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0-1 laws for kernels of a linear uniform measure

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§1. Introduction

The notions of the Lusin affine kernel and the Lusin kernel were first introduced by Hoffmann-Jϕrgensen [6] for the product measure $\mu=\prod\mu_n$ on $\boldsymbol{R}^{\infty}$. The Lusin affine kernel $A_L(\mu)$ is defined by $A_L(\mu)=\cap\{A$; *A is an affine subspace with* $\sup\{\mu(K); K\subset A,$ *K is compact convex*$\}=1\}$. The Lusin kernel $K_L(\mu)$ is defined by $K_L(\mu)=\cap\{S$; *S is a linear subspace with* sup $\{\mu(K)$; $K\subset S$, *K is compact convex*$\}=1\}$. Hoffmann-Jørgensen [6] determined explicitly the Lusin affine kernel $A_L(\mu)$ and the Lusin kernel $K_L(\mu)$ for the product measure $\mu=\prod\mu_n$ on $\boldsymbol{R}^{\infty}$.

Borell [3] considered the Lusin affine kernel $A_L(\mu)$ and the Lusin kernel $K_L(\mu)$ on the dual locally convex Hausdorff space E' instead of $\boldsymbol{R}^{\infty}$. Let μ be a Radon probability measure on E', where we put the weak* topology on E'. The definitions of $A_L(\mu)$ and $K_L(\mu)$ are the same as those of Hoffmnn-Jϕrgensen. Borell [3] proved a 0-1 law for $A_L(\mu)$ in the case where μ is an s-convex measure with $s>-1$. If μ is an s-convex measure with $s>-1$, then $\mu(A_L(\mu))=0$ or 1 according as $\dim K_L(\mu)=\infty$ or $<\infty$, see Borell [3], Theorem 2.4(c).

Similar 0-1 law was obtained by Zinn [12] for a p-stable measure on a separable Banach space E. Let μ be a probability measure on E and set $A_\mu=\{x$; $\tau_x(\mu)\sim\mu$*(equivalent)*$\}$, where $\tau_x(\mu)(C)=\mu(C-x)$. Then for a p-stable measure μ on E, it holds that $\mu(A_\mu-A_\mu)=0$ or the linear span of A_μ is finite dimensional, see Zinn [12], Proposition 3 and Corollary 5.3.

In this paper, we introduce several notions of kernels of a probability measure on a locally convex Hausdorff space and investigate the 0-1 laws of kernels for a uniform probability measure. The uniformness of a measure on a linear space was first introduced by Dudley [5] and studied in Takahashi and Okazaki [11]. The p-stable measures and the convex measures are uniform.

We introduce the following kernels $K(\mu)$ (the kernel), $A(\mu)$ (the affine kernel), $SK(\mu)$ (the strict kernel), $SA(\mu)$ (the strict affine kernel), $C(\mu)$ (the centered kernel) and $SC(\mu)$ (the strict centered kernel). Also we consider A_μ (the admissible translates) and $A_{\tilde{\mu}}$ (the partially admissible translates).

Our definitions of $K(\mu)$ and $A(\mu)$ are equal to those of $K_L(\mu)$ and $A_L(\mu)$, respectively, in the case of Hoffmann-Jϕrgensen [6] or of Borell [3].

The kernel $K(\mu)$ naturally arises if we consider the translation subordination of μ. For example, it holds that $K(\mu)=\{x\in E\,;\,\tau_x(\mu)s\,\mu\}$, see section 2 for the definition of the subordination $\tau_x(\mu)s\,\mu$. $A(\mu)$ is closely related to the centeredness of μ as we describe below.

μ is called scalarly centered at 0 (resp. strictly scalarly centered at 0) if for every measurable linear subspace S of the form $S=\{x\in E\,;\,x_n'(x)\to 0\}$, $x_n'\in E'$, $\tau_x(\mu)(S)=1$ implies $x\in S$ (resp. $\tau_x(\mu)(S)>0$ implies $x\in S$). The centeredness was introduced by Hoffmann-Jϕrgensen [6], see also Chevet [4]. We have the following characterization. μ is scalarly centered at 0 (resp. strictly scalarly centered at 0) iff $0\in A(\mu)$ (resp. $0\in SA(\mu)$) and iff $C(\mu)=K(\mu)$ (resp. $SC(\mu)=SK(\mu)$). If we consider the translation heredit of the centeredness, the kernels $K(\mu)$ and $SK(\mu)$ arise as follows. If μ is scalarly centered at 0, then $\tau_x(\mu)$ is scalarly centered at 0 iff $x\in K(\mu)$ (the strict case is $SK(\mu)$ instead of $K(\mu)$). In general, $\tau_x(\mu)$ is scalarly centered at 0 iff $C(\mu)=x+K(\tau_x(\mu))$ (the strict case is $SC(\mu)=x+SK(\tau_x(\mu))$.

