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On Riemannian manifolds admitting infinitesimal projective transformations

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§ 1. Introduction.

Let M be a connected Riemannian manifold of dimension $n\geqq 2$. If a transformation of M preserves geodesics then it is called a projective transformation. Let X be a vector field in M and let $\{\phi_t\}$ be a 1-parameter group of transformations generated by X. For each t, if ϕ_t is a projective transformation then X is called an infinitesimal projective transformation. A vector field X is an infinitesimal projective transformation if and only if there exists a 1-form φ_i such that $\mathcal{L}_X\Gamma_{ji}^{h}=\delta_j^h\varphi_i+\delta_i^h\varphi_j$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to the vector field X, Γ_{ji}^{h} are the components of the Christoffel symbols of the metric tensor g of M and δ_i^h mean the Kronecker deltas. If φ_i vanishes identically then X is called an infinitesimal affine transformation. We denote ∇_k and K_{ji} the covariant differentiation with respect to the Riemannian connection and the components of the Ricci tensor.

The main purpose of the present paper is to show the following theorem.

THEOREM 1. *Let M be a connected complete Riemannian manifold with positive constant scalar curvature and satisfying the condition $\nabla_k K_{ji}=\nabla_j K_{ki}$. If M admits a non-affine infinitesimal projective transformation then M is a space of positive constant curvature.*

REMARK. The condition $\nabla_k K_{ji}=\nabla_j K_{ki}$ in Theorem 1 geometrically has the important meaning that it implies a constancy of the scalar curvature of M due to the second Bianchi identity. And the examples of Riemannian manifold which satisfy $\nabla_k K_{ji}=\nabla_j K_{ki}$ but not $\nabla_k K_{ji}=0$ are already known (cf. [1], [2], [3]). Furthermore it is well known that a conformally flat Riemannian manifold with constant scalar curvature satisfies $\nabla_k K_{ji}=\nabla_j K_{ki}$. Conversely, in the case of lower dimensions, it is easily shown that if M of dim $M=3$ (resp. dim $M=2$) satisfies $\nabla_k K_{ji}=\nabla_j K_{ki}$ then M is a conformally flat space (resp. a space of constant curvature). Many mathematicians studied Riemannian manifolds with the condition $\nabla_k K_{ji}=\nabla_j K_{ki}$ and they obtained many interesting results (cf. [1], [2], [3], [4], [8], [9], [10], [11], etc.).

In the case that M satisfies the assumption of compactness, the following results are known.

THEOREM A. (Yamauchi. [15]). *Let M be a compact connected Riemannian manifold with constant scalar curvature* (dim $M \geqq 3$). *If the scalar curvature is non-positive then an infinitesimal projective transformation is an isometry.*

THEOREM B. (Yamauchi. [16]). *Let M be a compact connected and simply connected Riemannian manifold with constant scalar curvature K* (dim $M \geqq 3$). *If M admits a non-isometric infinitesimal projective transformation, then M is isometric to a sphere of radius $\sqrt{n(n-1)/K}$.*

In the case without the assumption of compactness, the following results are known.

THEOREM C. (Solodovnikov. [12]). *Let M be a complete connected analytic Riemannian manifold* (dim $M \geqq 3$). *If M admits a non-affine infinitesimal projective transformation then M is a space of positive constat curvature.*

THEOREM D. (Nagano. [6]). *Let M be a complete connected Riemannian manifold with parallel Ricci tensor. If M admits a non-affine infinitesimal projective transformation then M is a space of positive constant curvature.*

Our Theorem 1 is a generalization of the above Theorem D. For the proof of Theorem 1, the following Theorem 2 and Theorem E play important roles.

THEOREM 2. *Let M be a connected Riemannian manifold and let $T(M)$ be the tangent bundle of M with complete lift metric. $T(M)$ admits a fibre-preserving infinitesimal conformal transformation if and only if M admits an infinitesimal projective transformation.*

THEOREM E. (Tanno. [14]). *Let M be a complete connected Riemannian manifold. If M admits a non-trivial solution f satisfying the following differential equations*

$$(*) \quad \nabla_k \nabla_j f_i + \alpha(2f_k g_{ji} + f_j g_{ik} + f_i g_{kj}) = 0, \ \alpha = const. > 0,$$

then M is a space of positive constant curvature, where $f_i = \nabla_i f$.

