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Infinitesimal Deformations of Cusp Singularities

To the memory of Professor Takeshi Onodera

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Introduction. The purpose of this article is to compute infinitesimal deformations $\boldsymbol{T}^1$ of cusp singularities of two dimension. It is known that cusp singularities of three or higher dimension are rigid [3]. Let T be a cusp singularity of two dimension. The minimal resolution of T has a cycle of rational curves as its exceptional set. Let C be the (reduced) cycle of rational curves, r the number of irreducible components of C. Then our main consequence is that $\dim \boldsymbol{T}^1$ is equal to $r-C^2$ (if $C^2 \leqq -5$). This solves a conjecture of Behnke [1] in the affirmative. It should be mentioned that he himself solved his this conjecture in [2] in a manner different from ours.

Our method is just the same as in [1], where Behnke showed that $\boldsymbol{T}^1$ is the space of solutions of a system of infinitely many linear equations in infinitely many variables. We reduce this to finitely many linear equations in finitely many variables (§ 2). After preparing some lemmas about support points (§ 3), we solve the system of finitely many linear equations in the case $C^2 \leqq -5$ (§ 4). The same method yields a complete description of $\boldsymbol{T}^1$ in the case $-1 \geqq C^2 \geqq -4$ except for $r=-C^2=1$. The consequence leads us to a conjecture about deformations of $T_{p,q,r}$ and $\Pi_{p,q,r,s}$ in the cases $C^2=-3, -4$ (§ 5).

Contents.

Index of notations and terminologies

$\boldsymbol{C}$	complex numbers	$B(n)$	(1.7)
$\boldsymbol{H}$	$:=\{z \in \boldsymbol{C};\ \mathrm{Im}(z)>0\}$	χ, χ_n	(2.2), (2.5), (4.2), (5.4)
M	a complete module	*wt*	(3.1), (4.3)
$U^+(M)$	(1.1)	*internal*	(3.1), (4.3)

V	(1.1)	*extremal*	(3.1), (4.3)
$X'(M, V)$	(1.1)	ξ_{int}	(4.3)
$X, X(M, V)$	(1.1)	ξ_{ext}	(4.3)
$Y, Y(M, V)$	(1.1)	$h(B)$	(4.3)
$C, \mathscr{C}, \mathscr{D}$	(1.1)	$\boldsymbol{e}(x)$	$\exp(2\pi ix)$
w, A_k, a_k	(1.2)	$\theta(\mu)$	$\boldsymbol{e}(\mu z_1+\mu' z_2)$, where μ' is the conjugate of μ
w_*, B_k, b_k	(1.3)		
M^*	the dual of M	$\delta_{i,j}$	(4.1)
$(M^*)^+$	(1.3)	∂_k	$\partial/\partial z_k$
$i((M^*)^+)$	(1.3)	$\boldsymbol{E}(l)$	(4.5)
$\partial\Sigma^+(M)$	(1.3)	$\boldsymbol{F}$	(4.6), (4.7), See the case $l=h(B)+2$
r, s	(1.3)		

§ 1 Definitions.

(1.1) Let M be a complete module in a real quadratic field K, $U^+(M)$ the group of all totally positive units keeping M invariant by multiplication, V an infinite cyclic subgroup of $U^+(M)$. We define a subgroup $G(M, V)$ of $SL(2, \boldsymbol{R})$ by

$$G(M, V)=\left\{\begin{pmatrix} v & m \\ 0 & 1\end{pmatrix}\in SL(2, \boldsymbol{R})\ ;\ v\in V,\ m\in M\right\}.$$

We define an action of $G(M, V)$ on the product $\boldsymbol{H}\times\boldsymbol{H}$ of two upper half planes by

$$\begin{pmatrix} v & m \\ 0 & 1\end{pmatrix}:\ (z_1, z_2)\to(vz_1+m,\ v'z_2+m')$$

where v' and m' denote the conjugates of v and m respectively. The action of $G(M, V)$ on $\boldsymbol{H}\times\boldsymbol{H}$ is free and properly discontinuous. We have a nonsingular surface $X'(M, V)$ as quotient. This $X'(M, V)$ is partially compactified by adding a point ∞ into a normal complex space $X(M, V)$. Let $f:\ Y(M, V)\to X(M, V)$ be the minimal resolution of $X(M, V)$, C the exceptional set of f, $\pi:\ \mathscr{D}\to Y(M, V)$ the universal covering of $Y(M, V)$, $\mathscr{C}=\pi^{-1}(C)$. For brevity we denote $X(M, V)$ and $Y(M, V)$ by X and Y respectively. The space X has a unique isolated singularity at ∞, which we call a cusp singularity. The exceptional set C is a (reduced) cycle of rational curves.

(1.2) Without loss of generality, we may assume that $M=\boldsymbol{Z}+\boldsymbol{Z}w$, $w>1>w'>0$. Let $[[\overline{a_0, \dots, a_{r-1}}]]$ be the modified continued fraction expansion of w, and let $a_{j+kr}=a_j$ for $j=0, \dots, r-1$, $k\in\boldsymbol{Z}$. We define inductively $w_0=w$, $w_n=1/(a_{n-1}-w_{n-1})$ $(n>0)$, $w_{n-1}=a_{n-1}-(1/w_n)$ $(n\leqq 0)$,

$A_n=1/w_1w_2\dots w_n$ $(n>0)$, $A_0=1$, $A_n=w_{n+1}\dots w_{-1}w_0$ $(n<0)$. Following [4] we define complex variables x_n, y_n by

$$2\pi\sqrt{-1}\,z_1=A_n\log x_n+A_{n-1}\log y_n$$
$$2\pi\sqrt{-1}\,z_2=A_n'\log x_n+A_{n-1}'\log y_n.$$

Then by [4], $\mathscr{D}$ is covered by infinitely many charts $U_n=\{(x_n, y_n)\in\boldsymbol{C}^2\,;\ |x_n^{A_n}y_n^{A_{n-1}}|<1,\ |x_n^{A_n'}y_n^{A_{n-1}'}|<1\}$. The charts U_n and U_{n+1} are patched by $x_{n+1}=y_n^{-1}$, $y_{n+1}=x_ny_n^{a_n}$.

By choosing a suitable integral multiple of r instead of r if necessary, we may assume that a generator v of V transforms U_n onto U_{n+r} by $(x_n, y_n)\to(x_{n+r}', y_{n+r}')=(x_n, y_n)$ and $vA_n=A_{n+r}$ for any n.

(1.3) Let M^* be the dual of M, i. e. by definition $M^*=\{x\in K\,;\ tr\cdot(xy)\in\boldsymbol{Z}$ for any $y\in M\}$. Define a mapping i of K into $\boldsymbol{R}^2$ by $i(x)=(x, x')$, $x\in K$. Let $(M^*)^+=\{x\in M^*\,;\ x>0,\ x'>0\}$, and let $\Sigma^+(M^*)$ be the convex closure of $i((M^*)^+)$, $\partial\Sigma^+(M^*)$ the boundary of $\Sigma^+(M^*)$. Then we number "lattice points" lying on $\partial\Sigma^+(M^*)$ in a consecutive order. Namely we let $i^{-1}(\partial\Sigma^+(M^*)\cap i(M^*))=\{B_j\,;\ j\in\boldsymbol{Z}\}$ with $B_j<B_k$ for $j>k$. The group V acts on M^*, $\Sigma^+(M^*)$ and $\partial\Sigma^+(M^*)$. Let v be a generator of V with $0<v<1$. Then there exist r and s such that for any k

$$vA_k=A_{k+r},\quad vB_k=B_{k+s}.$$

We know that $s=-C^2$ by [6]. Moreover there are positive integers b_k $(\geqq 2)$ $(k\in\boldsymbol{Z})$ and $w_*\in K$ such that

$$M^* \text{ is strictly equivalent to } \boldsymbol{Z}+\boldsymbol{Z}w_*,$$
$$w_*=[[\overline{b_0,\dots,b_{s-1}}]],$$
$$b_{k+s}=b_k,\quad b_kB_k=B_{k-1}+B_{k+1} \text{ for any } k\in\boldsymbol{Z}.$$

If $M=\boldsymbol{Z}+\boldsymbol{Z}w$, $w=[[\overline{p_1, \underbrace{2,\dots,2}_{(q_1-3)}, p_2,\dots,\dots, p_n, \underbrace{2,\dots,2}_{(q_n-3)}}]]$ for some p_j, $q_j\geqq 3$, then

$$w_*=[[\overline{q_n, \underbrace{2,\dots,2}_{(p_n-3)}, q_{n-1},\dots, q_1, \underbrace{2,\dots,2}_{(p_1-3)}}]] \qquad (\textit{see } [1]).$$

In the same way as above we can define M^+, $\Sigma^+(M)$ for a complete module $M=\boldsymbol{Z}+\boldsymbol{Z}w$ in (1.2). With the notations in (1.2), we have $i^{-1}(\partial\Sigma^+(M)\cap i(M^+))=\{A_j,\ j\in\boldsymbol{Z}\}$. We call A_j and B_k *support points* of M and M^* respectively.