For stable or convex measure μ, the following 0-1 law is known. For every measurable linear subspace S, $\mu(S)=0$ or 1. The uniform measure satisfies the 0-1 law only for closed subspaces, see Proposition 2. In the 0-1 law for the kernel $K(\mu)$ of a convex measure due to Borell [3], the most interesting point is "$\mu(K(\mu))>0$ implies that $\dim K(\mu)<+\infty$". To prove this 0-1 law, Borell [3] used particular properties of s-convex measures ($s>-1$) such as the 0-1 law, integrability of measurable seminorm, $K(\mu)$ is a dual Banach space and so on, which are not valid for uniform measures. The 0-1 law of Zinn [12] is concerned with A_μ which says that if $\mu(A_\mu-A_\mu)>0$ then $\dim(\text{span}\,A_\mu)<+\infty$, where μ is a stable measure. This result depends on the Sudakov-Feldman's theorem of the non-existence of an E-quasi-invariant measure on E with $\dim E=\infty$. In this paper, we shall prove that if μ is uniform, then $\mu^*(K(\mu))>0$ imply that $\dim K(\mu)<+\infty$ and $\mu^*(K(\mu))=1$ (since we consider the cylindrical σ-algebra, the measurability of $K(\mu)$ is not assured, so we take the outer measure μ^*). We also prove the similar 0-1 laws for the kernels $A(\mu)$, $SK(\mu)$, $C(\mu)$ and $SC(\mu)$. For $L=A_\mu-A_\mu$ or $A_{\tilde{\mu}}-A_{\tilde{\mu}}$, we prove that $\mu^*(L)>0$ implies that $\dim(\text{span}\,L)<+\infty$. Our proof is completely different from that of Borell [3] and Zinn [12]. We use the fact that every nuclear Banach space is finite-dimensional. We start with the following lemma: for a measure μ (not necessarily uniform), if $\mu^*(K(\mu))=1$, then (E',τ_μ) is nuclear and locally convex, see Lemma 1.

The main results are as follows.

(1) *It holds that* $A(\mu)=x+K(\tau_{-x}(\mu))$ *for every* $x\in A(\mu)$ *and* $SA(\mu)=x+SK(\tau_{-x}(\mu))$ *for every* $x\in SA(\mu)$.

(2) *Suppose that* μ *is uniform and let* L *be each one of* $K(\mu)$, $A(\mu)$, $SK(\mu)$, $C(\mu)$ *or* $SC(\mu)$. *Then it holds that* $\mu^*(L)=0$ *or* 1. *If* $\mu^*(L)=1$, *then* $\dim(\operatorname{span} L)<\infty$, *where* μ^* *is the outer measure and* span L *is the linear span of* L.

(3) *Suppose that* μ *is uniform and* $\mu^*(SA(\mu))>0$, *then we have* $\dim(\operatorname{span} SA(\mu))<\infty$. *The* 0-1 *law is not valid for* $SA(\mu)$.

(4) *Suppose that* μ *is uniform and let* $L=A_\mu-A_\mu$ *or* $A_{\tilde{\mu}}-A_{\tilde{\mu}}$. *If* $\mu^*(L)>0$, *then we have* $\dim(\operatorname{span} L)<\infty$.

§ 2. Uniform measure

Let E be a real locally convex Hausdorff space and $C(E, E')$ be the cylindrical σ-algebra generated by the topological dual E'. Let μ be a probability measure on $C(E, E')$ and τ_μ be the topology of convergence in measure on E' (regarding each $x'\in E'$ as a μ-measurable function) semi-metrized by

$$d(x', y')_\mu=\int_E |x'(x)-y'(x)|/(1+|x'(x)-y'(x)|)d\mu(x).$$

Let μ and ν be probability measures on $C(E, E')$. After Dudley [5], we say μ is subordinate to ν (denoted by μ s ν) if the identity $(E', \tau_\nu)\to(E', \tau_\mu)$ is continuous, that is, τ_ν is finer than τ_μ. For each A and B in $C(E, E')$, we set $\mu_A(B)=\mu(A\cap B)/\mu(A)$. The measure μ is said to be uniform if it holds that μ s μ_A whenever $\mu(A)>0$.

The uniformness is characterized as follows. The measure μ is uniform if and only if for every sequence x'_n in E', $\mu(x\,;x'_n(x)\to 0)>0$ implies that $x'_{n_j}(x)\to 0$ μ-almost everywhere for a suitable subsequence x'_{n_j}, see Takahashi and Okazaki [11]. In particular, the convex measures of Borell [2] and the p-stable measures are uniform.

Proposition 1. *Let* E, F *be locally convex Hausdorff spaces and* $\Pi: E\to F$ *be a continuous linear mapping. If* μ *is a uniform measure on* $C(E, E')$, *then the image measure* $\Pi(\mu)$ *is a uniform measure on* $C(F, F')$.

