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Perfect Sets and Sets of Multiplicity

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§ **1.** The class of closed, uncountable sets and the class of closed sets of multiplicity (M-sets) have been very extensively investigated; each occurs as the class of removable sets for some problems in analysis. We describe some examples in which the first class is represented, roughly speaking, by the second. Let F be a compact set in Euclidean space and f a continuous map of F onto a Cantor set C. We require that

(a) For each element x of C, $f^{-1}(x)$ is a U-set (that is, not an M-set). The same is then true for $f^{-1}(A)$, whenever A is a closed, countable subset of C.

(b) For each perfect set P in C, $f^{-1}(P)$ is an M-set. We call such functions c. m. mappings (cardinality-multiplicity mappings) simply to have a name for them. The thee examples of c. m. mappings that follow are based on different principles and have specific properties that cannot be attained by a single construction. The idea of representing classes of sets by inverse images is from [3]; further comparisons are postponed to § 5.

§ **2. A simple example in R^2.** In this example and the next one, property (b) occurs in a stronger, somewhat peculiar form:

(b′) For each perfect set P in C, $f^{-1}(P)$ carries a probability measure μ, such that $\mu*\mu$ is absolutely continuous. Thus $f^{-1}(P)$ is an M_0-set.

Let C be represented as a closed set on the arc $0\leq\theta\leq\pi$ of the unit circle, let F be the set $\{re^{i\theta}: \theta\in C, 1\leq r\leq 2\}$ and let $f(re^{i\theta})=\theta$, so that (a) is obvious. As for (b′), let P be a perfect set in C, so that P carries a continuous probability measure λ. Then $f^{-1}(P)$ carries the measure $\mu=\lambda(d\theta)dr$, and assertion (b′) is simply an observation about the convolution of the linear measures on line segments which aren't parallel.

§ **3. An example in R^1.** We begin with an outline. To each element x in C we attach a probability measure $\mu(x)$, whose support, say $F(x)$, is a U-set; moreover $\mu(x)*\mu(y)$ is absolutely continuous whenever $x\neq y$. Then the function f is defined so that $f^{-1}(x)=F(x)$ for each x in C. The proof that f is single-valued and continuous is the most difficult point, and requires a detour in the method.

Step I. We suppose that C is contained in $[1,2]$ and contains no rational number. For $k=1,2,3,\dots$, we define $\phi_k(x)=[4^k x]$ so that $4^k\le\phi_k<2\cdot 4^k$ and ϕ^k is continuous on C. If $\phi_k(x)=\phi_l(y)$, for integers k, l and x, y in C, then $k=l$ and $|x-y|<4^{-k}$. We define a set $\sigma(x)$ of natural numbers as follows. Let $m=2^r+s$, with $0\le s<2^r$; then m belongs to $\sigma(x)$ if r is not one of the numbers $\phi_k(x)$. The set $\sigma(x)$ is enumerated $n_1(x)<n_2(x)<\dots<n_j(x)<\dots$, so that the numbers $n_j(x)$ are continuous functions on C.

Step II. Let B be a Cantor set whose wlements z are sequences $(\varepsilon_1, \varepsilon_2,\dots,\varepsilon_j,\dots)$, $\varepsilon_j=0,\ 1$. We define g on $C\times B$ by the formula

$$g(x,z)=\sum_1 \varepsilon_j 2^{-n_j(x)}.$$

For each x in C, $g(x,B)$ is a U-set. Indeed, there are infinitely many numbers $q=q(x)$, such that $\sigma(x)$ omits q and $q+1$. For any choice of the sequence z, $2^{q-1}g(x,z)\in[0,1/2]$ (modulo 1), since the numbers 2^{-q}, 2^{-1-q} are absent from the binary expansion of $g(x,z)$. Therefore $g(x,B)$ is a U-set; the details for sets on the torus are in [4I, pp. 317–345].

We shall determine a special closed set $B_1\subseteq B$, such that g is a homeomorphism on $C\times B_1$. Let λ be the usual product measure in B, so that $\sum_1^N\varepsilon_j=N/2+o(N)$ for almost all sequences (Borel's theorem on normal numbers). For B_1 we take any closed set, of positive λ-measure, on which this relation is valid with no exception.