(1.4) We denote by $\Omega_Y^1(\log C)$ the sheaf over Y of germs ω of meromorphic one forms such that the poles of ω and $d\omega$ are contained in C $(=C_{red})$. Since C is of normal crossing, $\Omega^1_Y(\log C)$ is locally free. In fact, $\Omega^1_Y(\log C)$ is isomorphic to $\mathcal{O}_Y(F)\oplus\mathcal{O}_Y(-F)$ for a flat line bundle F on Y. This can be seen by using natural extensions of two sections dz_1 and dz_2 to $\mathscr{D}$. Let $\tilde{\theta}_Y=\mathscr{H}om_{\mathcal{O}_Y}(\Omega^1_Y(\log C), \mathcal{O}_Y)$.

Let Y_R be the interior of the closure in Y of Y'_R, $(R\geqq 0)$

$$Y'_R:=\{(z_1, z_2)\in \boldsymbol{H}\times\boldsymbol{H}\ ;\ (\mathrm{Im}\ z_1)(\mathrm{Im}\ z_2)>R\}/G(M, V).$$

We notice that any open neighborhood of the cycle C contains Y_R for R large enough.

LEMMA (1.5) *Let Z be a strongly pseudoconvex open neighborhood of C in Y. Then*

1) $H^q(Z, \tilde{\Theta}_Z)=0$ $(q>0)$, $H^1(Z, \mathcal{O}_Z)\cong H^1(C, \mathcal{O}_C)$, $H^2(Z, \mathcal{O}_Z)=0$,
2) $H^1(Z, \tilde{\Theta}_Z(nC))=H^1(Z, \tilde{\Theta}_Z(nC)\otimes\mathcal{O}_{nC})$ *for any* $n>0$,
3) *the following is exact for any* $n>0$,
$0\to H^1(Z, \mathcal{O}_Z)\to H^1(Z, \mathcal{O}_Z(nC))\to H^1(Z, \mathcal{O}_{nC}(nC))\to 0.$

PROOF By [5], $H^q(Z, \mathcal{O}_Z)=\lim_n H^q(Z, \mathcal{O}_{nC})=H^q(C, \mathcal{O}_C)$ $(q>0)$. Similarly $H^q(Z, \tilde{\Theta}_Z)=\lim_n H^q(Z, \tilde{\Theta}_Z\otimes\mathcal{O}_{nC})=\lim_n H^q(Z, \mathcal{O}_{nC}(F))\oplus\lim_n H^q(Z, \mathcal{O}_{nC}(-F))$ $(q>0)$. It is easy to see $H^q(Z, \mathcal{O}_{nC}(F))=0$ for any nontrivial flat line bundle F. Thus we infer 1). By 1) and the exact sequence

$$H^1(Z, \tilde{\Theta}_Z)\to H^1(Z, \tilde{\Theta}_Z(nC))\to H^1(Z, \tilde{\Theta}_Z(nC)\otimes\mathcal{O}_{nC})\to H^2(Z, \tilde{\Theta}_Z)$$

we infer 2). Finally we shall show 3). First we consider the exact sequence

$$0\to\mathcal{O}_{(n-1)C}((n-1)C)\to\mathcal{O}_{nC}(nC)\to\mathcal{O}_C(nC)\to 0.$$

Since $H^0(C, \mathcal{O}_C(nC))=0$ for any $n>0$,
we have $H^0(Z, \mathcal{O}_{(n-1)C}((n-1)C)))=H^0(Z, \mathcal{O}_{nC}(nC))$ for any n, hence $H^0(Z, \mathcal{O}_{nC}(nC))=H^0(C, \mathcal{O}_C(C)))=0$. Therefore from the exact sequence

$$0\to\mathcal{O}_Z\to\mathcal{O}_Z(nC)\to\mathcal{O}_{nC}(nC)\to 0$$

we infer an exact sequence

$$\to H^0(Z, \mathcal{O}_{nC}(nC))\to H^1(Z, \mathcal{O}_Z)\to H^1(Z, \mathcal{O}_Z(nC))$$
$$\to H^1(Z, \mathcal{O}_{nC}(nC))\to H^2(Z, \mathcal{O}_Z)$$

which is just 3) because $H^2(Z, \mathcal{O}_Z)=0$. q. e. d.

COROLLARY (1.6) *For any* $n>0$,

1) $h^1(Z, \tilde{\Theta}_Z(nC))=(n^2+n)s,$
2) $h^1(Z, \mathcal{O}_Z(nC))=(n^2+n)s/2+1.$

PROOF. By Riemann-Roch and $h^0(Z, \mathcal{O}_{nC}(nC+G))=0$ for any flat line bundle G, we have $h^1(Z, \mathcal{O}_{nC}(nC+G))=-(n^2C^2-nK_ZC)/2$. Hence $h^1(Z, \tilde{\Theta}_Z(nC))=(n^2+n)s$ by (1.5). q. e. d.

LEMMA (1.7) *For any* $n>0$,

1) $H^1(Z, \tilde{\Theta}_Z(nC))\cong H^1(V, H^0(\mathscr{D}, \tilde{\mathcal{O}}_{\mathscr{D}}(n\mathscr{C}))),$
2) $H^1(Z, \mathcal{O}_Z(nC))\cong H^1(V, H^0(\mathscr{D}, \mathcal{O}_{\mathscr{D}}(n\mathscr{C}))).$

PROOF. Let $\pi: \mathscr{D}\to Y$ be the natural projection, $\mathscr{D}_R=\pi^{-1}(Y_R)$. Let $B(n)=\{-aB_{k-1}-bB_k\ (\neq -aB_s);\ 0\leqq a, b, 1\leqq a+b\leqq n, 1\leqq k\leqq s\}$. Since $\theta(B_k)$ is contained in $H^0(\mathscr{D}, \mathcal{O}_{\mathscr{D}}(-\mathscr{C}))$, but not in $H^0(\mathscr{D}, \mathcal{O}_{\mathscr{D}}(-2\mathscr{C}))$, $\theta(\mu)\partial_j$ $(\mu\in B(n))$ belongs to $H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C}))$. Then by the same argument as in [1], $H^1(V, H^0(\mathscr{D}_R, \tilde{\Theta}_{\mathscr{D}_R}(n\mathscr{C})))$ is generated by $\theta(\mu)\partial_1$ and $\theta(\mu)\partial_2$, $\mu\in B(n)$. Hence $H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$ is a subspace of $H^1(V, H^0(\mathscr{D}_R, \tilde{\Theta}_{\mathscr{D}_R}(n\mathscr{C})))$ for any $R>0$. By [11, Thèoréme 5.2.1, 5.3.1] $H^1(V, H^0(\mathscr{D}_R, \tilde{\Theta}_{\mathscr{D}_R}(n\mathscr{C})))$ is a subspace of $H^1(Y_R, \tilde{\Theta}_{Y_R}(nC))$ for any $R\geqq 0$. By choosing R large enough so that Z contains Y_R, we have a commutative diagram

$$\begin{array}{ccc} H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C}))) & \xrightarrow{\text{injective}} & H^1(V, H^0(\mathscr{D}_R, \tilde{\Theta}_{\mathscr{D}_R}(n\mathscr{C}))) \end{array}$$

$$H^1(Y, \tilde{\Theta}_Y(nC)) \to H^1(Z, \tilde{\Theta}_Z(nC)) \to H^1(Y_R, \tilde{\Theta}_{Y_R}(nC)).$$

Hence $H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$ is a subspace of $H^1(Z, \tilde{\Theta}_Z(nC))$. Since $\sharp(B(n))=(n^2+n)s$, we have 1) by (1.6). Similarly we infer 2). q. e. d.

§2 T^1 of a cusp singularity.

In (2.1)-(2.3) we work in the category of algebraic varieties over $\boldsymbol{C}$.