Proof. Suppose that $\Pi(\mu)(A)>0$ and $y'_n\to 0$ in $\tau_{\Pi(\mu)_A}$. Then $y'_n\circ\Pi\to 0$ in $\tau_{\mu_{\Pi^{-1}(A)}}$ with $\mu(\Pi^{-1}(A))>0$. By the uniformness of μ it follows that $y'_n\circ\Pi\to 0$ in τ_μ, that is, $y'_n\to 0$ in $\tau_{\Pi(\mu)}$.

The next result was proved in Takahashi and Okazaki [11]. We give a proof for the sake of completeness.

Proposition 2. *Let* μ *be a uniform Radon probability measure on the*

Borel field on E. Then for every closed linear subspace F of E, we have $\mu(F)=0$ *or* 1.

PROOF. Let F be a closed linear subspace and let $D=\{x'\in E'; F\subset \ker x'\}$, where $\ker x'=\{x\in E; x'(x)=0\}$, $x'\in E'$. Then the net $F_\alpha=\bigcap_{x'\in\alpha}\ker x'$ (α be a finite subset of D) is decreasing, closed and $\bigcap_\alpha F=F$. We have $\mu(F_\alpha)=0$ or 1 by the uniformness. Hence it holds that $\mu(F)=0$ or 1, remarking that $\mu(F)=\inf_\alpha \mu(F_\alpha)$ since μ is Radon.

§ 3. Kernels

Let E be a real locally convex Hausdorff space and μ be a probability measure on $C(E, E')$. We set $\tau_x(\mu)(A)=\mu(A-x)$ for $A\in C(E, E')$ and $x\in E$.

NOTATIONS

$$K(\mu)=\bigcap\{Z; \mu(Z)=1, Z=\{x; x'_n(x)\to 0\}, x'_n\in E'\}$$
$$A(\mu)=\bigcap_{x\in E}(x+K(\tau_{-x}(\mu)))$$
$$SK(\mu)=\bigcap\{Z; \mu(Z)>0, Z=\{x; x'_n(x)\to 0\}, x'_n\in E'\}$$
$$SA(\mu)=\bigcap_{x\in E}(x+SK(\tau_{-x}(\mu)))$$

We shall call $K(\mu)$, $A(\mu)$, $SK(\mu)$ and $SA(\mu)$ the kernel, the affine kernel, the strict kernel and the strict affine kernel, respectively. The spaces $K_L(\mu)$ and $A_L(\mu)$ of Hoffmann-Jørgensen and Borell are same to $K(\mu)$ and $A(\mu)$, see Hoffmann-Jørgensen [6], Theorem 4.4 and Borell [3], Theorem 2.1.

PROPOSITION 3. (1) $SK(\mu)\subset K(\mu)$ *and* $SA(\mu)\subset A(\mu)$.

(2) *For every fixed* $x\in E$, *it holds that* $A(\mu)=x+A(\tau_{-x}(\mu))$ *and* $SA(\mu)=x+SA(\tau_{-x}(\mu))$.

PROOF. (1) is obvious. (2) By the definition of the affine kernel, we have $x+A(\tau_{-x}(\mu))=x+\bigcap_{y\in E}(y+K(\tau_{-y}(\tau_{-x}(\mu))))=x+\bigcap_{y\in E}(y+K(\tau_{-(x+y)}\mu)))=\bigcap_{y\in E}(x+y+K(\tau_{-(x+y)}(\mu)))=\bigcap_{z\in E}(z+K(\tau_{-z}(\mu)))=A(\mu)$. The case for $SA(\mu)$ is analogous.

PROPOSITION 4. (1) *It holds that* $A(\mu)=\bigcap\{x+Z; x\in E, Z=\{y; x'_n(y)\to 0\}, \mu(Z+x)=1, x'_n\in E'\}$.

(2) *If* $0\in A(\mu)$, *then we have* $A(\mu)=K(\mu)$.

PROOF. (1) By the definition of $K(\tau_{-x}(\mu))$, we have $A(\mu)=\bigcap_{x\in E}(x+\bigcap\{Z; \tau_{-x}(\mu)(Z)=1, Z=\{y; x'_n(y)\to 0\}, x'_n\in E'\})=\bigcap_{x\in E}\bigcap_Z\{x+Z; \mu(Z+x)=1,$

$Z=\{y\,;x_n'(y)\to 0\}$, $x_n'\in E'\}$, which proves (1). (2) By (1) it follows that if $0\in A(\mu)$, then for every linear subspace Z of the form $Z=\{y\,;x_n'(y)\to 0\}$ with $\mu(x+Z)=1$ for some $x\in E$, we have $0\in x+Z$, that is, $x+Z=Z$. Hence we have $A(\mu)=K(\mu)$, by the definition of $K(\mu)$.

By Proposition 4, we can see that $A(\mu)$ is the intersection of all affine subspaces of measure 1 of the form $x+Z$, where $x\in E$ and $Z=\{y\,;x_n'(y)\to 0\}$, $x_n'\in E'$. A similar characterization for $SA(\mu)$ is obtained analogously.

