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H-separable Extensions and Torsion Theories

In memory of Professor Akira Hattori

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Introduction. Let A be a ring with identity and B a subring of A with common identity. We shall say that A is *H-separable* over B if $A\otimes_{B}A$ is isomorphic to a direct summand of a finite direct sum of copies of A as (A, A)-bimodules. Let C be the center of A and $V_{A}(B)$ the commutator of B in A. Then it is well-known that A is H-separable over B iff the maping η : $A\otimes_{B}A\rightarrow \mathrm{Hom}_{C}(V_{A}(B), A)$ given by $\eta(a\otimes a')(v)=ava'$ for a, a' in A and v in $V_{A}(B)$ is an isomorphism and $V_{A}(B)$ is a finitely generated projective C-module [7, Theorem 1.1].

Recently K. Sugano [8] has pointed out that H-separable extensions of B have close connections with Gabriel topologies on B. He showed, among other things, that if A is left flat and H-separable over B then $V_{A}(V_{A}(B))$ is isomorphic to the localization of B with respect to the right Gabriel topology consisting of all right ideals $\mathfrak{b}$ of B such that $\mathfrak{b}A=A$, where $V_{A}(V_{A}(B))$ denotes the double commutator of B in A. Using this he then showed that if A is H-separable over B and B is regular then $B=V_{A}(V_{A}(B))$.

Motivated by his results we shall study in this paper H-separable extensions of B from the point of view of torsion theories. We shall begin with the study of the torsion class

$$T=\{M_{B}\mid M\otimes_{B}A=0\}$$

of mod-B. If ${}_{B}A$ is flat, then T is hereditary. This assumption, however, is not necessary for T to be hereditary. We shall introduce the notion of weakly flat B-modules and show that the weakly flatness of A ensures T to be hereditary. We shall provide an example to show that not all weakly flat modules are flat. It is shown in case A is H-separable over B a necessary and sufficient condition for $B\rightarrow V_{A}(V_{A}(B))$ to be a right flat epimorphism (Theorem 3.9) and also one for $B=V_{A}(V_{A}(B))$ to hold (Theorem 3.12).

We shall use M_{B} to denote a right B-module M and $M'\leqq M$ a submodule M' of M. Consequently $\mathfrak{a}\leqq B_{B}$ means that $\mathfrak{a}$ is a right ideal of B. For undefined notions about torsion theory we shall refer to [6]. For a right

B-module M and a left B-module N we denote its tensor product by $M\otimes N$ instead of $M\otimes_B N$.

1. Preliminaries. Let A be a ring, B a subring of A with common identity and $\nu: B\rightarrow A$ the inclusion map. Let

$$T=\{M_B \mid M\otimes A=0\}.$$

Then T is a torsion class of mod-B. We shall denote by t the associated idempotent radical. It is easy to see that if ${}_BA$ is flat, then T is hereditary. The following proposition, however, shows that it is not necessary to assume ${}_BA$ being flat for T to be hereditary.

A B-module ${}_BN$ is said to be *t-weakly flat* if the functor $-\otimes_B N$ is exact on all the exact sequence of right B-modules

$$0\rightarrow L'\rightarrow L\rightarrow L''\rightarrow 0$$

with $L\in T$. Obviously flat modules are t-weakly flat. The converse, however, is not the case in general. In the next section we shall characterize t-weakly flat modules using the notion of weakly divisible modules. By this characterization we shall provide an example of modules which are t-weakly flat but not flat.

PROPOSITION 1.1. *If ${}_BA$ is t-weakly flat, then T is hereditary.*

PROOF. Let us put

$$L=\{\mathfrak{b}\leqq B_B \mid \mathfrak{b}A=A\}$$

and show that if $\mathfrak{b}\in L$ and $b\in B$, then $(\mathfrak{b}:b)\in L$. In fact, the canonical map $B/(\mathfrak{b}:b)\rightarrow B/\mathfrak{b}$ induces the exact sequence $0\rightarrow B/(\mathfrak{b}:b)\otimes A\rightarrow B/\mathfrak{b}\otimes A$ by assumption. Hence $B/\mathfrak{b}\otimes A=0$ implies $(\mathfrak{b}:b)A=A$.

