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On flat H-separable extensions and Gabriel topology

Dedicated to Professor Akira Hattori on his 60th birthday

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0. Introduction. Throughout this paper every ring will be associative and have the identity 1, and every subring of it will contain 1. Further, every module over a ring will be unital. All terminologies and notations are the same as those in [6] and [7].

In the case where A is an algebra over a commutative ring R, A in an H-separable extension of R if and only if A is central separable over C and $C\otimes_{R}C\cong C$ by the map π such that $\pi(a\otimes b)=ab$, for $a, b\in C$ (See Proposition 1.1 [4]). The aim of this paper is to generalize this result to the case of non commutative ring extensions. We will show that in the case where A is an H-separable extension of a subring B and flat as left B-module we have $V_{A}(V_{A}(B))\cong B_{(\mathfrak{F})}$, where $\mathfrak{F}$ is the Gabriel topology on B consisting of the right ideals $\mathfrak{a}$ of B such that $\mathfrak{a}A=A$ and $B_{(\mathfrak{F})}=\varinjlim \mathrm{Hom}(\mathfrak{a}_{R}, B_{R})$ (Theorem 2). Theorem 2 will induce an interesting result that if A is an H-separable extension of a regular ring B then $B=V_{A}(V_{A}(B))$ (Theorem 3). The rest of this paper will be devoted to the study on H-separable extension of a regular ring. We will give the complete improvement of Theorems 2 and 3 [9] concerning with H-separable extensions of full linear rings (Theorem 4).

1. Let R be a ring and M a flat left R-module. A right ideal $\mathfrak{a}$ of R satisfies $\mathfrak{a}M=M$ if and only if $R/\mathfrak{a}\otimes_{R}M=0$. Denote the class of all right ideals of R which satisfy this condition by $\mathfrak{F}$. $\mathfrak{F}$ is a Gabriel topology on R, namely, $\mathfrak{F}$ satisfies the following four conditions (see page 156 [6]).

(G. 1) If $\mathfrak{a}\in\mathfrak{F}$ and $\mathfrak{b}$ is a right ideal of R containing $\mathfrak{a}$, then $\mathfrak{b}\in\mathfrak{F}$.

(G. 2) If $\mathfrak{a}$, $\mathfrak{b}\in\mathfrak{F}$, then $\mathfrak{a}\cap\mathfrak{b}\in\mathfrak{F}$.

(G. 3) If $\mathfrak{a}\in\mathfrak{F}$, then $(\mathfrak{a} : a)\in\mathfrak{F}$ for any $a\in R$, where $(\mathfrak{a} : a)=\{r\in R\mid ar\in\mathfrak{a}\}$.

(G. 4) If $\mathfrak{a}$ is a right ideal of R, and if there exists $\mathfrak{b}\in\mathfrak{F}$ such that $(\mathfrak{a} : b)\in\mathfrak{F}$ for any $b\in\mathfrak{b}$, then $\mathfrak{a}\in\mathfrak{F}$.

Now for any Gabriel topology $\mathfrak{F}$ on R, we can construct rings $R_{(\mathfrak{F})}=\varinjlim \mathrm{Hom}(\mathfrak{a}_{R}, R_{R})$ and $R_{\mathfrak{F}}=\varinjlim \mathrm{Hom}(\mathfrak{a}_{R}, R/t(R)_{R})$, where $\mathfrak{a}$ runs over $\mathfrak{F}$, and $t(R)$ is the $\mathfrak{F}$-torsion submodule of R, which is an ideal of R. A left

R-module N is said to be $\mathfrak{F}$-divisible if $\mathfrak{a}N=N$ holds for each $\mathfrak{a}$ in $\mathfrak{F}$. Now let N be any $\mathfrak{F}$-divisible flat left R-module. We can give N a left $R_{(\mathfrak{F})}$-module structure as follows;