PROPOSITION 5. (1) *It holds that* $SA(\mu)=\bigcap\{x+Z\,;x\in E,\ Z=\{y\,;x_n'(y)\to 0\},\ \mu(Z+x)>0,\ x_n'\in E'\}$.

(2) *If* $0\in SA(\mu)$, *then we have* $SA(\mu)=SK(\mu)$.

The probability measure μ on $C(E,E')$ is called scalarly centered at 0 if for every linear subspace Z of the form $Z=\{y\,;x_n'(y)\to 0\}$, $x_n'\in E'$, $\tau_x(\mu)(Z)=\mu(Z-x)=1$ implies $x\in Z$. And μ is strictly scalarly centered at 0 if for every linear subspace Z of the form $Z=\{y\,;x_n'(y)\to 0\}$, $x_n'\in E'$, $\tau_x(\mu)(Z)=\mu(Z-x)>0$ implies that $x\in Z$. The scalarly centeredness was introduced by Hoffmann-Jørgensen and investigated by Chevet [4].

NOTATIONS

$C(\mu)=\{x\,;\tau_x(\mu)$ *is scalarly centered at* $0\}$

$SC(\mu)=\{x\,;\tau_x(\mu)$ *is strictly scalarly centered at* $0\}$

We shall call $C(\mu)$ and $SC(\mu)$ the centered kernel and the strict centered kernel, respectively.

PROPOSITION 6. (1) *It holds that* $C(\mu)=-A(\mu)=\bigcap_{x\in E}(x+K(\tau_x(\mu)))$.

(2) *It holds that* $SC(\mu)=-SA(\mu)=\bigcap_{x\in E}(x+SK(\tau_x(\mu)))$.

PROOF. (1) We show that $C(\mu)=\bigcap_{x\in E}(x+K(\tau_x(\mu)))$. Let $y\in C(\mu)$, that is, $\tau_y(\mu)$ is scalarly centered at 0. For every $x\in E$ and every $Z=\{z\,;x_n'(z)\to 0\}$ such that $\tau_x(\mu)(Z)=\mu(Z-x)=1$, we have $\tau_y(\mu)(Z+y-x)=\mu(Z-x)=1$. Since $\tau_y(\mu)$ is scalarly centered at 0, it follows that $y-x\in Z$. This implies that $y\in x+K(\tau_x(\mu))$ for every $x\in E$, since Z is arbitrary such as $\tau_x(\mu)(Z)=1$. Hence we have $C(\mu)\subset\bigcap_{x\in E}(x+K(\tau_x(\mu)))$. Conversely suppose that $y\in\bigcap_{x\in E}(x+K(\tau_x(\mu)))$. Assume that $\tau_x(\tau_y(\mu))(Z)=1$ for $Z=\{z\,;x_n'(z)\to 0\}$. We must prove that $x\in Z$. Since $y\in(x+y)+K(\tau_{(x+y)}(\mu))$ by the assumption, we have $y\in(x+y)+Z$. In fact, by $\tau_{(x+y)}(\mu)(Z)=\tau_x(\tau_y(\mu)(Z))=1$, $K(\tau_{(y+y)}(\mu))$ is contained in Z. Thus we have proved that $x\in Z$ as desired. (2) The proof is analogous to (1).

COROLLARY 1. (1) *μ is scalarly centered at* 0 *if and only if* $0\in A(\mu)$.
(2) *μ is strictly scalarly centered at* 0 *if and only if* $0\in SA(\mu)$.

THEOREM 1. (1) *For every x in $A(\mu)$ (resp. $C(\mu)$), it colds that $A(\mu)=x+K(\tau_{-x}(\mu))$ (resp. $C(\mu)=x+K(\tau_x(\mu))$).*
(2) *For every x in $SA(\mu)$ (resp. $SC(\mu)$), it holds that $SA(\mu)=x+SK(\tau_{-x}(\mu))$ (resp. $SC(\mu)=x+SK(\tau_x(\mu))$).*

PROOF. (1) Since $A(\mu)=-C(\mu)$ by Proposition 6, we shall only prove that $C(\mu)=x+K(\tau_x(\mu))$ for every $x\in C(\mu)$. The inclusion $C(\mu)\subset x+K(\tau_x(\mu))$ is obvious by Proposition 6. Now let $y\in K(\tau_x(\mu))$ is arbitrary, where $x\in C(\mu)$ is fixed. We prove that $x+y\in C(\mu)$, that is, $\tau_{(x+y)}(\mu)$ is scalarly centered at 0. Take any $u\in E$ and $Z=\{v\,;x'_n(v)\to 0\}$, $x'_n\in E'$ such that $\tau_u(\tau_{(x+y)}(\mu))(Z)=1$. We must show that $u\in Z$. Since $\tau_x(\mu)(Z-y-u)=\tau_u(\tau_{(x+y)}(\mu))(Z)=1$ and $x\in C(\mu)$, it follows that $y+u\in Z$. Thus we have $\tau_x(\mu)(Z)=1$. In particular $K(\tau_x(\mu))\subset Z$ by the definition of $K(\tau_x(\mu))$, which implies that $y\in Z$. Consequently we have $u\in Z$ as desired. The proof of (2) is completely analogous to (1).
This completes the proof.