In fact, if M admits an infinitesimal projective transformation then $T(M)$ admits a fibre-preserving infinitesimal conformal transformation by Theorem 2. Using this fact and $\nabla_k K_{ji} = \nabla_j K_{ki}$, we can show that there exists a non-trivial solution of the differential equation (∗) in Tanno's Theorem E.

Following this introductory section, in § 2, we shall recall the complete lift metric of the tangent bundle $T(M)$ of M. In § 3, we shall give the proof of Theorem 2 and find a tensor equation which plays important roles in the proof of Theorem 1. In § 4, we shall prove Theorem 1. Lastly, in § 5, we shall consider infinitesimal projective transformations in $T(M)$ and prove the following theorem.

THEOREM 3. *Let M be a complete connected Riemannian manifold. If the tangent bundle $T(M)$ with complete lift metric of M admits a non-affine infinitesimal projective transformation then M is a locally flat.*

§ 2. The complete lift metric in the tangent bundle of a Riemannian manifold.

In this section we shall recall definitions and properties concerning the complete lift metric in the tangent bundle of a Riemannian manifold following Yano-Ishihara [19].

Let π be the natural projection of $T(M)$ to M and $\{U, x^h\}$ be a local coordinate neighborhood of M, then each $\pi^{-1}(U)$ admits the induced coordinates (x^h, y^h). If $\{U', x^{h'}\}$ is another coordinate neighborhood of M and $U\cap U'\neq\phi$, then the induced coordinates $(x^{h'}, y^{h'})$ in $\pi^{-1}(U')$ will be given by

$$(2.1)\qquad x^{h'}=x^{h'}(x),\ y^{h'}=\frac{\partial x^{h'}}{\partial x^h}y^h.$$

Putting $x^{\bar{h}}=y^h$, $x^{\bar{h}'}=y^{h'}$, we often write the equation (2.1) as $x^{P'}=x^P(x)$. The indices $a, b, c, \dots, h, i, j, \dots$ run over the range $1, 2, 3, \dots, n$ and the indices $A, B, C, \dots, P, Q, R, \dots$ run over the range $1, 2, 3, \dots, n, \bar{1}, \bar{2}, \bar{3}, \dots, \bar{n}$. The summation convention will be used with respect to this system of indices.

Suppose that we are given a Riemannian metric $\tilde{g}$ in $T(M)$ having local expression $\tilde{g}_{CB}dx^C dx^B=2g_{ji}dx^j\delta y^i$ with respect to the induced coordinates (x^h, y^h), where $\delta y^h=dy^h+N^h_a dx^a$, $N^h_i=y^a\Gamma^h_{ai}$. We call this metric the complete lift metric of g. Thus $\tilde{g}$ has components

$$(2.2)\qquad (\tilde{g}_{CB})=\begin{pmatrix}\partial g_{ji} & g_{ji}\\ g_{ji} & 0\end{pmatrix}$$

and contravariant components

$$(2.3)\qquad (\tilde{g}^{CB})=\begin{pmatrix}0 & g^{ji}\\ g^{ji} & \partial g^{ji}\end{pmatrix}$$

with respect to the induced coordinates in $T(M)$, where g^{ji} denote the contravariant components of g and $\partial = y^a \partial_a$.

We have already known that the Riemannian connection defined by $\tilde{g}$ coincides with the complete lift of the Riemannian connection of g. Thus the components $\tilde{\Gamma}^A_{BC}$ of the Christoffel symbols of $\tilde{g}$ are given by

$$
(2.4) \qquad \begin{aligned}
&\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji},\ \tilde{\Gamma}^h_{\bar{j}i} = 0,\ \tilde{\Gamma}^h_{j\bar{i}} = 0,\ \tilde{\Gamma}^{\bar{h}}_{ji} = \partial \Gamma^h_{ji},\\
&\tilde{\Gamma}^{\bar{h}}_{\bar{j}i} = \Gamma^h_{ji},\ \tilde{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji},\ \tilde{\Gamma}^h_{\bar{j}\bar{i}} = 0,\ \tilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0,
\end{aligned}
$$

with respect to the induced coordinates in $T(M)$.

§ 3. Fibre-preserving infinitesimal conformal transformations in T(M).

In this section we shall prove Theorem 2 and find a tensor equation which plays important roles in the proof of Theorem 1.