To apply [2, Theorem 3.5], we have to prove that for each $M(\neq 0)\in T$, there exists $x(\neq 0)$ in M such that $xB\in T$. Suppose that $0\neq M\in T$. Then there exists $x(\neq 0)$ in M, and the sequence $0\rightarrow xB\rightarrow M$ is exact. By assumption $0\rightarrow xB\otimes A\rightarrow M\otimes A$ is also exact. Hence $M\otimes A=0$ implies $xB\otimes A=0$. Thus $xB\in T$.

Now throughout this section assume

$$T=\{M_B \mid M\otimes A=0\}$$

is hereditary. Then t is left exact and we have

LEMMA 1.2. (1) *The corresponding right Gabriel topology is given by*

$$L=\{\mathfrak{b}\leqq B_B \mid \mathfrak{b}A=A\}.$$

(2) *L has a basis consisting of finitely generated right ideals of B.*

Proof. (1) This is clear.

(2) Let $\mathfrak{b} \in L$. Then $\mathfrak{b}A = A$ and hence

$$1 = \sum b_i a_i$$

for some $b_i \in \mathfrak{b}$ and $a_i \in A$. The right ideal $\sum b_i B$ is contained in $\mathfrak{b}$ and belongs to L.

Lemma 1.3. (1) *For each A-module N_A,*

$$t(N) = 0$$

regarding as a B-module via ν.

(2) *For each B-module M_B,*

$$t(M) = \mathrm{Ker}(f_M)$$

where f_M: $M \to M \otimes A$ is given by $x \to x \otimes 1$.

Proof. (1) Let $x \in t(N)$. Then $r_B(x)A = A$ and hence $xA = x \cdot r_B(x)A = 0$. Thus we have $x = 0$.

(2) First by (1) $t(M \otimes A) = 0$. Hence $t(M) \leqq \mathrm{Ker}(f_M)$. On the other hand, for each $x \in \mathrm{Ker}(f_M)$ and $a \in A$, we have $x \otimes a = (x \otimes 1)a = 0$. Thus $\mathrm{Ker}(f_M)$ is torsion and $\mathrm{Ker}(f_M) \leqq t(M)$.

It follows from this lemma that A_B is torsionfree. Furthermore, for each B-module M_B, the diagrm

$$\begin{array}{ccc} & M & \\ \cong \swarrow & & \searrow f_M \\ M \otimes B & \xrightarrow[1 \otimes \nu]{} & M \otimes A \end{array}$$

is commutative. It follows that if M_B is flat, then f_M must be a monomorphism. Hence M_B is torsionfree. In particular, if B is a regular ring, t must be zero.

Since

$$\begin{array}{ccc} B \otimes A & \xrightarrow{\nu \otimes 1} & A \otimes A \to A/B \otimes A \to 0 \\ \cong \searrow & & \swarrow \sigma \\ & A & \end{array}$$

is a commutative diagram with exact row, where σ is given by $a\otimes a'\to aa'$, it follows that σ is an isomorphism iff $A/B\otimes A=0$, i. e., $(A/B)_B$ is torsion. This also means, as is well-known, ν is an epimorphism in the category of rings [6, Proposition XI. 1 2].

Note that σ is an isomorphism iff

$$a\otimes 1=1\otimes a \text{ in } A\otimes A$$

holds for all $a\in A$. More generally we have

LEMMA 1. 4. *Let B' be a submodule of A_B such that $B\leqq B'\leqq A$. Then the following conditions are equivalent:*

(1) *B'/B is torsion.*

(2) *The canonical mapping $B'\otimes A\to A$ given by $b'\otimes a\to b'a$ is an isomorphism.*

(3) *For each $b'\in B'$,*

$$b'\otimes 1=1\otimes b' \text{ in } B'\otimes A$$

holds.