COROLLARY 2. *Suppose $\tau_{-x}(\mu)$ is scalarly centered at* 0. *Then we have $A(\mu)=x+K(\tau_{-x}(\mu))$.*

PROOF. $\tau_{-x}(\mu)$ is scalarly centered at 0 if and only if $x\in A(\mu)$, see Proposition 6. Thus the assertion follows by Theorem 1.

Let μ be an s-convex measure or a p-stable measure such that $s>-1$, $p>1$ and such that μ is Radon satisfying $\sup\{\mu(K)\,; K$ *is compact and convex*$\}=1$. Then the mean vector $m\in E$ exists. In fact, it is well-known that τ_μ is equivalent to the L^1-metric, see Borell [2] and de Acosta [1]. Moreover τ_μ is weaker than the Mackey topology as easily seen, which implies that the natural mapping $i:(E', \tau_k)\to L^1(E, \mu)$ is continuous, where τ_k denotes the Mackey topology. Taking the adjoint $i^*: L^\infty(E, \mu)\to E$, $m=i^*(1)$ is the mean vector, that is $x'(m)=\int_E x'(x)d\mu(x)$ for every $x'\in E'$.

COROLLARY 3. *Let μ be an s-convex or p-stable ($s>-1$, $p>1$) probability measure satisfying* $\sup\{\mu(K)\,; K$ *is compact convex*$\}=1$. *Let m be the mean vector of μ. Then we have $A(\mu)=m+K(\tau_{-m}(\mu))$.*

PROOF. It is sufficient to see that $\tau_{-m}(\mu)$ is scalarly centered at 0 by Corollary 2. Since $\tau_{-m}(\mu)$ is a centered s-convex or p-stable measure ($s>-1$, $p>1$), the assertion follows by Chevet [4], (2.3), Example 2.

NOTATIONS

$A_\mu=\{x\,;\,\mu\sim\tau_x(\mu)$ *(equivalent)*$\}$

$A_\mu^\sim=\{x\,;\,\mu\not\perp\tau_x(\mu)$ *(not singular)*$\}$

The subset A_μ (resp. $A_\mu^\sim$) is called the admissible translates (resp. the partially admissible translates) of μ, see Takahashi [10].

PROPOSITION 7. $A_\mu\subset A_\mu^\sim\subset K(\mu)$.

PROOF. The first inclusion is obvious. Suppose that $x\in A_\mu^\sim$ and $x\notin K(\mu)$ for some $x\in E$. Since $x\notin K(\mu)$, there exists a sequence x_n' in E' such that $\mu(y\,;\,x_n'(y)\to 0)=1$ and $x\notin\{y\,;\,x_n'(y)\to 0\}$, see the definition of $K(\mu)$. So it follows that $\mu(Z)=1$, $\tau_x(\mu)(Z+x)=1$ and $Z\cap(Z+x)=\phi$, where $Z=\{y\,;\,x_n'(y)\to 0\}$. This means that μ and $\tau_x(\mu)$ are singular, which contaradicts to $x\in A_\mu^\sim$.

§ 4. 0-1 laws for kernels

Let E be a locally convex Hausdorff space, μ be a probability measure on $C(E, E')$ and τ_μ be the topology of convergence in measure restricted on E'. Let $(E')^a$ be the algebraic dual of E'. Then the dual $(E', \tau_\mu)'$ is a linear subspace of $(E')^a$. We may regard μ a probability measure on $C((E')^a, E')$ naturally by the embedding $E\to(E')^a$. Let μ^* be the outer measure derived by μ.

The next lemma was proved in Okazaki and Takahashi [8], Theorem 2, but we give a proof for the sake of completeness. See also Kwapien and Smolenski [7].

LEMMA 1. *Suppose that $\mu^*((E', \tau_\mu)')=1$. Then (E', τ_μ) is a locally convex nuclear semi-metric space.*

PROOF. Let $V_n=\{x'\,;\,\mu(x\,;\,x'(x)>1/n)<1/n\}$ be the basis of neighborhoods of 0 in τ_μ, $V_n^\circ=\{z\in(E')^a\,;\,|x'(z)|\leqq 1$ *for every* $x'\in V_n\}$. First we show that τ_μ equals the uniform convergence topology on each V_n° (the local convexity of τ_μ). Assume that $x_n'\to 0$ in τ_μ. For every m and j, there exists $N=N(m, j)$ such that $jx_n'\in V_m$ for every $n>N$, that is, $\sup\{|x_n'(x)|\,;\,x\in V_m^\circ\}\leqq 1/j$ for $n>N$. Thus τ_μ is stronger than the uniform convergence topology on each V_n°. Note that $(E', \tau_\mu)'=\cup V_n^\circ$. Since $\mu^*(\cup V_n^\circ)=1$, the converse is obvious.