(2.1) Let T be a cusp singularity, isomorphic to $(X(M, V), \infty)$, a germ of $X(M, V)$ at ∞. Let U be an affine variety with a unique singularity T at p. We notice that such a U really exists. Let $f: W\to U$ be the minimal resolution of U, C the exceptional set of f. Let d be the embedding dimension of T. It is known that $d=\max(3, s)$. We may assume that U is an affine subvariety of the affine d-space $\boldsymbol{A}^d$.

LEMMA (2.2) [10] *There is an exact sequence*

$$0\to \boldsymbol{T}^1\to H^1(U-p, \Theta_{U-p})\overset{\chi}{\to} H^1(U-p, \mathcal{O}_{U-p})^d.$$

Behnke [1] gave a description of χ : $H^1(X-\infty, \Theta_{X-\infty})\to H^1(X-\infty, \mathcal{O}_{X-\infty})^d$ in terms of B_k $(k\in \boldsymbol{Z})$, where the kernel of χ is $\boldsymbol{T}^1$. We shall improve his description in the subsequent lemmas (2.3)-(2.5).

LEMMA (2.3) *Let* $\tilde{\Theta}_W := \mathcal{H}om_{\mathcal{O}_W}(\Omega^1_W(\log C), \mathcal{O}_W)$. *Then for any sufficiently large* $n>0$,

$$0\to \boldsymbol{T}^1\to H^1(W, \tilde{\Theta}_W(nC))\overset{\chi}{\to} H^1(W, \mathcal{O}_W(nC))^d$$

is exact.

PROOF. Since $\tilde{\Theta}_W=\Theta_W$ outside C, $i_*\Theta_{W-C}=\lim\limits_{n>0}\tilde{\Theta}_W(nC)$, $i_*\mathcal{O}_{W-C}=\lim\limits_{n>0}\mathcal{O}_W(nC)$ for the inclusion i : $W-C\to W$, we have

$$H^q(W-C, \Theta_{W-C})=H^q(W, i_*\tilde{\Theta}_{W-C})=\lim_{n>0} H^q(W, \Theta_W(nC))$$
$$H^q(W-C, \mathcal{O}_{W-C})=H^q(W, i_*\mathcal{O}_{W-C})=\lim_{n>0} H^q(W, \mathcal{O}_W(nC)).$$

So by (2.2) $0\to \boldsymbol{T}^1\to\lim\limits_{n>0} H^1(W, \tilde{\Theta}_W(nC))\to\lim\limits_{n>0} H^1(W, \mathcal{O}_W(nC))^d$ is exact. $H^1(W, \tilde{\Theta}_W((n-1)C))$ is a subspace of $H^1(W, \tilde{\Theta}_W(nC))$ because $H^0(W, \tilde{\Theta}_W\otimes\mathcal{O}_C(nC))=0$. Since $\boldsymbol{T}^1$ is finite-dimensional for any isolated singularity, there exists n_0 such that $\boldsymbol{T}^1$ is contained in $H^1(W, \tilde{\Theta}_W(nC))$ for any $n>n_0$. Hence $0\to \boldsymbol{T}^1\to H^1(W, \tilde{\Theta}_W(nC))\overset{\chi}{\to} H^1(W, \mathcal{O}_W(nC))^d$ is exact. q. e. d.

LEMMA (2.4) *Let Z be a strongly pseudoconvex open neighborhood of C, contained in W. Then we have for* q, $n>0$.

$$H^q(Z, \tilde{\Theta}_Z(nC))\cong H^q(W, \tilde{\Theta}_W(nC)),\ H^q(Z, \mathcal{O}_Z(nC))\cong H^q(W, \mathcal{O}_W(nC))$$

PROOF We have exact sequences of $\mathcal{O}_Z$ modules and $\mathcal{O}_W$ modules;

$$0\to\mathcal{O}_Z\to\mathcal{O}_Z(nC)\to\mathcal{O}_{nC}(nC)\to 0$$
$$0\to\tilde{\Theta}_Z\to\tilde{\Theta}_Z(nC)\to\tilde{\Theta}_Z(nC)\otimes\mathcal{O}_{nC}\to 0$$

$$0 \to \mathcal{O}_W \to \mathcal{O}_W(nC) \to \mathcal{O}_{nC}(nC) \to 0$$
$$0 \to \tilde{\Theta}_W \to \tilde{\Theta}_W(nC) \to \tilde{\Theta}_W(nC) \otimes \mathcal{O}_{nC} \to 0.$$

Therefore we have commutative diagrams of exact sequences

$$H^1(Z, \mathcal{O}_Z) \to H^1(Z, \mathcal{O}_Z(nC)) \to H^1(Z, \mathcal{O}_{nC}(nC)) \to H^2(Z, \mathcal{O}_Z) = 0$$

$$H^1(W, \mathcal{O}_W) \to H^1(W, \mathcal{O}_W(nC)) \to H^1(W, \mathcal{O}_{nC}(nC)) \to H^2(W, \mathcal{O}_W)$$

$$H^1(Z, \tilde{\Theta}_Z) \to H^1(Z, \tilde{\Theta}_Z(nC)) \to H^1(Z, \tilde{\Theta}_Z(nC) \otimes \mathcal{O}_{nC}) \to H^2(Z, \tilde{\Theta}_Z) = 0$$

$$H^1(W, \tilde{\Theta}_W) \to H^1(W, \tilde{\Theta}_W(nC)) \to H^1(W, \tilde{\Theta}_W(nC) \otimes \mathcal{O}_{nC}) \to H^2(W, \tilde{\Theta}_W)$$

By (1.5) $H^1(Z, \tilde{\Theta}_Z) = 0$, $H^1(Z, \mathcal{O}_Z) \cong \lim_n H^1(Z, \mathcal{O}_{nC}) \cong H^1(C, \mathcal{O}_C)$. Since U is affine, we have $H^q(W, \tilde{\Theta}_W) = H^0(U, R^q f_* \tilde{\Theta}_W)$. We shall show that $R^q f_* \tilde{\Theta}_W = 0$ for $q > 0$. Since f is projective, it suffices to show that $R^q(f_{\text{anal}})_*(\tilde{\Theta}_W)_{\text{anal}} = 0$ for $q > 0$. We have $H^0(U', R^q f_{\text{anal}_*}(\tilde{\Theta}_W) = \lim_n H^q(Z', \tilde{\Theta}_{Z'} \otimes \mathcal{O}_{nC}) = 0$ by (1.5) where $Z' = f^{-1}(U')$ is strongly pseudoconvex. Hence $R^q f_* \tilde{\Theta}_W = 0$ $(q > 0)$. Similarly $H^1(W, \mathcal{O}_W) \cong H^1(C, \mathcal{O}_C)$, $H^2(W, \mathcal{O}_W) = 0$. Moreover it is clear that for $n > 0$. $H^q(Z, \mathcal{O}_{nC}(nC)) \cong H^q(W, \mathcal{O}_{nC}(nC))$, $H^q(Z, \tilde{\Theta}_Z(nC) \otimes \mathcal{O}_{nC}) \cong H^q(W, \tilde{\Theta}_W(nC) \otimes \mathcal{O}_{nC})$ Hence from the above commutative diagrams we infer the isomorphisms $H^q(Z, \tilde{\Theta}_Z(nC)) \cong H^q(W, \tilde{\Theta}_W(nC))$, $H^q(Z, \mathcal{O}_Z(nC)) \cong H^q(W, \mathcal{O}_W(nC))$ for $q, n > 0$.

LEMMA (2.5) *For any sufficiently large $n > 0$,*

$$0 \to \boldsymbol{T}^1 \to H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C}))) \to H^1(V, H^0(\mathscr{D}, \mathcal{O}_{\mathscr{D}}(n\mathscr{C})))^d$$

is exact.

PROOF. Clear from (1.7), (2.3) and (2.4). q. e. d.

§ 3 Lemmas.

(3.1) Let μ be in $(M^*)^+$. Then there exist k, a and b such that $\mu = aB_k + bB_{k+1}$, $a > 0$, $b \geqq 0$. These k, a and b are uniquely determined by μ. We call μ *internal* if $a, b > 0$ and call μ *k-extremal* if $a > 0$, $b = 0$. We say that μ is *extremal* if μ is *k-extremal* for some k. We define the weight of μ by $wt\ \mu = a + b$, $wt(0) = 0$. It $\mu(\neq 0)$ is not in $(M^*)^+$, then we define wt

$\mu=-\infty$. We notice that if $V\mu_1=V\mu_2$, then $wt\ \mu_1=wt\ \mu_2$.