In case ${}_BA$ is flat, the above conditions are also equivalent to:

(4) *For each $b'\in B'$,*

$$b'\otimes 1=1\otimes b' \text{ in } A\otimes A$$

holds.

PROOF. Straightforward.

Let $\bar{B}$ be the closure of B_B in A_B, *i. e.*

$$\bar{B}=\{a\in A\mid a+B\in t(A/B)\}$$
$$=\{a\in A\mid (B:a)\in L\}.$$

Then $B\leqq \bar{B}\leqq A$ and $\bar{B}$ is a subring of A.

A B-module M_B is called *t-injective* if, given $\mathfrak{b}\in L$ and $f\in \mathrm{Hom}_B(\mathfrak{b}, M)$, there exists $\bar{f}\in \mathrm{Hom}_B(B, M)$ such that $\bar{f}|_{\mathfrak{b}}=f$.

LEMMA 1. 5. (1) *$\bar{B}/B$ is torsion and $A/\bar{B}$ is torsionfree.*

(2) *A_B is t-injective in case ${}_BA$ is flat.*

(3) *$\bar{B}_B$ is also t-injective in case ${}_BA$ is flat.*

PROOF. (1) follows from definition. Indeed these conditions characterize the closure $\bar{B}$.

(2) Given $\mathfrak{b}\in L$ and $f\in Hom_B(\mathfrak{b}, A)$. Since $B/\mathfrak{b}$ is torsion and ${}_BA$ is flat,

$$\mu\otimes 1:\mathfrak{b}\otimes A\to B\otimes A$$

is an isomorphism where $\mu : \mathfrak{b} \to B$ is the inclusion map. Hence, for each $b \in B$, there exist $b_i \in \mathfrak{b}$ and $a_i \in A$ such that

$$b \otimes 1 = (\mu \otimes 1)\,(\sum b_i \otimes a_i).$$

Define $\bar{f}: B \to A$ to be $b \to \sum f(b_i) a_i$. It is easy to see that $\bar{f}$ is well-defined and is a B-homomorphism. Particularly for $b \in \mathfrak{b}$, $(\mu \otimes 1)(b \otimes 1) = b \otimes 1$. Thus we have $\bar{f}(b) = f(b)$.

(3) [4, Proposition 0.6].

By this lemma and [3, Proposition 3] we have

PROPOSITION 1.6. *If ${}_B A$ is flat, then there is a unique ring isomorphism $h : \bar{B} \to B_t$ such that the diagram*

$$\begin{array}{ccc} B & \leqq & \bar{B} \\ {\scriptstyle \phi_B}\downarrow & \swarrow {\scriptstyle h} & \\ B_t & & \end{array}$$

is commutative, where ϕ_B denotes the canonical homomorphism with respect to the localization.

2. Weakly flat modules. Let R be a ring with identity. Apart from the torsion class T in Section 1, let t be an arbitrary preradical of mod-R and $T(t) = \{ M_R \mid t(M) = M \}$.

Recall that ${}_R M$ is t-weakly flat if $- \otimes_R M$ is exact on all the exact sequences

$$0 \to L' \to L \to L'' \to 0$$

of right R-modules with $L \in T(t)$. On the other hand, following Sato [5], we call N_R *t-weakly divisible* if $\mathrm{Hom}_R(-, N)$ is exact on all the exact sequences

$$0 \to L' \to L \to L'' \to 0$$

of right R-modules with $L \in T(t)$.

First we shall characterize t-weakly flat R-modules by using the notion of weakly divisibility.