Remark that each V_n° is $\sigma((E')^a, E')$-compact,so we may assume that μ is a $\sigma((E')^a, E')$-Radon measure concentrated on $\cup V_n^\circ$ since $\mu^*(\cup V_n^\circ)=1$. Let $U_n=\{x'\in E'\,;\,|x'(x)|\leqq 1$ *for every* $x\in V_n^\circ\}$. Then $\{U_n\}$ is a basis of neighborhoods of 0 in τ_μ and $V_n\subset U_n$. For every but fixed n, take $m, j>n$

such as $\mu(U_j^\circ)\geqq 1-1/m$. We shall show that the natural mapping $E_{U_j}\to E_{U_n}$ is p-summing for every $p>0$, where E_{U_n} is the seminormed space with the unit ball U_n. For every $x'\in U_n$ we have

$$\int_{U_j^\circ\cap\{x\,;\,|x'(x)|>1/n\}}|x'(x)|^p d\mu(x)\geqq n^{\frac{1}{p}(\frac{1}{n}-\frac{1}{m})},$$

which implies that

$$|x'|_{U_n}^p\leqq n^{p+1}m/(m-n)\int_{U_j^\circ}|x'(x)|^p d\mu(x),$$

where $|\ |_{U_n}$ is the gauge seminorm of U_n. Thus the natural mapping $E_{U_j}\to E_{U_n}$ is p-summing by Pietsch [9], Theorem 2.3.3. By Pietsch [9], Theorem 4.1.5, it follows that (E', τ_μ) is nuclear. This completes the proof.

THEOREM 2. *Suppose that μ is uniform. Then it holds that $\mu^*(K(\mu))=0$ or 1. If $\mu^*(K(\mu))=1$, then* $\dim K(\mu)<\infty$.

PROOF. Assume that $\mu^*(K(\mu))>0$. Let $V_n=\{x'\in E'\,;\,\mu(x\,;\,x'(x)>1/n)<1/n\}$ and $B_n=\{x\in E\,;\,|x'(x)|\leqq 1$ for every $x'\in V_n\}=V_n^\circ\cap E$. Remark that $K(\mu)\subset\cup B_n=(E', \tau_\mu)'\cap E$. In fact, for each $x\in K(\mu)$, if $x_n'\to 0$ in τ_μ, then for every subsequence $\{x_{n_j}'\}$ such that $x_{n_j}'\to 0$ μ-almost everywhere, it follows that $x_{n_j}'(x)\to 0$ by the definition of $K(\mu)$. Hence $x'\to x'(x)$ is τ_μ-continuous for every $x\in K(\mu)$. Since $\mu^*(\cup B_n)>0$, there exists an n such that $\mu^*(B_n)>0$. Take $C\in C(E, E')$ such that $B_n\subset C$ and $\mu(C)=\mu^*(B_n)>0$. Let ν be the resriction of μ to C, that is, $\nu(A)=\mu(A\cap C)/\mu(C)$. By the uniformness of μ, it follows that $\tau_\nu\sim\tau_\mu$ (equivalent). We have $\nu^*((E', \tau_\nu)')=\nu^*((E', \tau_\mu)')\geqq\nu^*(B_n)=1$. Consequently by Lemma 1, it follows that (E', τ_ν) and (E', τ_μ) are nuclear locally convex spaces. We show further that (E', τ_μ) is a seminormed space. We prove that τ_μ is equivalent to the uniform convergence topology on B_n. Suppose that $x_n'\to 0$ uniformly on B_n. Then $\mu(x\,;\,x_n'(x)\to 0)\geqq\mu^*(B_n)>0$, which implies $x_n'\to 0$ in τ_μ by the uniformness. Conversely, if $x_n'\to 0$ in τ_μ, then $x_n'\to 0$ uniformly on each V_n° as proved in the proof of Lemma 1, in particular, $x_n'\to 0$ uniformly on B_n. Thus we have proved that if $\mu^*(K(\mu))>0$, then (E', τ_μ) is a nuclear seminormed space. So we have $\dim(E', \tau_\mu)'<\infty$. Since $K(\mu)\subset(E', \tau_\mu)'$, it follows also $\dim K(\mu)<\infty$. Now we show that $\mu^*(K(\mu))=1$. Take any $D\in C(E, E')$ such that $K(\mu)\subset D$. By the definition of the cylindrical σ-algebra $C(E, E')$, there exists a sequence $\{x_n'\}$ and a Borel subset B in $\boldsymbol{R}^\infty$ such that $D=\Pi^{-1}(B)$, where $\Pi: E\to\boldsymbol{R}^\infty$ be $\Pi(x)=\{x_n'(x)\}$. Let $\Pi(\mu)$ be the image measure. Then $\Pi(\mu)$ is uniform by Proposition 1. Since $\Pi(K(\mu))$ is a finite dimensional subspace of $\boldsymbol{R}^\infty$, it is a closed subspace. If we remark that

$\Pi(\mu)(\Pi(K(\mu)))=\mu(\Pi^{-1}(\Pi(K(\mu)))\geqq\mu^*(K(\mu))>0$, it holds that $\Pi(\mu)(\Pi(K(\mu)))=1$ by Proposition 2. Since $B\supset\Pi\ \Pi^{-1}(B)=\Pi(D)\supset\Pi(K(\mu))$, it follows that $\mu(D)=\Pi(\mu)(B)=1$, which proves the assertion.