Let $x\in R_{(\mathfrak{F})}$, $n\in N$, and let $\xi\in \mathrm{Hom}(\mathfrak{a}_R, R_R)$ with $\mathfrak{a}\in\mathfrak{F}$ represent x. Since $\mathfrak{a}N=N$, we have $n=\sum a_i n_i$ with $a_i\in\mathfrak{a}$ and $n_i\in N$. Now put $xn=\sum\xi(a_i)n_i$. If x is represented by $\eta\in \mathrm{Hom}(\mathfrak{b}_R, R_R)$ with $\mathfrak{b}\in\mathfrak{F}$, too, then there exists $\mathfrak{c}\in\mathfrak{F}$ such that $\mathfrak{c}\subset\mathfrak{a}\cap\mathfrak{b}$ and $\xi|\mathfrak{c}=\eta|\mathfrak{c}$. Here $\xi|\mathfrak{c}$ means the restriction of ξ on $\mathfrak{c}$. Now we have in $R\otimes_R N$ $1\otimes n=\sum a_i\otimes n_i=\sum b_j\otimes m_j=\sum c_k\otimes p_k$ with $a_i\in\mathfrak{a}$, $b_j\in\mathfrak{b}$, $c_k\in\mathfrak{c}$ and m_j, n_i, $p_k\in N$, and consequently, $\sum\xi(a_i)\otimes n_i=\sum\xi(c_k)\otimes p_k=\sum\eta(c_k)\otimes p_k=\sum\eta(b_j)\otimes m_j$. Then $\sum\xi(a_i)n_i=\sum\eta(b_j)m_j$, which implies that xn does not depend on the choice of the representation of x nor $a_i\in\mathfrak{a}$, $n_i\in N$ with $\sum a_i n_i=n$. Thus this multiplication is well defined, and it is easy to see that this gives N an $R_{(\mathfrak{F})}$-module structure which is compatible with the original R-module structure of N. The formulae $(xy)n=x(yn)$, for x, $y\in R_{(\mathfrak{F})}$, $n\in N$, follows from Lemma IX 1.1 [6].

For any right R-module X denote the canonical R-homomorphism of X to $X_{(\mathfrak{F})}$ by φ_X. Then Lemma IX 1.4 [6] shows that if n in $X_{(\mathfrak{F})}$ is represented by $\xi\in\mathrm{Hom}(\mathfrak{a}_R, X_R)$ with $\mathfrak{a}\in\mathfrak{F}$, then $\varphi_X\xi(a)=na$ for each $a\in\mathfrak{a}$.

Now let $x\in R_{(\mathfrak{F})}$, and suppose that x is represented by $\xi\in\mathrm{Hom}(\mathfrak{a}_R, R_R)$ whth $\mathfrak{a}\in\mathfrak{F}$. For any $n\in N$, let $n=\sum a_i n_i$ with $a_i\in\mathfrak{a}$, $n_i\in N$. Then in $R_{(\mathfrak{F})}\otimes_R N$ we have $1\otimes xn=\sum 1\otimes\xi(a_i)n_i=\sum\varphi_R\xi(a_i)\otimes n_i=\sum xa_i\otimes n_i=x\otimes\sum a_i n_i=x\otimes n$. This means that $R_{(\mathfrak{F})}\otimes_R N\cong N$ by $x\otimes n\rightarrow xn$. By this isomorphism we have $\mathrm{Hom}({}_RN, {}_RN')=\mathrm{Hom}({}_{R_{(\mathfrak{F})}}N, {}_{R_{(\mathfrak{F})}}N')$ for any left $R_{(\mathfrak{F})}$-module N'. Next suppose that $xN=0$. Then for any $a\in\mathfrak{a}$ we see that $0=x(aN)=\xi(a)N$. Thus we have $\xi\in\mathrm{Hom}(\mathfrak{a}_R, Ann({}_RN)_R)$, and we can easily see that $Ann({}_{R_{(\mathfrak{F})}}N)=(Ann({}_RN))_{(\mathfrak{F})}$.