THEOREM 2.1. *Let ${}_R M$ be an R-module. Then M is t-weakly flat iff M^* is t-weakly divisible, where $M^* = \mathrm{Hom}_Z(M, Q/Z)$ denotes the character module of M.*

PROOF. Let

$$0 \to L' \to L \to L'' \to 0$$

be an exact sequence of right R-modules wih $L \in T(t)$. Suppose that M is t-weakly flat. Then by definition

$$0 \to L' \otimes_R M \to L \otimes_R M \to L'' \otimes_R M \to 0$$

is exact. Since Q/Z is injective over Z, it follows that

$$0 \to (L'' \otimes_R M)^* \to (L \otimes_R M)^* \to (L' \otimes_R M)^* \to 0$$

is exact and hence so is

$$0 \to \mathrm{Hom}_R(L'', M^*) \to \mathrm{Hom}_R(L, M^*) \to \mathrm{Hom}_R(L', M^*) \to 0.$$

Thus M^* is t-weakly divisible. This argument may be reversed using the fact that Q/Z is a cogenerator over Z.

Using this theorem we now show that not all t-weakly flat modules are flat.

EXAMPLE. Let S be a left Artinian ring and I an ideal of S which is not a direct summand of ${}_S S$. Let $\bar{S} = S/I$ and put

$$R = \begin{pmatrix} S & \bar{S} \\ 0 & \bar{S} \end{pmatrix}.$$

Then this is a left Artinian ring and the mapping $f: R \to S$ given by $\begin{pmatrix} c & \bar{a} \\ 0 & \bar{b} \end{pmatrix} \to c$ is a ring homomorphism with $\mathrm{Ker}(f) = \begin{pmatrix} 0 & \bar{S} \\ 0 & \bar{S} \end{pmatrix}$, where $\bar{a}$ and $\bar{b}$ denote cosets containing a and b respectively. The left S-module $\bar{S}$ can be regarded as a left R-module via f and is not projective. Since $\bar{S}$ is R-isomorphic to $\begin{pmatrix} 0 & \bar{S} \\ 0 & 0 \end{pmatrix}$, it follows that ${}_R\begin{pmatrix} 0 & \bar{S} \\ 0 & 0 \end{pmatrix}$ is not projective and hence is not flat. On the other hand, $\mathrm{Ker}(f)$ is an idempotent ideal of R and is projective as a left R-module. Hence we can define a hereditary 3-fold torsion theory

$$(C_{\mathrm{Ker}(f)}, T_{\mathrm{Ker}(f)}, F_{\mathrm{Ker}(f)})$$

for mod-R [1, Theorem 6]. It is easy to see that the character module of ${}_R\begin{pmatrix} 0 & \bar{S} \\ 0 & 0 \end{pmatrix}$ is torsionfree with respect to $(C_{\mathrm{Ker}(f)}, T_{\mathrm{Ker}(f)})$. Thus ${}_R\begin{pmatrix} 0 & \bar{S} \\ 0 & 0 \end{pmatrix}$ is weakly flat with respect to this torsion theory by Theorem 2. 1.

3. H-separable extensions. Let A be a ring, B a subring of A with common identity and $\nu: B \to A$ the inclusion map as before. We will use the same notations as in Section 1.

We say $a \in A$ is *dominated* by ν [6, p. 225] if, for any ring S and ring homomorphisms $\alpha, \beta : A \to S$, $\alpha\nu = \beta\nu$ always implies $\alpha(a) = \beta(a)$. The set of elements of A dominated by ν is called the *dominion* of ν and is denoted by $\mathrm{Dom}(\nu)$. This is a subring of A containing B.

Applying [6, Proposition XI. 1. 1] we have

PROPOSITION 3. 1. *The following conditions on $a \in A$ are equivalent*:

(1) $a \in \mathrm{Dom}(\nu)$.

(2) *If N is an (A, A)-bimodule and $x \in N$ has the property that $bx = xb$ for all $b \in B$, then $ax = xa$.*

(3) $a \otimes 1 = 1 \otimes a$ *in* $A \otimes A$.