LEMMA (3.2)
1) *Let* $\mu_1, \mu_2 \in (M^*)^+$. *Then* $wt(\mu_1+\mu_2)\geqq wt(\mu_1)+wt(\mu_2)$.
2) *Suppose that* $j_1\leqq j_2\leqq \dots \leqq j_l$. *Then* $wt(B_{j_1}+B_{j_2}+\dots+B_{j_l})\geqq \sum_{\lambda=j_1+1}^{j_l-1}(b_\lambda-2)+l$. *Equality holds only when* $b_\lambda=2$ *for any* $\lambda\,(j_1+2\leqq\lambda\leqq j_l-2)$.
3) *Suppose that* $j_1\leqq j_2\leqq \dots \leqq j_l$. *Then* $wt(B_{j_1}+B_{j_2}+\dots+B_{j_l})=l$ *iff* $b_\lambda=2$ *for* $j_1+1\leqq\lambda\leqq j_l-1$.

PROOF. First we shall prove 1). Let $A=aB_j+bB_{j+1}$, $B=cB_k+dB_{k+1}$, $a, b, c, d\geqq 0$, $k\geqq j$, $n=k-j$. We shall prove an inequality $E(n, d)$: $wt(A+B)\geqq wt(A)+wt(B)$ by the double induction on n and d. $E(0, d)$ is clearly true. $E(1, d)$ is proved by noticing that $B_j+B_{j+2}=b_{j+1}B_{j+1}$. In fact, $aB_j+bB_{j+1}+cB_{j+1}+dB_{j+2}=(b+c+ab_{j+1})B_{j+1}+(d-a)B_{j+2}$ or $(a-d)B_j+(b+c+db_{j+1})B_{j+1}$ so that $wt(A+B)\geqq a+b+c+d$. Now we assume $n>1$. We assume $E(n', d')$ for any $n'\leqq n-1$ and any d' or for $n'=n$ and $d'\leqq d-1$. We shall prove $E(n, d)$. Since $E(n, 0)$ follows from $E(n-1, c)$, we may assume $d\geqq 1$. Let $B'=B-B_{k+1}=cB_k+(d-1)B_{k+1}$. Then there exist i, a' and b' such that $A+B'=a'B_i+b'B_{i+1}$, $a', b'\geqq 0$. Clearly $j\leqq i\leqq k$. If $k-i<n$, then $wt(A+B)\geqq wt(A+B')+1$ by $E(k-i, 1)$, hence by $E(n, d-1)$ $wt(A+B)\geqq wt(A+B')+1\geqq wt(A)+wt(B')+1=wt(A)+wt(B)$. So we consider the case $k-i=n$, $i=j$. We may assume $a'>0$. Therefore $A+B=(a'-1)B_j+b'B_{j+1}+(B_j+B_{k+1})$. Let $B_j+B_{k+1}=fB_l+gB_{l+1}$ for $f\geqq 0$, $g>0$, $j\leqq l\leqq k$. Clearly $f+g\geqq 2$. If $l=k$, then $B_j=fB_k+(g-1)B_{k+1}$ which is absurd. Hence $l<k$. Hence $l-j<n$ and by $E(l-j, g)$ $wt(A+B)=wt((a'-1)B_j+b'B_{j+1}+(fB_l+g\,B_{l+1}))\geqq wt((a'-1)B_j+b'B_{j+1})+wt(fB_l+g\,B_{l+1})=a'+b'-1+f+g\geqq a'+b'+1=wt(A+B')+1$. Hence by $E(n, d-1)$ we have $wt(A+B)\geqq wt(A+B')+1\geqq wt(A)+wt(B')+1\geqq wt(A)+wt(B)$.

Next we shall show 2). First we notice for $k\geqq j$,

$$B_j-B_{j+1}-B_k+B_{k+1}=(b_{j+1}-2)B_{j+1}+(b_k-2)B_k+(B_{j+1}-B_{j+2}-B_{k-1}+B_k),$$

hence by the induction on $k-j$, we have

$$B_j+B_{k+1}=B_{j+1}+B_k+\sum_{\lambda=j+1}^{k}(b_\lambda-2)B_\lambda .$$

By 1) $wt(B_j+B_{j+n})\geqq 2+\sum_{\lambda=j+1}^{j+n-1}(b_\lambda-2)$, equality holding only if $wt(B_{j+1}+B_{j+n-1})=2$. But since $B_{j+1}+B_{j+n-1}=B_{j+2}+B_{j+n-2}+\sum_{\lambda=j+2}^{j+n-2}(b_\lambda-2)B_\lambda$, $wt(B_{j+1}+B_{j+n-1})=2$ implies that $b_\lambda=2$ for $j+2\leqq\lambda\leqq j+n-2$.

The proof of 3) is easy. q. e. d.

LEMMA (3. 3) *Suppose* $n, a, b>0$.

1) $wt(nB_i-B_j)\leqq n-b_i+1$ *if* $i\neq j$. *If equality holds, then one of the following holds*;

1-1) $j=i+1$ *or* $i-1$,

1-2) $j\geqq i+2$, $b_k=2$ *for* $i\leqq k\leqq j-1$

1-3) $j\leqq i-2$, $b_k=2$ *for* $j+1\leqq k\leqq i$.

2) $wt(aB_{i-1}+bB_i-B_j)\leqq a+b-1$ *if* $a, b>0$. *If equality holds, then one of the following holds*;

2-1) $j=i-1$ *or* i,

2-2) $i+1\leqq j$, $b_i=\ldots=b_{j-1}=2$

2-3) $j\leqq i-2$, $b_{j+1}=\ldots=b_{i-1}=2$.

PROOF. Let $nB_i-B_j=\mu$. Suppose $\mu\in(M^*)^+$. Then there exist m, a $(\geqq 0)$ and b (>0) such that $nB_i=B_j+aB_{m-1}+bB_m$. We consider the following three cases separately;

Case 1. $i<j$, $a, b>0$. Case 2. $i<j$, $a=0$, $b>0$.

Case 3. $i>j$, $b>0$.

CASE 1. Then $m\leqq i$. By (3. 2), $n\geqq a+b+1+(b_m-2)+\ldots+(b_{j-1}-2)\geqq a+b-1+b_i=wt(\mu)+b_i-1$. Equality holds only if $b_{m+1}=\ldots=b_{j-2}=2$ and $b_m+b_{j-1}=b_i+2$. Suppose the equality holds. If $j=i+1$, then 1-1) occurs. If $j-2\geqq i\geqq m+1$, then $b_i=2$, $b_{j-1}=b_m=2$, hence 1-2) occurs. If $j-2\geqq i=m$, then $b_{j-1}=2$, $(n-b-b_i)B_i+B_{i+1}=B_j+(a-1)B_{i-1}$. If $a=1$, then it is absurd. If $a>1$, then $wt(B_j+(a-1)B_{i-1})\geqq a+b_i-2$, hence $n-b-b_i+1\geqq a+b_i-2$, so $b_i=2$. Therefore 1-2) occurs.

CASE 2. Then $m\leqq i-1$. Suppose $j\geqq i+2$. By (3. 2) $n\geqq b+1+(b_{m+1}-2)+\ldots+(b_{j-1}-2)\geqq b+b_i-1=wt(\mu)+b_i-1$. If equality holds, then $b_{m+2}=\ldots=b_{j-2}=2$ and $b_{m+1}+b_{j-1}=b_i+2$. If $m\leqq i-2$, then $b_i=2$, $b_{m+1}=b_{j-1}=2$. If $m=i-1$, then $b_{j-1}=2$, $(b+b_i-1)B_i=B_j+bB_{i-1}$. Hence $B_j+(b-1)(b_i-1)B_i=bB_{i+1}$. Since $b_{i+1}=\ldots=b_{j-1}=2$, we have $wt(B_j+(b-1)(b_i-1)B_i)=1+(b-1)(b_i-1)=wt(bB_{i+1})=b$. Therefore $(b-1)(b_i-2)=0$, so $b=1$ or $b_i=2$. If $b=1$, then $B_j+B_{i-1}=b_iB_i$, hence $j=i+1$ which contradicts $j\geqq i+2$. Hence $b_i=2$. This is the case 1-2). If $j=i+1$, then 1-1).

CASE 3. The proof in this case is similar to the above. Thus the proof of 1) is complete.

Next we consider 2). Let $\mu=aB_{i-1}+bB_i-B_j$. Suppose $\mu\in(M^*)^+$. Then there exist m, c $(\geqq 0)$ and d (>0) such that $aB_{i-1}+bB_i=B_j+cB_{m-1}+dB_m$, $\mu=cB_{m-1}+dB_m$. We consider the following three cases separately;

Case 1. $i<j,\ c>0,\ d>0$. Case 2. $i<j,\ c=0,\ d>0$,
Case 3. $i-1>j,\ d>0$.