Now since $t(R)\subset Ann({}_RN)$, N is also a left $R/t(R)$-module. Then by the same method as the above argument we can give N a left $R_{\mathfrak{F}}$-module structure, and have $R_{\mathfrak{F}}\otimes_R N\cong N$, $\mathrm{Hom}({}_RN, {}_RN')=\mathrm{Hom}({}_{R_{\mathfrak{F}}}N, {}_{R_{\mathfrak{F}}}N')$ for any left $R_{\mathfrak{F}}$-module N' and $Ann({}_{R_{\mathfrak{F}}}N)=(Ann({}_RN))_{\mathfrak{F}}$. If $t(R)=Ann({}_RN)$, $R_{\mathfrak{F}}$ can be regarded as a subring of $\mathrm{Bicom}({}_RN)$.

Now we can obtain a little more explicite proof of Theorems 1.4 and 1.6 [5].

THEOREM 1. *$\mathfrak{F}=\{\mathfrak{a}_R\subset R_R\,|\,\mathfrak{a}M=M\}$ is a Gabriel topology, and M is a flat and faithful left $R_{\mathfrak{F}}$-module such that $R_{\mathfrak{F}}\otimes_R M\cong M$ and $Hom({}_RM, {}_RM)=Hom({}_{R_{\mathfrak{F}}}M, {}_{R_{\mathfrak{F}}}M)$. Thus $R_{\mathfrak{F}}$ can be regarded as a subring of $Bicom({}_RM)$* (See Theorem 1.4 and Theorem 1.6 [5]).

PROOF. Since M is R-flat, we have $0=Ann({}_RM)M=Ann({}_RM)\otimes_R M$. Then $Ann({}_RM)\subset t(R)$ (See page 156 [6]). Hence we have $Ann({}_RM)=t(R)$. Then the proof of this Theorem is immediate from the above argument.

2. Now we will consider the case where A is a ring and B is a subring of A such that A is left B-flat. Let $\mathfrak{F}$ be the Gabriel topology on B consisting of right ideals $\mathfrak{a}$ of B such that $\mathfrak{a}A=A$. Note that the functor L of Mod-B to Mod-$B_{(\mathfrak{F})}$ defined by $L(N)=N_{(\mathfrak{F})}$ for each N_B is left exact (see page 199 [6]). Thus we can regard $B_{(\mathfrak{F})}$ as a subring of A by virtue of the next lemma, which has already been proved by Y. Kurata in [4].

LEMMA 1. *A is an $\mathfrak{F}$-closed right B-module.*

PROOF. Let $\mathfrak{a}\in\mathfrak{F}$. Since $\mathfrak{a}A=A$ and ${}_BA$ is flat, the map $\pi_{\mathfrak{a}}$ of $\mathfrak{a}\otimes_B A$ to A defined by $\pi_{\mathfrak{a}}(a\otimes x)=ax$, for $a\in\mathfrak{a}$, $x\in A$, is an isomorphism. Let $1=\sum a_i x_i$ with $a_i\in\mathfrak{a}$, $x_i\in A$. Then $a\otimes 1=\sum a_i\otimes x_i a$ for each $a\in\mathfrak{a}$. Hence for any $\xi\in\mathrm{Hom}(\mathfrak{a}_B, A_B)$ we have $\xi(a)=\sum\xi(a_i)x_i a$. This means that A is $\mathfrak{F}$-injective. On the other hand if $x\mathfrak{a}=0$ for $x\in A$ and $\mathfrak{a}\in\mathfrak{F}$, then we have $x\mathfrak{a}A=xA=0$ and $x=0$. Thus A is $\mathfrak{F}$-torsion free.