(4) *If N and N' are right A-modules and $f : N \to N'$ is a B-homomorphism, then $f(xa) = f(x) \cdot a$ for all $x \in N$.*

We see in particular from this proposition that if we take $N = A$, then (2) means that

$$\mathrm{Dom}(\nu) \leqq V_A(V_A(B)).$$

Also by (3) we have

$$\mathrm{Dom}(\nu) = \{a \in A \mid a \otimes 1 = 1 \otimes a \text{ in } A \otimes A\}.$$

Consider the torsion class

$$T = \{M_B \mid M \otimes A = 0\}$$

again and throughout this section assume T is hereditary. Then, as a consequence of Lemma 1. 4 we have $\bar{B} \leqq \mathrm{Dom}(\nu)$, since $\bar{B}/B$ is torsion and (1)⇒(4) in Lemma 1. 4 can be shown without the assumption that ${}_BA$ is flat. However, we shall prove this fact by using the following two lemmas, because it seems that Lemma 3. 2 may be of interest by itself.

LEMMA 3. 2. *$A/\mathrm{Dom}(\nu)$ is torsionfree.*

PROOF. Let $a + \mathrm{Dom}(\nu) \in t(A/\mathrm{Dom}(\nu))$. Then $(\mathrm{Dom}(\nu) : a)A = A$ and there exist some $b_i \in (\mathrm{Dom}(\nu) : a)$ and $a_i \in A$ such that $\sum b_i a_i = 1$. Since $ab_i \otimes 1 = 1 \otimes ab_i$ for each i, $a \otimes b_i a_i = ab_i \otimes a_i = (ab_i \otimes 1)a_i = (1 \otimes ab_i)a_i = 1 \otimes ab_i a_i$ for each i. Hence we have $a \otimes 1 = \sum a \otimes b_i a_i = \sum 1 \otimes ab_i a_i = 1 \otimes a$. Thus we see that $a \in \mathrm{Dom}(\nu)$.

LEMMA 3. 3. *Let B' be a submodule of A_B such that $B \leqq B' \leqq A$. If A/B' is torsionfree, then we have $\bar{B} \leqq B'$.*

PROOF. Obvious.

Summarizing the discussion above we obtain

PROPOSITION 3.4. $B \leqq \bar{B} \leqq \mathrm{Dom}(\nu) \leqq V_A(V_A(B)) \leqq A$.
However, we have

LEMMA 3.5. *If A is H-separable over B, then*

$$\mathrm{Dom}(\nu) = V_A(V_A(B)).$$

PROOF. Let $a \in V_A(V_A(B))$ and consider the isomorphism $\eta: A \otimes A \to \mathrm{Hom}_C(V_A(B), A)$ mentioned in Introduction. Then $\eta(a \otimes 1) = \eta(1 \otimes a)$ and hence $a \otimes 1 = 1 \otimes a$. Thus we have $a \in \mathrm{Dom}(\nu)$.

LEMMA 3.6. *If $_BA$ is flat, then*

$$\bar{B} = \mathrm{Dom}(\nu).$$

PROOF. By Lemma 1.4, $\mathrm{Dom}(\nu)/B$ is torsion. On the other hand, $A/\mathrm{Dom}(\nu)$ is torsionfree by Lemma 3.2. Thus $\mathrm{Dom}(\nu)$ has to coincide with $\bar{B}$.

THEOREM 3.7. *If A is H-separable over B and $_BA$ is flat, then we have*

$$B \leqq \bar{B} = \mathrm{Dom}(\nu) = V_A(V_A(B)) \leqq A.$$

Combining this theorem with Proposition 1.6, we have

COROLLARY 3.8 ([8, Theorem 2]). *If A is H-separable over B and $_BA$ is flat, then we have*

$$B_t \cong V_A(V_A(B)).$$

Sugano [8, Proposition 2] has shown that if A is H-separable over B, $_BA$ is flat and $V_A(V_A(B))$ is a direct summand of $_BA$, then the inclusion map $B \to V_A(V_A(B))$ is a right flat epimorphism. Concerning this, we shall give the following theorem which follows from [6, Theorem XI.2.1].