Case 1. Then $m\leqq i$. By (3.2) $a+b\geqq c+d+1+(b_m-2)+\ldots+(b_{j-1}-2)=wt(\mu)+1+(b_m-2)+\ldots+(b_{j-1}-2)$. If $wt\ \mu=a+b-1$, then $b_m=\ldots=b_{j-1}=2$, hence 2-2) occurs.

Case 2. Then $m\leqq i-1$. By (3.2) $a+b\geqq d+1+(b_{m+1}-2)+\ldots+(b_{j-1}-2)$. If $wt\ \mu=a+b-1$, then $b_{m+1}=\ldots=b_{j-1}=2$, 2-2) occurs.

Case 3. Then $m\geqq i$. Similarly it follows that $b_{j+1}=\ldots=b_{m-1}=2$, 2-3) occurs. q. e. d.

Corollary (3.4) *Suppose* $m\neq 0$, $n\geqq b_i$, $0\leqq i\leqq s-1$, $0\leqq j\leqq s-1$.

1) *If* $s\geqq 2$, *then* $wt(nB_i-B_{i+ms})\leqq n-b_i$.
2) *If* $s\geqq 3$, *then* $wt(nB_i-B_{i+1+ms})\leqq n-b_i+1$, *equality holding only if* $m=-1$, $b_l=2$ *for* $l\neq i+1$ *mod* s.
3) *If* $s\geqq 3$, *then* $wt(nB_i-B_{i-1+ms})\leqq n-b_i+1$, *equality holding only if* $m=1$, $b_l=2$ *for* $l\neq i-1$ *mod* s.
4) *If* $s\geqq 4$, $i+2\leqq j\leqq i+s-2$, *then* $wt(nB_i-B_{j+ms})\leqq n-b_i+1$, *equality holding only if* $m=-1$, $b_l=2$ *for* $j-s+1\leqq l\leqq i$.
5) *It* $s\geqq 4$, $i-s+2\leqq j\leqq i-2$, *then* $wt(nB_i-B_{j+ms})\leqq n-b_i+1$, *equality holding only if* $m=1$, $b_l=2$ *for* $i\leqq l\leqq j+s-1$.

Proof. 1) Since $m\neq 0$, $s\geqq 2$, we have $i+ms\geqq i+2$ or $i+ms\leqq i-2$. Hence if $wt(nB_i-B_{i+ms})=n-b_i+1$, then by (3.3) $m>0$, $b_i=b_{i+1}=\ldots=b_{i+s-1}=2$ or $m<0$, $b_{i-s+1}=b_{i-s+2}=\ldots=b_i=2$, which contradicts that at least one of b_j $(j=0,\ldots, s-1)$ is greater that two. Hence 1) follows. 2) Since $m\neq 0$, $s\geqq 3$, we have $i+1+ms\geqq i+2$ or $i+1+ms\leqq i-2$.
Hence if $wt(nB_i-B_{i+1+ms})=n-b_i+1$, then by (3.3) $m>0$, $b_i=b_{i+1}=\ldots=b_{i+s}=2$ or $m=-1$, $b_{i-s+2}=\ldots=b_{i-1}=b_i=2$, or $m<-1$, $b_{i-2s+2}=b_{i-2s+1}=\ldots=b_i=2$. Hence 2) follows. The assertions 3)-5) are proved similarly.
q. e. d.

Corollary (3.5) *Suppose* $m\neq 0$, $a,\ b>0$, $0\leqq i\leqq s-1$, $0\leqq j\leqq s-1$.

1) *If* $s\geqq 2$, $wt(aB_{i-1}+bB_i-B_{i-1+ms})=a+b-1$, *then* $m=1$, $b_l=2$ *for* $l\neq i-1$ *mod* s.
2) *If* $s\geqq 2$, $wt(aB_{i-1}+bB_i-B_{i+ms})=a+b-1$, *then* $m=-1$, $b_l=2$ *for* $l\neq i$ *mod* s.
3) *If* $s\geqq 3$, $i-s+1\leqq j\leqq i-2$, $wt(aB_{i-1}+bB_i-B_{j+ms})=a+b-1$, *then* $m=1$, $b_l=2$ *for* $i\leqq l\leqq j+s-1$.
4) *If* $s\geqq 3$, $i+1\leqq j\leqq i+s-2$, $wt(aB_{i-1}+bB_i-B_{j+ms})=a+b-1$, *then* $m=-1$ $b_l=2$ *for* $j-s+1\leqq l\leqq i-1$.

PROOF. Clear from (3. 3) 2). q. e. d.

LEMMA (3. 6) *Let* $s\geqq 4$, $0\leqq i\leqq s-1$, $0\leqq j\leqq s-1$, $\mu=\alpha B_i+\beta B_{i+1}\in(M^*)^+$, $\alpha>0$, $\beta\geqq 0$, $m>1$, a, $b>0$.

1) *If* $\mu-B_{j+ks}=(m-1)B_{j+hs}$, *then* $wt\ \mu\geqq m$, *equality holding only if*

1-1) $k=h=0$, *or*

1-2) $b_l=2$ *for* $l\neq j$ *mod* s *and* (k, h) *is one of the pairs* $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.

2) *If* $\mu-B_{j+ks}=aB_{j-1+hs}+(b-1)B_{j+hs}$, *then one of the following holds;*

2-1) μ *is internal and* $wt\ \mu\geqq a+b+1$,

2-2) μ *is i-extremal and* $wt\ \mu\geqq a+b+b_i-1$

2-3) $k=h=0$ *or* $k=h=1$

2-4) $b_l=2$ *for* $l\neq j-1$, j *mod* s, b_{j-1}, $b_j\geqq 3$, $b=1$, $(k, h)=(0, \mathrm{a})$, *or* $(-1,0)$,

2-5) $b_l=2$ *for* $l\neq j$ *mod* s *and* $(k, h)=(0,1)$ *or* $(-1,0)$

3) *It* $\mu-B_{j+ks}=(a-1)B_{j+hs}+bB_{j+1+hs}$, *then one of the following holds;*

3-1) μ *is internal and* $wt\ \mu\geqq a+b+1$,

3-2) μ *is i-extremal and* $wt\ \mu\geqq a+b+b_i-1$,

3-3) $k=h=0$, *or*

3-4) $b_l=2$ *for* $l\neq j$, $j+1$ *mod* s, b_j, $b_{j+1}\geqq 3$, $a=1$, $(k, h)=(0,-1)$ *or* $(1,0)$,

3-5) $b_l=2$ *for* $l\neq j$ *mod* s, $(k, h)=(0,-1)$ *or* $(1,0)$.

PROOF. 1) is clear from (3. 2). We shall prove 2). We first assume that μ is internal. Since $\mu=B_{j+ks}+aB_{j-1+hs}+(b-1)B_{j+hs}$, we have $wt\ \mu\geqq a+b$ by (3. 2). If $|h-k|\geqq 2$, then $wt\ \mu\geqq a+b+\sum_{\lambda=j-2s+1}^{j-2}(b_\lambda-2)$ or $wt\ \mu\geqq a+b+\sum_{\lambda=j}^{j+2s-1}(b_\lambda-2)$, hence $wt\ \mu\geqq a+b+1$. Suppose $wt\ \mu=a+b$. Since $\mu=\alpha B_i+\beta B_{i+1}$, (k, h) is one of $(0,0)$, $(\pm 1,0)$, $(0,\pm 1)$, $(1,1)$. Suppose $(k, h)=(1,0)$. Then $b_\lambda=2$ for $j\leqq\lambda\leqq j+s-1$, hence $b_\lambda=2$ for any λ, which is absurd. Suppose $(k, h)=(0,-1)$. Then $b_\lambda=2$ for $j-s\leqq\lambda\leqq j-1$, which is again absurd. If $(k, h)=(0,1)$, then it follows from (3. 2) that $b_\lambda=2$ for $j+1\leqq\lambda\leqq j+s-1$ or that $b_\lambda=2$ for $\lambda\neq j-1$, j mod s and $b=1$. If $(k, h)=(-1,0)$, then by (3. 2) $b_\lambda=2$ for $\lambda\neq j$ mod s or $b_\lambda=2$ for $\lambda\neq j-1$, j mod s and $b=1$.