This completes the proof.

PROPOSITION 8. *Let μ be a Radon probability measure such that τ_μ is locally convex and weaker than the Mackey topology. Then if* $\dim K(\mu)<\infty$, *it holds that $\mu(K(\mu))=1$.*

PROOF. Since τ_μ is weaker than the Mackey topology, we have $(E', \tau_\mu)'\subset E$ and $K(\mu)=(E', \tau_\mu)'$. In fact, the inclusion $K(\mu)\subset(E', \tau_\mu)'$ is always true, see the proof of Theorem 2, and the converse is proved as follows. Let $Z=\{y\,;x_n'(y)\to 0\}$ be $\mu(Z)=1$. Then for every $x\in(E', \tau_\mu)'$, $x_n'(x)\to 0$ since x_n' converges to 0 in τ_μ. Thus we have $x\in Z$, which implies the assertion. By the assumption $K(\mu)=(E', \tau_\mu)'$ is a closed subspace. We have $K(\mu)=\cap\{\ker x'\,;x'\in K(\mu)^\perp\}$ where $K(\mu)^\perp=\{x'\in E'\,;x'(y)=0$ *for every* $y\in K(\mu)\}$. For every $x'\in K(\mu)^\perp$, $x'(y)=0$ for every $y\in(E', \tau_\mu)'$ and τ_μ is locally convex, so it follows that $x'=0$ in (E', τ_μ), that is, $x'(x)=0$ μ-almost everywhere. We have proved that $\mu(\ker x')=1$ for every $x'\in K(\mu)^\perp$. Thus by the argument similar to the proof of Proposition 2, it follows that $\mu(K(\mu))=1$.

COROLLARY 4. *Suppose that μ is uniform and let $L=A_\mu-A_\mu$ or $L=A_{\tilde\mu}-A_{\tilde\mu}$. If $\mu^*(L)>0$, then we have* $\dim(\text{span } L)<\infty$, *where* span L *is the linear span of L.*

PROOF. The assertion follows by $L\subset K(\mu)$ (Proposition 7).

REMARK 1. There exists a measure (not uniform) such that span $A_{\tilde\mu}=E$, and $\dim E=\infty$, see Takahashi and Okazaki [11].

The 0-1 law for $K(\mu)$ is valid for $\tau_x(\mu)$, where $x\in E$ is arbitrary.

THEOREM 3. *Suppose that μ is uniform and $x\in E$ be arbitrary. Then it holds that $\tau_x(\mu)^*(K(\mu))=0$ or 1. If $\tau_x(\mu)^*(K(\mu))=1$, then* $\dim K(\mu)<\infty$.

PROOF. We show in fact that if $\tau_x(\mu)^*(K(\mu))=\mu^*(K(\mu)-x)>0$, then $x\in K(\mu)$. Then the assertion follows by Theorem 2. Assume that $x\notin K(\mu)$. Then there exists a linear subspace Z of the form $Z=\{y\,;x_n'(y)\to 0\}$ such that $x\notin Z$ and $\mu(Z)=1$. Since $Z\cap(Z+x)=\phi$, we have $\mu(Z+x)=0$, which contradicts to $\mu^*(K(\mu)+x)>0$.

This completes the proof.

In the sequel, we examine the 0-1 laws for $A(\mu)$, $SK(\mu)$, $C(\mu)$, $SC(\mu)$ and $SA(\mu)$.

THEOREM 4. *Suppose that* μ *is uniform. Then it holds that* $\mu^*(A(\mu))=0$ *or* 1. *If* $\mu^*(A(\mu))=1$, *then* $\dim(\mathrm{span}\, A(\mu))<\infty$.

PROOF Since $A(\mu)\subset K(\mu)$, if $\mu^*(A(\mu))>0$, then it follows that $\mu^*(K(\mu))=1$ and $\dim K(\mu)<\infty$ by Theorem 2. We may regard μ a probability measure concentrated on the finite dimensional subspace $K(\mu)$, in particular μ is Radon. By Proposition 4, we have $A(\mu)=\cap\{x+Z\,;\,x\in E,\ Z=\{y\,;\,x_n'(y)\to 0\},\ x_n'\in K(\mu)',\ \mu(Z+x)=1$ and $Z+x\subset K(\mu)\}$. For every decreasing net F_α of closed subsets we have $\mu(\cap F_\alpha)=\inf_\alpha \mu(F_\alpha)$ since μ is a Radon measure on the finite dimensioal space $K(\mu)$. Remark that $x+Z\subset K(\mu)$ is closed since $\dim K(\mu)<\infty$. Thus by the way similar to the proof of Proposition 2, we have $\mu(A(\mu))=1$.