Suppose now that there is an equality

$$\sum_1^\infty\varepsilon_j 2^{-n_j(x)}=\sum_1^\infty\varepsilon_j' 2^{-n_j(y)}$$

with x, y in C, $z=(\varepsilon_j)$, $z'=(\varepsilon_j')$ in B_1. Since $\sigma(x)$ and $\sigma(y)$ have infinite complements in the set of natural numbers, it must be true that $\varepsilon_j=0$ whenever $n_j\in\sigma(x)\setminus\sigma(y)$. If $x\neq y$, and $4^{-k}<|x-y|$, then $p=\phi_k(y)$ is different from all the numbers $\phi_l(x)$. The numbers $m=2^p+s$, $0\le s<2^p$ are contained in $\sigma(x)\setminus\sigma(y)$, and therefore $\varepsilon_j=0$ for $2^p\le j<2^{p+1}$. By the choice of the set B_1, this is impossible for large k, and this contradiction proves that $x=y$ (and consequently $z=z'$).

Thus g is a homeomorphism on $C\times B_1$ and f is then defined by $f(g(x,z))\equiv x$ whenever $x\in C$, $z\in B_1$. Now $f^{-1}(x)=g(x,B_1)$ and we have seen that this is a U-set.

Step III. Property (b′) is verified in two stages. For each fixed x, the mapping of B onto $g(x,B)$ transforms the measure λ onto a measure $\nu=\nu(x)$ such that, for all real u

$$\int \exp(-2\pi iu)\,d\nu \equiv \prod_{1}^{\infty}(1+\exp -2\pi i 2^{-n_j(x)}u)/2.$$

Now $\nu(x)*\nu(y)$ is absolutely continuous whenever $x \neq y$. In fact $\sigma(x) \cup \sigma(y)$ contains all natural numbers beginning with some $N \geq 1$. Therefore $\nu(x)*\nu(y)$ contains, as a convolution factor, the measure whose Fourier-Stiltjes transform is $\prod_{N}^{\infty}(1+\exp -2\pi i \cdot 2^{-n}u)/2$, and that measure is absolutely continuous.

Now B_1 carries a probability measure λ_1, such that $\lambda_1 < \lambda$; writing $\mu(x)$ for the transform of λ_1 by g, we see that $\mu(x)*\mu(y) < \nu(x)*\nu(y)$, whence $\mu(x)*\mu(y)$ is absolutely continuous. The conclusion of the proof of (b') then follows the lines of the first example.

§ **4. A further example.** Let H be a compact M-set in R^1. We construct a c. m. mapping whose domain F is contained in H. It is clear that the method cannot be the same as in the first two examples ; it can be adapted easily to all Euclidean spaces and even some totally disconnected groups. By hypothesis H carries a distribution $S \neq 0$, such that $S^\wedge$ is bounded and vanishes at $\pm\infty$. Now $S^\wedge$ is uniformly continuous, because H is compact, and it is no loss of generality to suppose that $S^\wedge(0) = \|S^\wedge\|_\infty = 1$. The example is explained through one Lemma and two constructive steps.

LEMMA. *To each* $0 < \delta < 1$ *there are functions* $f_1 \geq 0$, $f_2 \geq 0$ *with disjoint supports, both having a Fourier expansion*

$$f = \sum a_k \exp 2\pi ikx, \quad a_0 = 1$$
$$\sum |a_k| < +\infty, \quad |a_k| < \delta \text{ for } k \neq 0.$$

PROOF. Let μ be a probability measure on the torus of length 1, such that $\hat{\mu} = 0$ at $\pm\infty$, and the closed support of μ has Lebesgue measure 0. Then $\mu \sim \sum b_k \exp 2\pi ikx$ and $|b_k| < \delta$ for $|k| \geq p > 1$. Therefore $\lambda \sim \sum b_{kp} \exp 2\pi ikpx$ has the same properties as μ and $|\hat{\lambda}(k)| < \delta$ when $k \neq 0$. To find f_1, we convolve λ with a smooth function of period 1, taking care that the closed support of f_1 has Lebesgue measure $< \delta/2$. We can then find f_2, with support disjoint from that of f_1, and $\int_0^1 |f_2(t) - |dt < \delta$, whence $|\hat{f}_2(k)| < \delta$ for $k \neq 0$.