Let the tangent bundle $T(M)$ be endowed with the complete lift metric of M. A transformation of $T(M)$ is said to be fibre-preserving if it sends each fibre of $T(M)$ into a fibre. Let $\tilde{X}$ be a vector field with components $\begin{pmatrix} X^h \\ X^{\bar{h}} \end{pmatrix}$ with respect to the induced coordinates in $T(M)$. If $\tilde{X}$ generates a local 1-parameter group of fibre-preserving transformations then it is called a fibre-preserving infinitesimal transformation. A vector field $\tilde{X}$ is a fibre-preserving infinitesimal transformation if and only if the components X^h of $\tilde{X}$ depend only on the variables x^h with respect to the induced coordinates (x^h, y^h) in $T(M)$. If there exists a function $\tilde{\rho}$ in $T(M)$ such that

$$
(3.1) \qquad \mathcal{L}_{\tilde{X}} \tilde{g} = 2\tilde{\rho}\tilde{g}
$$

then $\tilde{X}$ is called an infinitesimal conformal transformation in the tangent bundle $T(M)$, where $\mathcal{L}_{\tilde{X}}$ denotes the Lie derivation with respect to the vector field $\tilde{X}$.

Let $\tilde{X}$ be an infinitesimal conformal transformation. Then using (3. 1), we get following formulas (3. 2) and (3. 3) by the relation $\mathcal{L}_{\tilde{X}} \tilde{\Gamma}^A_{BC} = \frac{1}{2} \tilde{g}^{AE} \left\{ \tilde{\nabla}_B (\mathcal{L}_{\tilde{X}} \tilde{g}_{EC}) + \tilde{\nabla}_C (\mathcal{L}_{\tilde{X}} \tilde{g}_{BE}) - \tilde{\nabla}_E (\mathcal{L}_{\tilde{X}} \tilde{g}_{BC}) \right\}$ and by the definitions of Lie derivative of $\tilde{g}$ and $\tilde{\Gamma}$:

$$
(3.2) \qquad X^E \partial_E \tilde{g}_{BC} + \tilde{g}_{EC}\, \partial_B X^E + \tilde{g}_{BE}\, \partial_C X^E = 2\tilde{\rho}\tilde{g}_{BC} ,
$$

$$
(3.3) \qquad \begin{aligned}
&\partial_B \partial_C X^A + X^E \partial_E \tilde{\Gamma}^A_{BC} - \tilde{\Gamma}^E_{BC}\, \partial_E X^A + \tilde{\Gamma}^A_{EC}\, \partial_B X^E + \tilde{\Gamma}^A_{BE}\, \partial_C X^E \\
&= \delta^A_B \tilde{\rho}_C + \delta^A_C \tilde{\rho}_B - \tilde{g}_{BC}\, \tilde{\rho}^A ,
\end{aligned}
$$

where $\tilde{\rho}_C$ means $\tilde{\nabla}_C\tilde{\rho}$, $\tilde{\nabla}_C$ denotes the covariant differentiation with respect to the Riemannian connection defined by $\tilde{g}$ and $\tilde{\rho}^A=\tilde{g}^{AE}\tilde{\rho}_E$.

PROOF OF THEOREM 2. Let $T(M)$ admits a fibre-preserving infinitesimal conformal transformation $\tilde{X}$ with componens $\begin{pmatrix} X^h \\ X^{\bar{h}} \end{pmatrix}$. Then the first n components X^h of $\tilde{X}$ depend only on the variables x^h and the formula (3.3) holds. Putting $A=h$, $B=j$ and $C=\bar{i}$ in (3.3) and taking account of (2.2), (2.3) and (2.4), we can show the function $\tilde{\rho}$ in $T(M)$ depends only on the variables x^h. Thus we can regard $\tilde{\rho}$ as a function in M. When we regard it as a function in M we express it as ρ. Since $\tilde{X}$ is a fibre-preserving infinitesimal transformation in $T(M)$, it induces a vector field X with components X^h in the base space M. Next, putting $A=h$, $B=j$ and $C=i$ in (3.3) and taking account of (2.2), (2.3) and (2.4), we obtain $\mathfrak{L}_X\Gamma^h_{ji}=\delta^h_j\rho_i+\delta^h_i\rho_j$ where $\mathfrak{L}_X$ denotes the Lie derivation with respect to the induced vector field X in M and $\rho_i=\nabla_i\rho$. This shows the induced vector field X in M is an infinitesimal projective transformation. Therefore if $T(M)$ admits a fibre-preserving infinitesimal conformal transformation then M admits an infinitesimal projective transformation.