Next we consider the case μ is *i-extremal.* We shall prove that if μ is *i-extremal,* then $wt\ \mu\geqq a+b+b_i-1$ or 2-2), or 2-3) is true. Let $\mu=\alpha\mathrm{B}_i$. If $wt\ \mu=a+b+b_i-1$, then it is easy to see by using (3. 2) repeatedly that (k, h) is one of $(0,0)$, $(\pm 1,0)$, $(0,\pm 1)$. Suppose $(k, h)=(0,0)$. Then $\alpha B_i=aB_{j-1}+bB_j$ which is absurd.

CASE 1. *We assume* $(k, h)=(0, 1)$. Assume moreover $b>1$. Then by (3.2) and $j+1\leqq i\leqq j+s-1$,

$$\alpha=wt(B_j+aB_{j-1+s}+(b-1)B_{j+s})\geqq a+b+\sum_{\lambda=j+1}^{j+s-1}(b_\lambda-2)\geqq a+b+b_i-2.$$

If $\alpha=a+b+b_i-2$, then $b_\lambda=2$ for $\lambda\neq j$, i mod s. Since

$$B_j=(k+1)B_{j+k}-kB_{j+k+1}\quad(0\leqq k\leqq i-j-1)=(i-j)B_{i-1}-(i-j-1)B_i,$$

we have

$$\begin{aligned}\alpha B_i+(i-j)B_{i+1}&=(i-j)(B_{i-1}+B_{i+1})-(i-j-1)B_i+aB_{j-1+s}+(b-1)B_{j+s}\\&=((i-j)(b_i-1)+1)B_i+aB_{j+1+s}+(b-1)B_{j+s}.\end{aligned}$$

By (3.2) 3), we have $\alpha-i-j=(i-j)(b_i-1)+a+b$, so $(i-j-1)(b_i-2)=0$. Consequently $i=j+1$, or $b_i=2$. We assume $i=j+1$ to derive a contradiction. We have then $\alpha B_i-B_j=(\alpha-b_{j+1})B_{j+1}+B_{j+2}=aB_{j+s-1}+(b-1)B_{j+s}$. Since $s\geqq 3$, this is impossible. Hence $b_i=2$, and this is the case 2–3). When $b=1$, by the same argument we infer $b_i=2$ or $i=j+1$ and that $b_\lambda=2$ for $\lambda\neq j-1$, j, i mod s. If $i=j+1$, then we derive a contradiction by $s\geqq 4$ so that 2–2) occurs.

CASE 2. *We assume* $(k, h)=(-1, 0)$. Assume moreover $b>1$. Then by (3.2) $\alpha\geqq a+b+\sum_{\lambda=j-s+1}^{j-1}(b_\lambda-2)\geqq a+b+b_i-2$. If $\alpha=a+b+b_i-2$, then $b_\lambda=2$ for $\lambda\neq i$, j mod s and $B_{j-s}=((i-j+s)(b_i-1)+1)B_i-(i-j+s)B_{i-1}$. Hence $\alpha+i-j+s=(i-j+s)(b_i-1)+1+a+b-1$ so that $i=j-s+1$ or $b_i=2$. If $i=j-s+1$, then $i=0$, $j=s-1$, $\alpha B_i-B_{j-s}=\alpha B_0-B_{-1}=(\alpha-b_0)B_0+B_1\neq aB_{s-2}+(b-1)B_{s-1}$ which is absurd. Hence 2–3) occurs. When $b=1$, 2–2) occurs.

CASE 3. *We assume* $(k, h)=(1, 0)$. If $\alpha=a+b+b_i-2$, then $b_\lambda=2$ for $\lambda\neq i$ mod s, hence $b_i\geqq 3$. By the same argument as above $(i-j-s+1)(b_i-2)=0$ so that $i=j+s-1$, $i=s-1$, $j=0$. Hence $\alpha B_{s-1}-B_s=(\alpha-b_{s-1})B_{s-1}+B_{s-2}\neq aB_{-1}+(b-1)B_0$ by $s\geqq 4$, which is absurd. So $(k, h)=(1, 0)$ does not occur. *Similarly we can show that* $(k, h)=(0, -1)$ *does not occur.* Thus the proof of 2) is complete. The remaining assertions are proved similarly. q. e. d.

LEMMA (3.7) *Let* $s\geqq 5$, $l\geqq 2$, $0\leqq i\leqq s-1$, $0\leqq j\leqq s-1$, $\mu=\alpha B_i+\beta B_{i+1}\in(M^*)^+$, $\alpha>0$, $\beta\geqq 0$.

1) *If* $\mu-B_{j+ks}=(l-2)B_{j-1+hs}+B_{j-2+hs}$, *then one of the following holds ;*

1-1) μ *is internal and* $wt\ \mu\geqq l+1$

1-2) μ *is i-extremal and* $wt\ \mu\geqq l+b_i-1$,

1-3) $k=h=0$, $\mu=(l+b_{j-1}-2)B_{j-1}$, *or* $k=h=1$, $j=0$, $\mu=(l+b_{s-1}-2)B_{s-1}$,
1-4) $b_\lambda=2$ *for* $\lambda\neq j-1$, j *mod s*, $(k, h)=(0,1)$ *or* $(-1,0)$
1-5) $b_\lambda=2$ *for* $\lambda\neq j-2$, $j-1$, j *mod s*, $l=2$, $(k, h)=(0,1)$ *or* $(-1,0)$
2) *If* $\mu-B_{j+ks}=(l-2)B_{j+1+hs}+B_{j+2+hs}$, *then one of the following holds*;
2-1) μ *is internal and* $wt\ \mu\geqq l+1$,
2-2) μ *is i-extremal and* $wt\ \mu\geqq l+b_i-1$,
2-3) $k=h=0$, $\mu=(l+b_{j+1}-2)B_{j+1}$, *or* $k=h=-1$, $j=s-1$, $\mu=(l+b_0-2)B_0$.
2-4) $b_\lambda=2$ *for* $\lambda\neq j$, $j+1$ *mod s*, $(k, h)=(1,0)$ *or* $(0,-1)$
2-5) $b_\lambda=2$ *for* $\lambda\neq j$, $j+1$, $j+2$ *mod s*, $l=2$, $(k, h)=(1,0)$ *or* $(0,-1)$

PROOF. We shall prove 1). First we assume that μ is internal. Then by (3.2) $wt\ \mu\geqq l$. Suppose $wt\ \mu=l$. If $|h-k|\geqq 2$, then by (3.2) $b_\lambda=2$ for any λ which is absurd. Hence $(k, h)=(0,0),\pm(1,1),\pm(1,0),\pm(0,1)$. If $(k, h)=(1,0)$ or $(0,-1)$, then $b_\lambda=2$ for any λ which is absurd. Clearly $(k, h)\neq(-1,-1)$. If $(k, h)=(0,0)$ or $(1,1)$, then μ is extremal. Hence $(k, h)=(0,1)$ or $(-1,0)$, 1-4) or 1-5) occurs.

Next we consider the case μ is i-extremal. Let $\mu=\alpha B_i$. Then by (3.2) $\alpha\geqq l+b_i-2$. If $\alpha=l+b_i-2$, then $(k, h)=(0,0)$, $(1,1)$, $\pm(1,0)$, $\pm(0,1)$. Suppose $\alpha=l+b_i-2$.

CASE 1. Assume $(k, h)=(0,0)$ or $(1,1)$. Then 1-3) occurs.

CASE 2. Assume $(k, h)=(0,1)$. Assume moreover $l>2$. Then $b_\lambda=2$ for $\lambda\neq i$, $j-1$, j mod s. Then by the same argument as in (3.6), we infer $i=j+1$ or $b_i=2$. If $i=j+1$, then $(\alpha-b_{j+1})B_{j+1}+B_{j+2}=B_{j-2+s}+(l-2)B_{j-1+s}$ which contradicts $n\geqq 5$. Hence 1-4) occurs. When $wt\ \mu=l=2$, we have $b_\lambda=2$ for $\lambda\neq j-2$, $j-1$, j mod s in the same manner. Hence 1-4) or 1-5) occurs.

CASE 3. Assume $(k, h)=(-1,0)$. Assume $l>2$. Then $b_\lambda=2$ for $\lambda\neq i$, $j-1$, j mod s. We infer $b_i=2$ or $i=j-s+1$. If $i=j-s+1$, then we have a contradiction as above. So 1-4) occurs. If $l=2$, then 1-5) occurs.

CASE 4. Assume $(k, h)=(1,0)$. Then by (3.2) $b_\lambda=2$ for $\lambda\neq i$ mod s. We infer $i=j-1$ or $b_i=2$. In either case we have a contradiction. When $(k, h)=(0,-1)$, then we have a contradiction similarly. q. e. d.