THEOREM 3.9. *Let A be H-separable over B and $_BA$ flat. Then the inclusion map $B \to V_A(V_A(B))$ is a right flat epimorphism iff $(B:x)\bar{B} = \bar{B}$ for all $x \in \bar{B}$.*

Now consider

$$L' = \{\mathfrak{b} \leqq B_B \mid \mathfrak{b}\bar{B} = \bar{B}\}.$$

Then we have

LEMMA 3.10. $L' \subseteq L$.

PROOF. Let $\mathfrak{b}\in L'$. Then $\mathfrak{b}\bar{B}=\bar{B}$. For each $b\in B$, there exist some $b_i\in\mathfrak{b}$ and $x_i\in\bar{B}$ such that $b=\sum b_ix_i$. Since $\bar{B}/B$ is torsion, it follows that $\cap(B:x_i)\in L$. If $b'\in\cap(B:x_i)$, then $bb'=\sum b_i(x_ib')\in\mathfrak{b}$. This means that $\cap(B:x_i)\leqq(\mathfrak{b}:b)$. Thus $(\mathfrak{b}:b)\in L$ and $B/\mathfrak{b}$ is torsion.

Let A be H-separable over B and ${}_BA$ flat. Assume that $\bar{B}$ is a direct summand of ${}_BA$. Then there exists some $C'\leqq{}_BA$ such that $A=\bar{B}\oplus C'$. For each $\mathfrak{b}\in L$, $A=\mathfrak{b}A=\mathfrak{b}\bar{B}\oplus\mathfrak{b}C'$ and hence $\bar{B}=\mathfrak{b}\bar{B}\oplus(\bar{B}\cap\mathfrak{b}C')=\mathfrak{b}\bar{B}$. Thus we have $L\subseteqq L'$ and by Lemma 3.10 $L=L'$. Since $\bar{B}/B$ is torsion, for each $x\in\bar{B}$, $(B:x)\in L=L'$. Therefore, by Theorem 3.9, the inclusion map $B\to V_A(V_A(B))$ is a right flat epimorphism.

Sugano [8, Theorem 3] has shown that if B is regular and A is H-separable over B, then $V_A(V_A(B))=B$, i.e. B has the double commutator property. Also he has shown in [7, Proposition 1.2] that if A is H-separable over B such that B is a left (or right) direct summand of A, then $V_A(V_A(B))=B$.

By Lemma 1.4, A/B is torsion iff $A=\mathrm{Dom}(\nu)$. On the contrary, we have

LEMMA 3.11. *A/B is torsionfree iff $B=\mathrm{Dom}(\nu)$.*

PROOF. The "if" part is trivial by Lemma 3.2. Now suppose that A/B is torsionfree. Then, by Lemma 1.3, the mapping $f_{A/B}:A/B\to A/B\otimes A$ given by $\bar{a}\to\bar{a}\otimes 1$ is a monomorphism, where $\bar{a}$ denotes the coset containing a. Let $\pi:A\to A/B$ be the canonical homomorphism and consider the mapping $\pi\otimes 1:A\otimes A\to A/B\otimes A$. For $a\in\mathrm{Dom}(\nu)$, $\bar{a}\otimes 1=(\pi\otimes 1)(a\otimes 1)=(\pi\otimes 1)(1\otimes a)=\bar{1}\otimes a=0$. Hence $\bar{a}=0$ and we have $a\in B$.

In particular, we obtain

THEOREM 3.12. *Let A be H-separable over B. Then $B=V_A(V_A(B))$ iff A/B is torsionfree.*

If B is regular, as we have shown in Section 1, $t=0$ and hence A/B is torsionfree. Thus [8, Theorem 3] is a direct consequence of Theorem 3.12. Furthermore, if B is a direct summand of A_B, then A/B is torsionfree. Hence if, in addition, we assume that A is H-separable over B, Theorem 3.12 implies that $B=V_A(V_A(B))$. Likewise if we assume that A is H-separable over B and B is a direct summand of ${}_BA$, then we have $B=V_A(V_A(B))$.

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