Now φ_A is an isomorphism, and $B_{(\mathfrak{F})}$ can be identified with the subring $\mathrm{Im}(\varphi_A^{-1}L(i))$, where i is the inclusion map of B to A and L is the functor introduced above. Put $B^*=\mathrm{Im}(\varphi_A^{-1}L(i))$. Now we will investigate the structure of B*. Note that $B_{(\mathfrak{F})}$ and $A_{(\mathfrak{F})}$ can be identified with $B_{\mathfrak{F}}$ and $A_{\mathfrak{F}}$, respectively, since B and A are $\mathfrak{F}$-torsion free.

LEMMA 2. *Let $y\in A_{\mathfrak{F}}$ be represented by $\xi\in Hom(\mathfrak{a}_B, A_B)$ with $\mathfrak{a}\in\mathfrak{F}$. Then $\varphi_A^{-1}(y)=\sum\xi(a_i)x_i$, where $a_i\in\mathfrak{a}$, $x_i\in A$ with $\sum a_i x_i=1$.*

PROOF. By Lemma IX 1.4 [6] we have $\varphi_A\xi(a)=ya$ for each $a\in\mathfrak{a}$. *Let* $\sum a_i x_i=1$ with $a_i\in\mathfrak{a}$, $x_i\in A$. Then $\xi(a)=\sum\xi(a_i)x_i a$ for each $a\in\mathfrak{a}$ as is shown in the proof of Lemma 1. Then $ya=\varphi_A\xi(a)=\varphi_A(\sum\xi(a_i)x_i)a$, and we have $(\varphi_A(\sum\xi(a_i)x_i)-y)\mathfrak{a}=0$. But $A_{\mathfrak{F}}$ is $\mathfrak{F}$-closed by Proposition IX 1.8 [6]. Hence we have $y=\varphi_A(\sum\xi(a_i)x_i)$.

PROPOSITION 1. *$B^*(=Im(\varphi_A^{-1}L(i)))=\{\sum\xi(a_i)x_i \mid \mathfrak{a}\in\mathfrak{F},\ \xi\in Hom(\mathfrak{a}_B, B_B)\ \sum a_i x_i=1,\ a_i\in\mathfrak{a},\ x_i\in A\}\subset V_A(V_A(B))$. $B^*\otimes_B A\cong A$, and A is left B^*-flat.*

PROOF. If $\xi\in\mathrm{Hom}(\mathfrak{a}_B, B_B)$ with $\mathfrak{a}\in\mathfrak{F}$ and $\sum a_i x_i=1$ with $a_i\in\mathfrak{a}$, $x_i\in A$, then we have $\varphi_A(\sum\xi(a_i)x_i)=\sum L(i)(\varphi_B\xi(a_i))\varphi_A(x_i)=\sum L(i)(ya_i)\varphi_A(x_i)=L$

$(i)(y)\varphi_A(\sum a_i x_i)=L(i)(y)$ where y is the element of $B_{\mathfrak{F}}$ represented by ξ. Hence $\sum\xi(a_i)x_i\in\mathrm{Im}(\varphi_A^{-1}L(i))$. Conversely if $x\in\mathrm{Im}(L(i))$, x is represented by $i\xi$ for some $\xi\in\mathrm{Hom}(\mathfrak{a}_B, B_B)$, $\mathfrak{a}\in\mathfrak{F}$. Therefore the converse inclusion is immediate by Lemma 2. Next, let $\sum a_i x_i=1$ with $a_i\in\mathfrak{a}\in\mathfrak{F}$, $x_i\in A$ and $\xi\in\mathrm{Hom}(\mathfrak{a}_B, B_B)$, and put $D=V_A(B)$. Then we have $\sum a_i\otimes dx_i=\sum a_i\otimes x_i d$ in $\mathfrak{a}\otimes_B A$ for each $d\in D$, because both are mapped to d by the isomorphism $\pi_{\mathfrak{a}}:\mathfrak{a}\otimes_B A\to A$. Then $\sum\xi(a_i)x_i d=\sum\xi(a_i)dx_i=d\sum\xi(a_i)x_i$. This means that $B^*\subseteq V_A(D)$.