LEMMA (3.8) *Suppose* $s=4$, $b_iB_i-B_j\in(M^*)^+$. *Then*
1) $j=i$, $i+1$ *or* $i-1$ *or*
2) $j=i\pm 2$ *and* $b_{i-1}=b_i=b_{i+1}=2$.

PROOF. Assume $j\geqq i+2$. Let $b_iB_i=B_j+\alpha B_k+\beta B_{k+1}$ for $\alpha>0$, $\beta\geqq 0$. Then $k\leqq i-1$. Hence $b_i=wt(b_iB_i)\geqq\alpha+\beta+1+(b_{k+1}-2)+(b_{k+2}-2)+\dots+$

$(b_{j-1}-2)\geqq b_i$ by (3.2), equality holding only if $\alpha=1$, $\beta=0$, $b_\lambda=2$ for $k+1\leqq\lambda\leqq j-1$, except when $i=k+1$ or $j-1$. By our assumption $i\neq j-1$. If $i=k+1$, then $\alpha=1$, $\beta=0$ and $b_iB_i=B_j+B_k$, hence $B_{k+2}=B_j$ which is absurd. Hence we have $\alpha=1$, $\beta=0$, $b_\lambda=2$ for any $k+1\leqq\lambda\leqq j-1$ and $k\leqq i-2$, $j-k\geqq(i+2)-(i-2)=4$. If $j-k\geqq 5$, then $b_\lambda=2$ for any λ which is absurd. Therefore $j-k=4$, $j=i+2$, $k=i-2$. So we must have $B_{i-2}+B_{i+2}=b_iB_i$. Since $b_\lambda=2$ for $i-1\leqq\lambda\leqq i+1$, we have $B_{i-2}+B_{i+2}=2B_{i-1}-2B_i+2B_{i+1}=2B_i$. So we are done.

LEMMA (3.9) *Suppose* $s=3$, $0\leqq i\leqq 2$.

1) *If* $nB_i-B_j\in(M^*)^+$, $1\leqq n\leqq b_i$, *then*

1-1) $j-i$ *or*

1-2) $j=i\pm 1$, *and* $n=b_i$

2) *Suppose* $(b_i+1)B_i-B_j\in(M^*)^+$.

2-1) *Assume* b_0, $b_1\geqq 3$. *Then* $j=i$, $i-1$ *or* $i+1$.

2-2) *Assume* $b_0\geqq 4$, $b_1=b_2=2$. *Then* $j=i, i-1, i+1$ *or* $(i,j)=(1,3)$ *and* $3B_1=2B_0+B_3$, *or* $(i,j)=(2,0)$, $3B_2=B_0+2B_3$.

2-3) *Assume* $b_0=3$, $b_1=b_2=2$. *Then* $j=i, i-1, i+1$, *or* $(i,j)=(1,3)$ *and* $3B_1=2B_0+B_3$, *or* $(2,0)$ *and* $3B_2=B_0+2B_3$, *or* $(i,j)=(0,2)$, $(0,-2)$ *and* $4B_0=B_2+B_{-2}$.

PROOF, 1) follows from (3.3) 1). In fact, if $j\geqq i+2$, then $b_k=2$ for $i\leqq k\leqq j-1$, hence $b_i=b_{i+1}=2$ and $n=b_i$, $j=i+2$ and $b_iB_i-B_j=B_{i-1}$. This is absurd because $b_iB_i-B_{i+2}=(b_i+1)B_i-2B_{i+1}=B_{i-1}+B_i-B_{i+1}\neq B_{i-1}$. If $j\leqq i-2$, then we can derive a contradiction similarly.

Next we shall prove 2-1). First we assume $j\geqq i+2$ and $wt((b_i+1)B_i-B_j)\geqq 2$. Then by (3.3) it is equal to 2 and $b_\lambda=2$ for $i\leqq\lambda\leqq j-1$. Hence $i=j-1$ because b_2 only can be 2. But $i=j-1$ contradicts the choice of j. Hence if $j\geqq i+2$, then $wt(b_i+1)B_i-B_j)=1$. So we write $(b_i+1)B_i=B_j+B_k$ for some k $(\leqq i-1)$. So by (3.2) $b_i+1\geqq 2+(b_{k+1}-2)+\ldots+(b_{j-1}-2)$, hence $k+1=i$ or $i-1$, and $j=i+2$.

If $k=i-1$ or $i-2$, then $B_{i-1}+B_{i+2}=(b_i-1)B_i+(b_{i+1}-1)B_{i+1}\neq(b_i+1)B_i$, $B_{i-2}+B_{i+2}=(2b_i-2)B_i+B_{i\pm 1}\neq(b_i+1)B_i$. In either case we have a contradiction. Hence $j\leqq i+1$. Similarly we can prove $j\geqq i-1$. Thus the proof of 2-1) is complete. The remaining assertions can be proved similarly. q. e. d.

§ 4 Main theorem (the cases s≧5).

THEOREM (4.1) *Let* T *be a cusp singularity with embedding dimension* s $(\geqq 5)$. *Then the space* $\boldsymbol{T}^1$ *of infinitesimal deformations of* T *is, as a* *subspace of* $H^1(V, H^0(\mathscr{D}, i_*\Theta_{\mathscr{D}-\mathscr{C}}))$, *generated by*

$$\delta_{i,j} := \theta(-iB_j)\delta_j, \quad 0 \leqq j \leqq s-1, \quad 1 \leqq i \leqq b_j - 1$$

where $\delta_j = B'_j\partial_1 - B_j\partial_2$.

(4. 1) is proved in (4. 8).

(4.2) For the proof of (4. 1) we recall the description of χ from [1]. By the proof of (1. 7) $H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$ is a subspace of $H^1(V, H^0(\mathscr{D}, i_*\Theta_{\mathscr{D}-\mathscr{C}}))$. Let χ_n be the restriction of $\chi: H^1(V, H^0(\mathscr{D}, i_*\Theta_{\mathscr{D}-\mathscr{C}})) \to H^1(V, H^0(\mathscr{D}, i_*\mathscr{O}_{\mathscr{D}-\mathscr{C}})) \otimes T_0(\boldsymbol{C}^s)$ to the subspace $H^1(V, H^0(\mathscr{D}, \Theta_{\mathscr{D}}(n\mathscr{C})))$. Then by the definition of χ, the image of χ_n is contained in $H^1(V, H^0(\mathscr{D}, \mathscr{O}_{\mathscr{D}}(n\mathscr{C}))) \otimes T_0(\boldsymbol{C}^s)$. By choosing a suitable embedding T into $(\boldsymbol{C}^s, 0)$, hence by choosing a basis $e_0, \dots, e_{s-1}$ of $T_0(\boldsymbol{C}^s)$, we can express by [1, Th. 5. 1]

$$\chi_n(\theta(\mu)\partial_1) = \sum_{j=0}^{s-1} \sum_k{}' B_{j+ks}\theta(\mu + B_{j+ks})e_j$$

$$\chi_n(\theta(\mu)\partial_2) = \sum_{j=0}^{s-1} \sum_k B'_{j+ks}\theta(\mu + B_{j+ks})e_j$$

where $\sum'$ is the summation over the set of all k with $\mu + B_{j+ks} \in -(M^*)^+ \cup \{0\}$. If $V\mu_1 = V\mu_2$, then $\theta(\mu_1) = \theta(\mu_2)$ in $H^1(V, H^0(\mathscr{D}, \mathscr{O}_{\mathscr{D}}(n\mathscr{C})))$, $(\theta(\mu_1)/\mu_1)\partial_1 = (\theta(\mu_2)/\mu_2)\partial_1$, $(\theta(\mu_1)/\mu'_1)\partial_2 = (\theta(\mu_2)/\mu'_2)\partial_2$ in $H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$. We notice the converse is true.