Conversely, let X be an infinitesimal projective transformation with components X^h in M, that is, there exists a 1-form φ_i in M such that $\mathfrak{L}_X\Gamma^h_{ji}=\delta^h_j\varphi_i+\delta^h_i\varphi_j$. We put $\frac{1}{n+1}\nabla_aX^a=\rho$. Then from the classical relation on the Lie derivative of the Christoffel symbols: $\mathfrak{L}_X\Gamma^h_{ji}=\nabla_j\nabla_iX^h+K_{aji}{}^hX^a$, we get

$$\nabla_j\nabla_iX^h+K_{aji}{}^hX^a=\delta^h_j\varphi_i+\delta^h_i\varphi_j, \tag{3.4}$$

where $K_{kji}{}^h$ denote the componets of the curvature tensor of M. Contracting h and i in (3.4) we have $\varphi_j=\nabla_j\rho$, thus $(\varphi^h)=(g^{ha}\varphi_a)$ is a gradient vector field of ρ. Now we put $A^h_i=2\rho\delta^h_i-g^{ha}\mathfrak{L}_Xg_{ia}$ then they are the components of a (1,1) tensor field in M. Using this tensor field, we define $\tilde{X}=\begin{pmatrix} X^h \\ y^a(\partial_aX^h+A^h_a) \end{pmatrix}$. Then we can show, by using (2.1), $\tilde{X}$ is a vector field in $T(M)$. Thus $\tilde{X}$ is a fibre-preserving infinitesimal transformation in $T(M)$. Using (2.2) and the left hand equation of (3.2), we compute the Lie derivative of $\tilde{g}$ with respect to $\tilde{X}$. For example,

$$\begin{aligned}\mathfrak{L}_{\tilde{X}}\tilde{g}_{ji}&=X^m\partial_m(y^a\partial_ag_{ji})+y^a(\partial_aX^m+A^m_a)\partial_{\bar{m}}(y^b\partial_bg_{ji})\\&\quad+y^a\partial_ag_{mi}\partial_jX^m+g_{mi}\partial_j\{y^a(\partial_aX^m+A^m_a)\}+y^a\partial_ag_{jm}\partial_iX^m\\&\quad+g_{jm}\partial_i\{y^a(\partial_aX^m+A^m_a)\}\\&=y^a\{\partial_a\mathfrak{L}_Xg_{ji}+g_{mi}\nabla_jA^m_a+g_{jm}\nabla_iA^m_a+g_{mi}A^m_r\Gamma^r_{ja}+g_{jm}A^m_r\Gamma^r_{ia}\}\end{aligned}$$

$$
\begin{aligned}
&= y^a\{\nabla_a(\mathfrak{L}_X g_{ji}) + \Gamma^m_{aj}\mathfrak{L}_X g_{mi} + \Gamma^m_{ai}\mathfrak{L}_X g_{jm} + g_{mi}(2\rho_j\delta^m_a \\
&\quad - g^{mb}\nabla_j(\mathfrak{L}_X g_{ab})) + g_{jm}(2\rho_i\delta^m_a - g^{mb}\nabla_i(\mathfrak{L}_X g_{ab})) \\
&\quad + g_{mi}A^m_r\Gamma^r_{ja} + g_{jm}A^m_r\Gamma^r_{ia}\} \\
&= y^a\{\nabla_a(\mathfrak{L}_X g_{ji}) - \nabla_j(\mathfrak{L}_X g_{ai}) - \nabla_i(\mathfrak{L}_X g_{aj}) + 2\rho_j g_{ai} \\
&\quad + 2\rho_i g_{aj} + (\mathfrak{L}_X g_{mi} + g_{ri}A^r_m)\Gamma^m_{aj} + (\mathfrak{L}_X g_{jm} + g_{jr}A^r_m)\Gamma^m_{ai}\} \\
&= y^a\{-2g_{am}\mathfrak{L}_X\Gamma^m_{ji} + 2\rho_j g_{ai} + 2\rho_i g_{aj} + 2\rho\, g_{mi}\Gamma^m_{aj} + 2\rho\, g_{jm}\Gamma^m_{ai}\} \\
&= 2\rho y^a\partial_a g_{ji} \\
&= 2\rho\,\tilde{g}_{ji}.
\end{aligned}
$$

Then we obtain $\mathfrak{L}_{\tilde{X}}\tilde{g}=2\rho\tilde{g}$. Thus if M admits an infinitesimal projective transformation then $T(M)$ admits a fibre-preserving infinitesimal conformal transformation. This completes the proof of Theorem 2. Q. E. D.