Let C be the center of A, and denote $D=V_A(B)$ and $B'=V_A(V_A(B))$. A is an H-separable extension of B if and only if D is C-finitely generated projective and the map η of $A\otimes_B A$ to Hom $({}_C D, {}_C A)$ defined by $\eta(x\otimes y)(d)=xdy$, for $x, y\in A$, $d\in D$, is an isomorphism.

THEOREM 2 *If A is an H-separable extension of B and left B-flat, then we have $B^*=B'$.*

PROOF. Let $b\in B'=V_A(D)$, and consider the isomorphism η of $A\otimes_B A$ to $\mathrm{Hom}({}_C D, {}_C A)$ introduced above. Clearly $\eta(b\otimes 1)=\eta(1\otimes b)$, and we have $b\otimes 1=1\otimes b$ in $A\otimes_B A$. Then since ${}_B A$ is flat, $b\otimes 1=1\otimes b$ holds in $B'\otimes {}_B A$, too, for each $b\in B'$. This implies $B'\otimes_B A\cong A$ by $\pi_{B'}$ $(\pi_{B'}(b\otimes x)=bx$, $b\in B'$, $x\in A)$. Then $(B'/B)\otimes_B A=0$, and we see that B'/B is $\mathfrak{F}$-torsion (see page 156 [6]). Thus if $b\in B'$, there exists $\mathfrak{a}\in\mathfrak{F}$ such that $b\mathfrak{a}\subset B$, and we see that the left multiplication $b^{(l)}$ of b on $\mathfrak{a}$ yields a right B-homomorphism of $\mathfrak{a}$ to B. Then for $a_i\in\mathfrak{a}$ and $x_i\in A$ with $\sum a_i x_i=1$, we have $b=\sum(b^{(l)}a_i)x_i\in B^*$. Thus we have $B'\subset B^*$. The converse inclusion has been shown in Proposition 1.

Let $\mathfrak{F}'$ denote the class of right ideals $\mathfrak{a}$ of B such that $\mathfrak{a}B_{\mathfrak{F}}=B_{\mathfrak{F}}$. Obviously we have $\mathfrak{F}'\subset\mathfrak{F}$ in the present situation. Now the problem whether or not B' is a right perfect localization of B in this case comes out. This is affirmative if $\mathfrak{a}B'=B'$ holds for each $\mathfrak{a}\in\mathfrak{F}$ by Theorem IX 2.1 [6]. Thus we have

PROPOSITION 2. *Let A be an H-separble extension of B, and suppose that A is left B-flat and B' is a left B-direct summand of A. Then B' is a perfect right localization of B.*

PROOF. We can prove this without using Theorem XI 2.1 [6]. In the proof of Theorem 2 we have shown that $b\otimes 1=1\otimes b$ holds in $B'\otimes {}_B A$ for each $b\in B'$. But now $B'\otimes_B B'$ is a direct summand of $B'\otimes_B A$. Hence for each $b\in B'$, $b\otimes 1=1\otimes b$ holds in $B'\otimes_B B'$. This implies $B'\otimes_B B'\cong B'$. Of course,

B' is also left B-fllat.

3. In this section we will deal with H-separable extensions of von Neumann regular ring. First applying Theorem 2 we obtain

THEOREM 3. *If B is a regular ring, and A is an H-separable extension of B, then we have $B=V_A(V_A(B))$.*

PROOF. First, A is left B-flat, since B is a regular ring. Next let $\mathfrak{a}\in\mathfrak{F}$. There exist finite $a_i\in\mathfrak{a}$ and $x_i\in A$ such that $\sum a_ix_i=1$. Put $\mathfrak{b}=\sum a_iB$. Then $\mathfrak{b}$ is also in $\mathfrak{F}$ and generated by an idempotent e in B, since B is regular. Then $A=\mathfrak{b}A=eA$, and we see that $e=1$. Thus $\mathfrak{b}=\mathfrak{a}=B$, and we see that $\mathfrak{F}=\{B\}$, which implies that $B'=B^*=B$.

