{\circ}{W}{}^{1,2}(Q)$. It is sufficient to show that $\{u_k\}$ is bounded in $W^{1,2}(Q)$. Then a suitable subsequence is convergent weakly in $W^{1,2}(Q)$ and strongly in $L^2(Q)$ to a solution of (4), (5) with $\mu=\mathscr{H}_1(m)$. If we assume, contrary to the assertion, that $\{u_k\}$ is unbounded in $W^{1,2}(Q)$, then we may assume that $\|u_k\|_{W^{1,2}}\to\infty$ as $k\to\infty$ and consequently $v_k=u_k\|u_k\|_{W^{1,2}}^{-1}$ contains a subsequence convergent to a function v, weakly in $W^{1,2}(Q)$ and strongly in $L^2(Q)$. Using the fact that $a_{ij}(x, t)\to A_{ij}(x)$ and $D_x a_{ij}(x, t)\to D_x A_{ij}(x)$ as $|t|\to\infty$ uniformly on $\bar{Q}$, we show that v satisfies the equation

$$-\sum_{i,j=1}^{n} D_i(A_{ij}(x)D_j v)=\mathscr{H}_1(m)m(x)v$$

and moreover that $v_k\to v$ strongly in $\overset{\circ}{W}{}^{1,2}(Q)$. Therefore $\|v\|_{W^{1,2}}=1$ and this contradicts the fact that $\mathscr{H}_1(m)<\bar{\mu}(m)$. Details of the proof are similar to the argument used un [5]. If $\phi\geq 0$ on ∂Q and $h\geq 0$ on Q, then the maximum principle implies that $u\geq 0$ on Q in the case when $\mu<\mathscr{H}_1(m)$. If $\mu=\mathscr{H}_1(m)$, then the solutions u_k of (4_k), (5) are non-negative and hence $u\geq 0$ on Q.

LEMMA, 2. *Suppose that* $\mathscr{H}_1(m)<\bar{\mu}(m)$, $0<\mu<\leq\mathscr{H}_1(m)$ *and* $\phi\in L^2(\partial Q)$. *Let* $\{u_k\}$ *be a sequence of solutions of* (4), (5) *in* $W^{1,2}(Q)$ *with* $\phi=\phi_k$ *and* $\phi_k\in C^1(\partial Q)$. *If* $\lim_{k\to\infty}\phi_k=\phi$ *in* $L^2(\partial Q)$, *then a subsequence of* $\{u_k\}$ *converges in* $\tilde{W}^{1,2}(Q)$ *to a function* u *satisfying* (4), (5).

To prove our assertion it is sufficient to show that $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. The proof is similar to the argument used in the proof of Lemma 1. Again the assumption that $\lim_{|t|\to\infty} a_{ij}(x, t)=A_{ij}(x)$ and $\lim_{|t|\to\infty} D_x a_{ij}(x, t)=D_x A_{ij}(x)$ $(i, j=1,\dots, n)$ uniformly on $\bar{Q}$ is essential in the proof, as well as the compactness of the imbedding of $\tilde{W}^{1,2}(Q)$ in $L^2(Q)$ (Theorem 4.11 in [11]). All details can be found in [5] or [6] (Theorem 6).

We are now in a position to establish our main existence result.

THEOREM 1. *suppose that the nonlinearity* $f(x, u)$ *satisfies the follow-*

ing two conditions

(a) $f(x, 0)\geq 0$ *on* Q,

(b) *there exist functions* $g\in L^{\infty}(Q)$ *and* $c\in L^{2}(Q)$ *such that*

$$f(x, s)\leq g(x)s+c(x)$$

for all $s\geq 0$ *and* $x\in Q$; *moreover* $c(x)\geq 0$ *in* Q *and* $\mathscr{H}_1(g)\geq 1$.

If $\mathscr{H}_1(g)<\bar{\mu}(g)$, $\phi\in L^2(\partial Q)$ *and* $\phi\not\equiv 0$ *on* ∂Q, *then the Dirichlet problem* (1), (2) *admits at least one positive solution* $u\in\tilde{W}^{1,2}(Q)$.

PROOF. Let $\{\phi_k\}$ be a sequence of non-nnegative C^1- functions on ∂Q such that $\lim_{k\to\infty}\phi_k=\phi$ in $L^2(\partial Q)$.

By Lemma 1, for each $k>1$ the Dirichlet problem

(6) $Lu=g(x)u+c(x)$ in Q

(2_k) $u(x)=\phi_k(x)$ on ∂Q

admits a non-negative solution $\bar{u}_k\in W^{1,2}(Q)$. It follows from the assumption (b) that $\bar{u}_k$ is a supersolution of the problem (1), (2_k). Since, by the assumption (a), $\bar{u}_k\equiv 0$ on Q is a subsolution of (1), (2_k), the results of [9] (p. 51) yield the existence of a solution $u_k\in W^{1,2}(Q)$ of (1), (2_k) such that $0\leq u_k(x)\leq\bar{u}_k(x)$ on Q for each k. It follows from LEMMA 2 that a subsequence of $\{\bar{u}_k\}$ converges strongly in $\tilde{W}^{1,2}(Q)$ to a function $\bar{u}$ satisfying (6), (2). We now show that there exists a constant $C>0$ such that

$$(7)\quad \int_Q|Du_k(x)|^2r(x)\,dx\leq C\Big[\int_{\partial Q}\phi_k(x)^2dSx+\int_Q u_k(x)^2dx\Big]$$