Next we find a tensor equation which plays important roles in the proof of Theorem 1, that is, we show the following proposition.

PROPOSITION 3.1. *Let M admits an infinitesimal projective transformation X with components X^h. Then the following tensor equation holds*

$$
(3.5)\qquad A^a_k K_{aji}{}^h - A^h_a K_{kji}{}^a = -\delta^h_j\nabla_k\rho_i + g_{ki}\nabla_j\rho^h
$$

where A^h_i are the components of the $(1,1)$ tensor field defined in the proof of Theorem 2, $K_{kji}{}^h$ denote the components of the curvature tensor of M and $\rho^h=g^{ha}\rho_a$, $\rho_a=\nabla_a\rho$, $\rho=\frac{1}{n+1}\nabla_a X^a$.

PROOF. Since M admits an infinitesimal projective transformation $X=(X^h)$, the vector field $\tilde{X}$ in $T(M)$ with components $\begin{pmatrix} X^h \\ y^a(\partial_a X^h + A^h_a)\end{pmatrix}$ is a fibre-preserving infinitesimal conformal transformation by means of the proof of Theorem 2. Thus the formula (3.3) holds. Putting $A=\bar{h}$, $B=j$ and $C=i$ in (3.3) and taking account of (2.2), (2.3) and (2.4) we obtain

$$
\begin{aligned}
(3.6)\qquad & A^a_k K^h{}_{aji}{}^h - A^h_a K_{kji}{}^a + \nabla_j\nabla_i A^h_k + \Gamma^a_{jk}\nabla_i A^h_a + \Gamma^a_{ik}\nabla_j A^h_a \\
& + \partial_k(\mathfrak{L}_X\Gamma^h_{ji}) + (\Gamma^a_{kj}g_{ai} + \Gamma^a_{ki}g_{ja})\rho^h = 0.
\end{aligned}
$$

On the other hand, from the definition of A^h_i and the commutation formula for $\mathfrak{L}_X$ and ∇_k, we get (3.7) and (3.8), by using $\mathfrak{L}_X\Gamma^h_{ji}=\delta^h_j\rho_i+\delta^h_i\rho_j$,

$$
(3.7)\qquad \nabla_k A^h_i = -\delta^h_k\rho_i - g_{ki}\rho^h,
$$

$$
(3.8)\qquad \partial_k(\mathfrak{L}_X\Gamma^h_{ji}) = \delta^h_j(\nabla_k\rho_i + \Gamma^a_{ki}\rho_a) + \delta^h_i(\nabla_k\rho_j + \Gamma^a_{kj}\rho_a).
$$

Substituting (3.7) and (3.8) into (3.6), we obtain (3.5). Q. E. D.

§ 4. Proof of Theorem 1.

In this section we assume M be a complete Riemannian manifold with positive constant scalar curvature K and satisfying the condition $\nabla_k K_{ji}=\nabla_j K_{ki}$. And let M admits a non-affine infinitesiaml projective transformation X with components X^h. Then by definition, there exists a 1-form ρ_i such that

$$(4.1)\qquad \mathfrak{L}_X\Gamma^h_{ji}=\delta^h_j\rho_i+\delta^h_i\rho_j,$$

where $\rho_i=\nabla_i\rho$ and $\rho=\frac{1}{n+1}\nabla_a X^a$. Since X is a non-affine infinitesimal projective transformation, ρ is a non constant function in M. We shall show this function ρ is a solution of the differential equation (*) in Tanno's Theorem E.

Using the relation on the Lie derivative of the curvature tensor: $\mathfrak{L}_X K_{kji}{}^h=\nabla_k(L_X\Gamma^h_{ji})-\nabla_j(L_X\Gamma^h_{ki})$, we can prove

$$(4.2)\qquad \mathfrak{L}_X K_{ji}=-(n-1)\nabla_j\rho_i.$$

By means of the commutation formula for ∇_k and $\mathfrak{L}_X$ and by (4.1), (4.2) and the Ricci identity for $\nabla_j\nabla_k\rho_i-\nabla_k\nabla_j\rho_i$, we get

$$\begin{aligned}0&=\mathfrak{L}_X(\nabla_k K_{ji}-\nabla_j K_{ki})\\&=(n-1)K_{kji}{}^a\rho_a-\rho_k K_{ji}+\rho_j K_{ki}.\end{aligned}$$

Thus we obtain the following equations:

$$(4.3)\qquad K_{kji}{}^a\rho_a=\frac{1}{n-1}(\rho_k K_{ji}-\rho_j K_{ki}),$$

$$(4.4)\qquad K^a_k\rho_a=\frac{K}{n}\rho_k.$$

To prove Theorem 1 we prove the following proposition.