REMARK. It is already known that A is also a regular ring under the assumption of Theorem 3 (See [3] Proposition 5. 4).

LEMMA 3. *Let B be a regular ring and A an H-separable extension of B. Then if A is finitely generated as left B-module, A is left B-finitely generated projective. In partiqular if B is a right B-direct summand of A, A is left B-finitely generated projective.*

PROOF. By assumption we see that A is left B-flat and $A\otimes_BA$ is left A-projective. Thus, if A is left B-finitely generated, A is left B-projective by Proposition I 11. 6 [6]. If B is right B-direct summand of A, A is left B-finitely generated by (2. 2) [8]. Hence A is left B-projective.

PROPOSITION 3. *Let B be an indecomposable regular ring, and A an H-separable extension of B. If A is left B-finitely generated, then A is an indecomposable regular ring and D is a simple artinian ring.*

PROOF. Let Z be the center of B. Then Z is a field, and we have $C\subset Z=$the center of D, because $B=V_A(V_A(B))$ by Theorem 3. Thus A is an indecomposable regular ring, and C is a field. On the other hand we have a ring isomorphism $D\otimes_RA^0\cong\mathrm{Hom}({}_BA, {}_BA)$ (See e. g. (1. 5) [8]). Now, $\mathrm{Hom}({}_BA, {}_BA)$ is regular, since B is regular and A is left B-finitely generated projective by Lemma 3. Then D has no nilpotent ideal, and we see that D is semisimple artinian, because $[D:C]<\infty$. Then D is simple, since its center Z is a field.

A ring A is a left full linear ring if there exist a division ring K and a left K-vector space V such that $A=\mathrm{Hom}({}_KV, {}_KV)$. This is indecomposable regular and left self injective. In [9] left full linear ring is called right closed irreducible ring, and sometimes was called right closed primitive ring. Now by Lemma 3 we can improve Theorem 3 [8] as follows;

THEOREM 4. *Let B be a left full linear ring. Then A is an H-separable extension of B if and only if following three conditions are satisfied ;*

(1) *A is also a left full linear ring*

(2) $V_A(V_A(B))=B$

(3) *$D(=V_A(B))$ is a simple C-algebra with $[D:C]<\infty$.*

If these conditions are satisfied, A is a free Frobenius extension of B having a left (or right) free basis over B consisting of $[D:C]$ elements.