To construct the example we form $S(\varepsilon_1), \dots, S(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j)$, with $\varepsilon_j = 0, 1$. At each stage, intermediate approximations are denoted by T. For simplicity we give the details for $S(\varepsilon_1)$.

Step I. $T(0)$ will be $f_1(Nt)\cdot S$ for a suitable integer N, and, for example $\delta<1/2$. The Fourier transform of $f_1(Nt)\cdot S$ at u is $S^\wedge(u)+\sum' a_k S^\wedge(u-Nk)$, where $\sum'$is a sum over integers $k\neq 0$. Since $\sum|a_k|<+\infty$ and $S^\wedge(\pm\infty)=0$, the second sum can be estimated uniformly by $\delta+o(1)$, for large N. Doing the same for $T(1)=f_2(Nt)\cdot S$, we obtain distributions with disjoint supports.

Step II. We fix once and for all a smooth function $h\geq 0$, of mean 1 and period 1, vanishing on $(0, 1/2)$. We choose $S(0)=h(Mt)\cdot T(0)$ and $S(1)=h(M^2t)\cdot T(0)$ and let $M\to+\infty$. For $S(0)$ and $T(0)$ the Fourier transforms are uniformly close, outside a domain defined by $M^{3/4}<|u|<M^{5/4}$, for example, whereas for $S(1)$ and $T(1)$ the domain to be avoided is contained in $M^{7/4}<|u|<M^{9/4}$. Thus $(S(0)*S(1))^\wedge$ converges uniformly to $(T(0)*T(1))^\wedge$ as $M\to\infty$, and moreover $|S(0)^\wedge|<3/2$, $|S(1)^\wedge|<3/2$. The support of $S(0)$ is contained in H, and also in the set defined by $Mt\in[1/2, 1]$ (modulo 1), and a similar statement is valid for $S(1)$.

Steps 1 and II are repeated in alternation, leading to distributions $S(z)=S(\varepsilon_1, \varepsilon_2, \ldots)$. We denote the Fourier transforms by $S^\wedge(z, u)$. With a little care in estimating the errors at each step, we obtain this situation.

(i) The closed support $F(z)$ of each $S(z)$ is a U-set contained in H, the union F all the sets $F(z)$ is compact, and the mapping of F determined by the condition $f^{-1}(z)=F(z)$ is continuous.

(ii) $|S^\wedge(z, u)|<3/2$, $|S^\wedge(z, 0)-1|<1/2$, and $S^\wedge$ is continuous in both variables. Whenever $z\neq z'$, then $S^\wedge(z, u)\cdot S^\wedge(z', u)$ vanishes when $|u|\to\infty$. These properties yield (b) in the same way as before.

§ **5.** Let g be a real function, continuous on a closed set of R^1, and $N(g)$ the set of real numbers t, such that $g^{-1}(t)$ is uncountable. Mazurkiewicz and Sierpinski [3] prove that

(α) $N(g)$ is an analytic set in R^1.

(β) Each analytic set in R^1 can be realized as a set $N(g)$ for a certain g.

Since the Cantor set can be mapped continuously onto $[0, 1]$, and the mapping is at most 2—to—1, each of our examples yields a variant of (β), with " M " in place of " uncountable." Related results using much more from the theory of Fourier transforms appear in [1, 2] ; we have learned indirectly that R. Solovay (Berkeley) has obtained examples of the same nature.

The cited works, and the present one, show that the complexity of the class of M-sets (or uncountable sets) in the class of all closed sets is the same as the complexity of analytic sets. The examples show directly how the class of M-sets is at least as complex as the class of uncountable sets. Unlike uncountable closed sets, M-sets are not preserved by an extensive class of homeomorphisms, so examples of c. m. mappings with particular properties (for example (b′) in the first two examples) must be based on a variety of techniques. The classes of M-sets and U-sets contain many interesting subclasses, each presenting its particular analytical problems.

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