PROPOSITION 4.1. *The following equations hold*

$$(4.5)\qquad \Delta\rho+\frac{2(n+1)K}{n(n-1)}\rho=\textit{constant},$$

$$(4.6)\qquad \nabla_k\nabla_j\rho_h+\frac{(n+3)K}{n(n-1)}\rho_k g_{jh}-\frac{n+1}{n-1}\rho_k K_{jh}+\frac{1}{n-1}\rho_j K_{hk}+\frac{1}{n-1}\rho_h K_{kj}=0,$$

where $\Delta\rho=\nabla_a\rho^a$.

PROOF. Since M admits an infinitesimal projective transformation, from Proposition 3. 1, the equation (3. 5) holds. Transvecting (3. 5) with g^{ji}, we get $A_k^a K_a^h - A_a^h K_k^a = 0$. Applying ∇_h to the both sides of this equation and using (3. 7), (4. 4) and $\nabla_a K_k^a \left(= \frac{1}{2}\nabla_k K\right) = 0$, we obtain

$$(4.7) \qquad A^{ab}\nabla_a K_{bk} = 0,$$

where $A^{ji} = g^{ia} A_a^j$. On the other hand, from the definition of A_i^h we get

$$\begin{aligned}(4.8) \qquad A^{ab}\nabla_a K_{bk} &= (2\rho g^{ab} + \mathcal{L}_X g^{ab})\nabla_a K_{bk} \\ &= -g^{ab}\mathcal{L}_X(\nabla_a K_{bk}) \\ &= (n-1)\nabla_k\left(\Delta\rho + \frac{2(n+1)K}{n(n-1)}\rho\right).\end{aligned}$$

Thus from (4. 7) and (4. 8) we have (4. 5). Next, we apply $g^{ib}\nabla_b$ to the both sides of (3. 5) [the index h being lowered]. Then we can get (4. 6) by means of (3. 7), (4. 3), (4. 4), (4. 5) and $\nabla_i K_{kjh}{}^i (= \nabla_k K_{jh} - \nabla_j K_{kh}) = 0$.

Q. E. D.

PROOF OF THEOREM 1. By the Ricci identity for $\nabla_k \nabla_j \rho_h - \nabla_j \nabla_k \rho_h$ and by (4. 3) and (4. 6), we can prove

$$(4.9) \qquad \rho_k G_{jh} = \rho_j G_{kh},$$

where we define $G_{jh} = K_{jh} - \dfrac{K}{n} g_{jh}$.

Using (4. 4) and (4. 9), we can easily show $\rho_k G_{jh} = 0$. Thus (4. 6) is rewriten in the form

$$\nabla_k \nabla_j \rho_h + \frac{K}{n(n-1)}(2\rho_k g_{jh} + \rho_j g_{hk} + \rho_h g_{kj}) = 0.$$

This shows the function ρ is the solution of the differential equation (*) in Tanno's Theorem E. And this solution ρ is non-trivial because it is a non constant function. Therefore M is a space of positive constant curvature by Theorem E. This completes the proof of Theorem 1. Q. E. D.

§ 5. Infinitesimal projective transformations in tangent bundles.

Let the tangent bundle $T(M)$ be endowed with the complete lift metric of M. A vector field $\tilde{X}$ in $T(M)$ is called an infinitesimal projective transformation if there exists a function $\tilde{\rho}$ in $T(M)$ such that $\mathcal{L}_{\tilde{X}}\tilde{\Gamma}_{BC}^A = \delta_B^A \tilde{\rho}_C + \delta_C^A \tilde{\rho}_B$, where $\tilde{\rho}_C = \tilde{\nabla}_C \tilde{\rho}$. And if $\tilde{\rho}$ is a constant function then $\tilde{X}$ is called an infinitesimal affine transformation.