PROOF. Now we can give a proof of the "only if" part different from the one given in [9]. Suppose that A is an H-separable extension of B. Then by Proposition 3 we have already (2) and (3). Also we can see that A is an indecomposable regular ring and right B-finitely generated projective. Furthermore, A is left A-injective by (2. 3) [8], since B is left B-injective. It is already well known that the left socle of B coincides with the right socle of B and is contained in every two sided ideal of B (See e. g., Theorem 15. 1 [1]). In addition, we have $\mathfrak{A}=A(B\cap\mathfrak{A})$ for every two sided ideal $\mathfrak{A}$ of A by (2. 2) [8], since B is a left B-direct summand of A. Thus ASA is contained in every two sided ideal of A, where S is the socle of B. Then, since A has the Jacobson radical 0, there exists a maximal right ideal I of A such that $ASA\not\subset I$. Put $M=A/I$. If M is not faithful, we have $I\supset \mathrm{Ann}({}_AM)\supset ASA$, a contradiction. Thus M is a faithful simple right A-module. On the other hand, B has a faithful minimal left ideal $\mathfrak{l}$. Then since $\mathfrak{l}M\neq 0$, there exists an x in M such that $0\neq\mathfrak{l}x\cong\mathfrak{l}$. This isomorphism is extended to a left B-homomorphism of M to B, since B is left B-injective. Thus we have $0\neq\mathrm{Hom}({}_BM, {}_BA)\supset\mathrm{Hom}({}_BM, {}_BB)$. Then since A is an H-separable extension of B, for an A-A-module $X=\mathrm{Hom}({}_CM, {}_CA)$ we have $\mathrm{Hom}({}_BM, {}_BA)=X^B\cong D\otimes_C X^A$ (See (1. 3) [8]). Thus we have $X^A=\mathrm{Hom}({}_AM, {}_AA)\neq 0$. This means that A has a faithful minimal left ideal isomorphic to M. Then by Theorem 15. 3 [1] A has also a faithful minimal right ideal $\mathfrak{r}$. Set $L=\mathrm{Hom}(\mathfrak{r}_A, \mathfrak{r}_A)$ and $A^*=\mathrm{Hom}({}_L\mathfrak{r}, {}_L\mathfrak{r})$. Clearly $A\subset A^*$, and we can easily see that $\mathfrak{r}$ is also a faithful minimal right ideal of A^*. While, we have $A^*=A\oplus N$ for some left A-submodule N of A^*, since A is left A-injective. Then $\mathfrak{r}=\mathfrak{r}A^*=\mathfrak{r}\oplus\mathfrak{r}N$, and we have $\mathfrak{r}N=0$. Hence $N=0$, and $A=A^*$. Thus we have (1), and have finished the proof of the "only if" part. Conversely, suppose that A and B satisfy the conditions (1), (2) and (3). Let $[D:C]=n$. Now we will show that A has a right $V_A(D)$-free basis consisting of n elements. This assertion has already been shown in Theorem 36. 2 [1] by a little bit different proof. By (1) there exist division ring K and a left K-module $\mathfrak{m}$ such that $A=\mathrm{End}({}_K\mathfrak{m})$. As usual we regard ${}_K\mathfrak{m}_A$ as a left $K\otimes$

${}_CD^0$-module. Now $K\otimes_C D^0$ is a simple artinian ring, since C coincides with the center of K and $[K\otimes_C D^0 : K]=n<\infty$. Hence $\mathfrak{m}$ is a direct sum of simple $K\otimes_C D^0$-submodules $\{I_\alpha\}_{\alpha\in\Lambda}$, which are mutually isomorphic and finite dimensional over K, where Λ is an index set. Let I be one of $\{I_\alpha\}$ chosen arbitrarily, and set $L=\mathrm{End}({}_K I_D)$, $S=\mathrm{End}({}_K I)$. Then $A=\mathrm{End}({}_K\mathfrak{m})$ is isomorphic to $M_\Lambda(S)$, the ring of $\Lambda\times\Lambda$-row finite matrices over S, and $V_A(D)=\mathrm{End}({}_K\mathfrak{m}_D)$ is isomorphic to $M_\Lambda(L)$. On the other hand, since $K\otimes_C D^0=I_1\oplus I_2\oplus\dots\oplus I_r$ with $I\cong I_j(1\leqq j\leqq n)$ as left $K\otimes_C D^0$-module, we have $K\otimes_C\mathrm{End}({}_CD^0)\cong\mathrm{End}\,({}_KK\otimes_C D^0)\cong M_r(S)$ and $K\otimes_C D^0\cong\mathrm{End}({}_{K\otimes_C D^0}K\otimes_C D^0)\cong M_r(L)$. Here, $K\otimes_C\mathrm{End}({}_CD^0)$ is a right $K\otimes_C D^0$-free module (See Proposition 3.4 [10]), and its rank is $[K\otimes_C\mathrm{End}({}_CD):K]/[K\otimes_C D:K]=n$. Hence S must be a right L-free module of rank n, and we see that $M_\Lambda(S)$ has a right $M_\Lambda(L)$-free basis consisting of n elements. Thus we have the conclusion. Note that the above proof shows that this assertion holds for any left full linear ring A and any simple C-subalgebra S of A with $[S:C]<\infty$(Theorem 36.2 [1]). As for the rest of the proof, see Theorem 2 and Theorem 3 [9].

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