This completes the proof.

THEOREM 5. *Suppose that* μ *is uniform. Then it hold that* $SK(\mu)=K(\mu)$ *and* $\mu^*(SK(\mu))=0$ *or* 1. *If* $\mu^*(SK(\mu))=1$, *then* $\dim \mathrm{SK}(\mu)<\infty$.

PROOF. $SK(\mu)\subset K(\mu)$ is clear. To show the converse, let $Z=\{x\,;\,x_n'(x)\to 0\}$ be $\mu(Z)>0$. We prove $K(\mu)\subset Z$. If $y\notin Z$, then $x_n'(y)\not\to 0$. So there exists a subsequence $\{x_{n_k}'\}$ and $\varepsilon>0$ such that $|x_{n_k}'(y)|\geqq\varepsilon$ $(k=1,2,\ldots)$. Put $Z_1=\{x\,;\,x_{n_k}'(x)\to 0\}$, then by $Z\subset Z_1$, we have $\mu(Z_1)>0$. Since μ is uniform, it follows that $x_{n_k}'\to 0$ in τ_μ. We can take a subsequence $\{x_{n_{k(i)}}'\}$ such that $x_{n_{k(i)}}'\to 0$ μ-a. e.. If we set $Z_2=\{x\,;\,x_{n_{k(i)}}'(x)\to 0\}$, then $\mu(Z_2)=1$. Since $y\notin Z_2$, it follows that $y\notin K(\mu)$. Other assertions follow from Theorem 2.

This completes the proof.

LEMMA 2. *Suppose that* $\mu^*(C(\mu))>0$. *Then* μ *is scalarly centered at* 0.

PROOF. Take any $x\in E$ and any Z of the form $Z=\{y\,;\,x_n'(y)\to 0\}$ such that $\mu(Z-x)=1$. Since $K(\tau_x(\mu))\subset Z$, it follows that $\mathrm{C}(\mu)\subset x+Z$ and hence $\mu(Z+x)>0$. Thus we have $(Z-x)\cap(Z+x)\neq\phi$, that is $x\in Z$, which shows that μ is scalarly centered at 0. This proves the lemma.

THEOREM 6. *Suppose that* μ *is uniform. Then it holds that* $\mu^*(C(\mu))=0$ *or* 1. *In fact if* $\mu^*(C(\mu))>0$ *then we have* $C(\mu)=K(\mu)$.

PROOF. If $\mu^*(C(\mu))>0$, then μ is scalarly centered at 0 by Lemma 2. By Theorem 1 (1), we have $C(\mu)=K(\mu)$. Thus the assertion follows by Theorem 2.

This completes the proof.

REMARK 2. There is an example of a uniform measure μ such that $\mu(K(\mu))=1$ and $\mu(C(\mu))=0$. For example, let μ be a probability measure on $\boldsymbol{R}^2$ without point mass concentrated on the affine subspace $H=\{(t, 1); t\in \boldsymbol{R}\}$. Then μ is uniform since μ satisfies the 0-1 law for closed subspaces. In this example, we have $K(\mu)=\boldsymbol{R}^2$ and $C(\mu)=\{(t; -1); t\in\boldsymbol{R}\}=-H$.

LEMMA 3. *Suppose that* $\mu^*(SC(\mu))>0$. *Then* μ *is strictly scalarly centered at* 0.

PROOF. The proof is analogous to that of Lemma 2.

THEOREM 7. *Suppose that* μ *is uniform. Then it holds that* $\mu^*(SC(\mu))=0$ *or* 1. *In fact if* $\mu^*(SC(\mu))>0$ *then we have* $SC(\mu)=SK(\mu)$.

PROOF. If $\mu^*(SC(\mu))>0$, then μ is strictly scalarly centered at 0 by Lemma 3. By Theorem 1 (2), we have $SC(\mu)=SK(\mu)$. Thus the assertion follows by Theorem 5.

This completes the proof.

THEOREM 8. *Suppose that* μ *is uniform. If* $\mu^*(SA(\mu))>0$, *then* $\dim(\operatorname{span} SA(\mu))<\infty$.

PROOF. Since $SA(\mu)\subset A(\mu)$, the assertion follows by Theorem 4.

REMARK 3. There is an example of a uniform measure μ such that $0<\mu^*(SA(\mu))<1$. Let ν_1 be a probability measure on $\boldsymbol{R}^2$ without point mass concentrated on $H=\{(t, 1); t\in\boldsymbol{R}\}$ and $\nu_2(A)=\lambda_G(A\cap\{(t, s); t\in\boldsymbol{R}, s<0\})$ where λ_G is the centered Gaussian measure on $\boldsymbol{R}^2$ with covariance matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Then $\mu=\nu_1/2+\nu_2$ is uniform since μ satisfies the 0-1 law for linear subspaces. In this example, we have $SA(\mu)=H$ and $\mu(SA(\mu))=1/2$.

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