In this section we shall prove Theorem 3. And to prove Theorem 3, we show the following fundamental proposition.

PROPOSITION 5.1. *Let $\tilde{X}$ be an infinitesimal projective transformation with components* $\begin{pmatrix} X^h \\ X^{\bar{h}} \end{pmatrix}$. *Then X^h, $X^{\bar{h}}$ and $\tilde{\rho}$ are expressed in the following forms* :

(1) $X^h = y^a A_a^h + B^h$,

(2) $X^{\bar{h}} = y^a y^b(-\Gamma_{ar}^h A_b^r + \delta_a^h \varphi_b) + y^a(\partial_a B^h + C_a^h) + F^h$,

(3) $\tilde{\rho} = y^a \varphi_a + \psi$,

where A_i^h, C_i^h are the components of (1, 1) *tensor fields in M, B^h, F^h are the components of contravariant vector fields in M, ψ a function of x^h only and $\varphi_i = \nabla_i A_a^a$. And furthermore the following equations hold* :

(4) $\nabla_j A_i^h = \delta_j^h \varphi_i$,

(5) $\nabla_j C_i^h = -\delta_j^h \psi_i$,

(6) $\nabla_j \varphi_i = 0$,

(7) $\nabla_j \psi_i = 0$,

(8) $\mathcal{L}_F \Gamma_{ji}^h = 0$,

(9) $\mathcal{L}_B \Gamma_{ji}^h = \delta_j^h \psi_i + \delta_i^h \psi_j$,

(10) $A_k^a K_{aji}{}^h = A_a^h K_{kji}{}^a = A_i^a K_{kja}{}^h = 0$,

(11) $C_k^a K_{aji}{}^h = C_a^h K_{kji}{}^a = C_i^a K_{kja}{}^h$,

where $\psi_i = \nabla_i \psi$ and $\mathcal{L}_B$, $\mathcal{L}_F$ denote the Lie derivations with respect to $B = (B^h)$ and $F = (F^h)$, respectively. Conversely, let A_i^h, C_i^h are the components of (1, 1) *tensor fields in M, B^h, F^h are the components of contravariant vector fields in M and ψ a function in M. If they satisfy* (4)$\sim$(11) *then the vector field $\tilde{X}$ whose components are defined by* (1) *and* (2) *is an infinitesimal projective transformation in $T(M)$, that is, $\tilde{\rho}$ defined by* (3) *satisfies* $\mathcal{L}_{\tilde{X}} \tilde{\Gamma}_{BC}^A = \delta_B^A \tilde{\rho}_C + \delta_C^A \tilde{\rho}_B$.

This Proposition 5. 1 will be proved by simple but long computation of $\mathcal{L}_{\tilde{X}} \tilde{\Gamma}_{BA}^A$. The detailed proof is omitted here.

The following lemma is useful to prove Theorem 3.

LEMMA. (cf. [5]). *If M is a complete connected Riemannian manifold which is not locally flat, then every homothetic transformation of M is an isometry.*

PROOF OF THEOREM 3. Let M be a complete connected Riemannian manifold and $T(M)$ admits an infinitesimal projective transformation $\tilde{X}$. Then, by Proposition 5. 1, (1)$\sim$(11) hold. Now, we define the contravariant vector fields $Z = (Z^h)$ and $W = (W^h)$ as $Z^h = A_a^h \varphi^a$ and $W^h = -C_a^h \psi^a$, respectively, where $\varphi^h = g^{ha}\varphi_a$ and $\psi^h = g^{ha}\psi_a$. Using (4), (5), (6) and (7), we obtain

(5.1) $\mathfrak{L}_Z g_{ji} = 2(\varphi_a\varphi^a) g_{ji}$ and $\varphi_a\varphi^a = \text{constant}$,

(5.2) $\mathfrak{L}_W g_{ji} = 2(\psi_a\psi^a) g_{ji}$ and $\psi_a\psi^a = \text{constant}$.

Thus Z and W are infinitesimal homothetic transformations. Therefore, by Lemma, Z and W are infinitesimal isometries if M is not locally flat. If Z and W are infinitesimal isometries then by (5.1) and (5.2) we obtain $\varphi_i = 0$ and $\psi_i = 0$. This shows $\tilde{\rho} = \text{constant}$ by (3). Thus $\tilde{X}$ is an infinitesimal affine transformation. This completes the proof of Theorem 3. Q. E. D.

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