$k=1, 2,\dots$, To establish this estimate we take as test functions in (3)

$$v_k(x)=\begin{cases}u_k(x)(\rho(x)-\delta) & \text{for } x\in Q_\delta,\\ 0 & \text{for } x\in Q-Q_\delta,\end{cases}$$

where $0<\delta<\delta_0$. Letting $\delta\to 0$ we obtain

$$(8)\quad \int_Q\sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_kD_ju_k\rho dx+\int_Q\sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_ku_kD_j\rho dx$$
$$+\int_Q a_0(x)u_k^2\rho dx=\int_Q f(x, u_k)u_k\rho dx.$$

Integrating by parts we obtain

$$(9)\quad \int_Q\sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_ku_kD_j\rho dx=\frac{1}{2}\int_Q\sum_{i,j=1}^n D_i\Big(\int_0^{u_k^2}a_{ij}(x, s)ds\Big)\ D_j\rho dx$$
$$-\frac{1}{2}\int_Q\sum_{i,j=1}^n\int_0^{u_k^2}D_ia_{ij}(x, s)dsD_j\rho dx=-\frac{1}{2}\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\phi_k^2}a_{ij}(x, s)dsD_i\rho D_i\rho dS_x$$
$$-\frac{1}{2}\int_Q\sum_{i,j=1}^n\int_0^{u_k^2}a_{ij}(x, s)dsD_{ij}\rho dx-\frac{1}{2}\int_Q\sum_{i,j=1}^n\int_0^{u_k^2}D_ia_{ij}(x, s)dsD_j\rho dx.$$

The estimate (7) readily follows from (8), (9) and (b) and the ellipticity condition in(A). Since $0\leq u_k\leq\bar{u}_k$, the estimate (7) implies that the sequence $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. By Theorem 4.11 in [9], $\tilde{W}^{1,2}(Q)$ is compactly imbedded in $L^2(Q)$. Therefore we may assume that u_k converges weakly in $\tilde{W}^{1,2}(Q)$ and strongly in $L^2(Q)$ to a function u. It is easy to check that u is a solution of (1). By Theorem 1 in [4] there exists a function $\xi \in L^2(\partial Q)$ such that $u(x_\delta) \to \xi$ in $L^2(\partial Q)$ as $\delta \to 0$. Repeating the argument from Theorem 3 in [4], we show that $\xi = \phi$ a. e. on ∂Q. Finally we notice that $u(x) \leq \bar{u}(x)$ on Q.

We mention here that for semi-linear elliptic equations in the case of $C^{2,\alpha}$-solutions, the result of this type is essentially due to Amann [1].

We also observe that if $g(x) \leq 0$ on Q, then the assumption $\mathscr{H}_1(g) \geq 1$ should be dropped.

To obtain the existence result when $\mathscr{H}_1(g) = \bar{\mu}(g)$ we replace the inequality $\mathscr{H}_1(g) \geq 1$ in (b) by $\mathscr{H}_1(g) > 1$.

THEOREM 2. *Suppose that the nonlinearity* $f(x, u)$ *satisifies* (a) *and* (b) *with* $\mathscr{H}_1(g) > 1$. *If* $\mathscr{H}_1(g) = \bar{\mu}(g)$, $\phi \in L^2(\partial Q)$ *and* $\phi \not\equiv 0$, *then the problem* (1), (2) *admits at least one positive solution.*

The proof is based on modifications of LEMMAS 1 and 2 which continue to hold in the case $\mathscr{H}_1(m) = \bar{\mu}(m)$ provided $\mu < \mathscr{H}_1(m)$.

It is worthwhile to notice that in the case $\phi \equiv 0$ on ∂Q, the assumption (a) must be replaced by the stronger condition

$$g(x, s) \geq g_0(x)s \quad \text{for } 0 < s < s_0,$$

for some $s_0 > 0$, with $\mathscr{H}_2(g_0) \leq 1$, where $\mathscr{H}_2(g_0) = \sup_{v \in L_2(Q)} \lambda_1(v_1, g_0)$. Here $\lambda_1(v_1, g_0)$ denotes the first eigenvalue of (EVP) with m$= g_0$ (see [7]). Then according to [2] and [7], for each $r > 0$ there exists a positive eigenfunction w, with $\|w\|_{L^2} = r$, of the problem $Lu = \lambda g_0(x) u$ in Q, $u(x) = 0$ on ∂Q for some $\mathscr{H}_1(g_0) \leq \lambda \leq \mathscr{H}_2(g_0)$. It turns out that w, with r sufficiently small, is a suitable subsolution of the problem (1), (2). This also requires some stronger assumptions on a_{ij}, c and g_0 to ensure that the outward normal derivative $\frac{dw}{d\nu}$ is negative on ∂Q. Since $w > 0$ on Q, the corresponding non-nogative solution of (1), (2) is non-trivial (for details see [7]).

Examples of functions $f: Q \times R \to R$ satisfying the conditions (a) and (b) can be found in [7] and [8].

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