(4.3) Let $\mu \in -(M^*)^+$. We call μ *internal* if $\mu = -aB_i - bB_{i+1}$ for some i, $a(>0)$ and $b(>0)$. We call μ *i-extremal* if $\mu = -mB_i$ for some $m > 0$ and i. We say that μ is *extremal* if μ is *i-extremal* for some i. We define $wt^-\mu = wt(-\mu)$. Take $\xi \in H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$ and write

$$\xi = \textstyle\sum_{\mu \in B} \theta(\mu)(C(\mu)\partial_1 + D(\mu)\partial_2)$$

for a subset B of $B(n)$ and constants $C(\mu)$, $D(\mu)$. (See (1. 7) for the definition of $B(n)$.) We define

$$\xi_{int} = \textstyle\sum_{\mu \in B:\ internal} \theta(\mu)(C(\mu)\partial_1 + D(\mu)\partial_2),$$
$$\xi_{ext} = \textstyle\sum_{\mu \in B:\ extremal} \theta(\mu)(C(\mu)\partial_1 + D(\mu)\partial_2),$$
$$h(B) = max\ \{wt^-(\mu) - b_i\ ;\ \mu(\in B)\ i\text{-extremal},\ 0 \leqq i \leqq s-1\}.$$
$$[\xi]^m = \textstyle\sum_{\mu \in B,\ wt^-\mu \geqq m} \theta(\mu)(C(\mu)\partial_1 + D(\mu)\partial_2),$$
$$[\xi]_m = \textstyle\sum_{\mu \in B,\ wt^-\mu = m} \theta(\mu)(C(\mu)\partial_1 + D(\mu)\partial_2).$$

Take ρ, $\eta_j \in H^1(V, H^0(\mathscr{D}, \mathscr{O}_{\mathscr{D}}(n\mathscr{C})))$, $\eta \in H^1(V, H^0(\mathscr{D}, \mathscr{O}_{\mathscr{D}}(n\mathscr{C}))) \otimes T_0(\boldsymbol{C}^s)$, and suppose $\rho = \sum_{\mu \in B} E(\mu)\theta(\mu)$, $\eta = \eta_0 e_0 + \eta_1 e_1 + \dots + \eta_{s-1}e_{s-1}$ for a subset of B of $B(n)$. We define

$$[\rho]^{m}=\sum_{\mu\in B,\ wt^{-}\mu\geqq m} E(\mu)\theta(\mu),$$
$$[\rho]_{m}=\sum_{\mu\in B,\ wt^{-}\mu=m} E(\mu)\theta(\mu),$$
$$[\eta]^{m}=[\eta_0]^{m}e_0+[\eta_1]^{m}e_1+\ldots+[\eta_{s-1}]^{m}e_{s-1}$$
$$[\eta]_{m}=[\eta_0]_{m}e_0+[\eta_1]_{m}e_1+\ldots+[\eta_{s-1}]_{m}e_{s-1}.$$

Let $e_0^*,\ldots, e_{s-1}^*$ be a dual basis of $T_0(\boldsymbol{C}^s)^*=\mathrm{Hom}_{\boldsymbol{C}}(T_0(\boldsymbol{C}^s), \boldsymbol{C})$, then we have $\eta_j=(\eta, e_j^*)$, $[\eta_j]^m=([\eta]^m, e_j^*)$, $[\eta_j]_m=([\eta]_m, e_j^*)$.

LEMMA (4. 4) *Suppose* $s\geqq 5$, $\chi_n(\xi)=0$, $h(B)\geqq 0$. *Then*
1) $C(\mu)=D(\mu)=0$ *if* μ *is internal and if* $wt^{-}\mu\geqq h(B)+2$,
2) $C(\mu)=D(\mu)=0$ *if* μ *is i-extremal* $(0\leqq i\leqq s-1)$ *and if* $wt^{-}\mu\geqq h(B)+b_i$.

We consider $C(\mu)=D(\mu)=0$ if μ is not contained in B.

For the proof of (4. 4) we shall consider the following three cases separately ;
Case 1. there is no consecutive subsequence $b_j, b_{j+1},\ldots, b_{j+s-3}$ of b_i $(i\in\boldsymbol{Z})$ with $b_k=2$ for $j\leqq k\leqq j+s-3$,
Case 2. b_0, $b_1\geqq 3$, $b_l=2$ for $l\neq 0$, 1 mod s,
Case 3. $b_0\geqq 3$, $b_l=2$ for $l\neq 0$ mod s.

We notice that we may take $(b_0,\ldots, b_{s-1})$ for $(b_i, b_{i+1},\ldots, b_{i+s-1})$ for any i by a cyclic permutation.

(4.5) CASE 1. We shall prove, by induction on $l(\geqq h(B)+2)$ from the above, $\boldsymbol{E}(l)$: $C(\mu)=D(\mu)=0$ if μ is internal and if $wt^{-}\mu=l$. $\boldsymbol{E}(n+1)$ is clear. So we assume $\boldsymbol{E}(t)$ for $t\geqq l+1$ to prove $\boldsymbol{E}(l)$. Suppose $l\geqq h(B)+3$. By (3. 3) 1) and (3. 4) 1), if $-mB_i\in B$, we have $wt^{-}(-mB_i+B_{j+ks})\leqq m-b_i+1\leqq h(B)+1\leqq l-2$ for $i\neq j$, any m, and $wt^{-}(-mB_j+B_{j+ks})\leqq m-b_j<l-2$ for $k\neq 0$. Hence

A1) $$[\chi_n(\xi_{ext})]^{l-1}=\sum_{j}\sum_{m\geqq l}(C(-mB_j)B_j+D(-mB_j)B_j')\theta(-(m-1)B_j)e_j.$$

By $\boldsymbol{E}(t)$ for $t\leqq l+1$,

A2) $$[\chi_n(\xi_{int})]^{l-1}=\chi_n([\xi_{int}]_l)]_{l-1}$$
$$=\sum_{j}\sum_{k,\mu}^{2}(C(\mu)B_{j+ks}+D(\mu)B'_{j+ks})\theta(\mu+B_{j+ks})e_j$$

where $\sum^2$ is the summation for all k and μ such that $\mu\in B$, $wt^{-}\mu=l$, μ is internal and $\mu+B_{j+ks}\in-(M^*)^+\cup\{0\}$, $wt^{-}(\mu+B_{j+ks})=l-1$. We have

A3) $$([\chi_n(\xi)]^{l-1}, e_j^*)=([\chi_n(\xi_{ext})]^{l-1}, e_j^*)+([\chi_n(\xi_{int})]^{l-1}, e_j^*)$$

$$= \sum_{m \geqq l} (C(-mB_j)B_j + D(-mB_j)B'_j)\theta(-(m-1)B_j)$$
$$+ \sum_{k,\mu}^{2} (C(\mu)B_{j+ks} + D(\mu)B'_{j+ks})\theta(\mu + B_{j+ks}).$$

Let a, $b>0$, $a+b=l$. Then the coefficient of (the equivalence class of) $\theta(-aB_{j-1}-(b-1)B_j)$ in $([\chi_n(\xi)]^{l-1}, e_j^*)$ is equal to

$$\sum_{k,\mu}^{3} (C(\mu)B_{j+ks} + D(\mu)B'_{j+ks})$$

where $\sum^3$ is the summation for all k and μ such that $wt^-\mu = l$, $\mu = -\alpha B_i - \beta B_{i+1} \in B$ for some i $(0 \leqq i \leqq s-1)$, $\alpha > 0$, $\beta \geqq 0$, $\mu + B_{j+ks} = -aB_{j-1+hs} - (b-1) B_{j+ks}$ for some h. By (3.6) 2) and by $([\chi_n(\xi)]^{l-1}, e_j^*) = 0$, we have

A4) $0 = \sum^3 (C(\mu)B_{j+ks} + D(\mu)B'_{j+ks})$
$= C(-aB_{j-1} - bB_j)B_j + D(-aB_{j-1} - bB_j)B'_j.$

The coefficient of (the equivalence class of) $\theta(-(a-1)B_j - bB_{j+1})$ in $([\chi_n(\xi)]^{l-1}, e_j^*)$ is equal to

A5) $0 = \sum^4 (C(\mu)B_{j+ks} + D(\mu)B'_{j+ks})$
$= C(-aB_j - bB_{j+1})B_j + D(-aB_j - bB_{j+1})B'_j,$

where $\sum^4$ is the summation for all k, μ such that $\mu \in B$, $\mu + B_{j+ks} = -(a-1) B_{j+hs} - bB_{j+1+hs}$ for some h, $wt\ \mu = l$, μ is internal. Hence by substituting $j-1$ for j, we have

A6) $C(-aB_{j-1} - bB_j)B_{j-1} + D(-aB_{j-1} - bB_j)B'_{j-1} = 0.$

Since $[B_{j-1}, B_j] := B'_{j-1}B_j - B_{j-1}B'_j \neq 0$, we infer from A4) and A6) that

$$C(-aB_{j-1} - bB_j) = D(-aB_{j-1} - bB_j) = 0 \text{ for } a+b = l \geqq h(B)+3.$$

This completes the proof of $\boldsymbol{E}(l)$ for $l \geqq h(B)+3$.

Next consider the case $l = h(B)+2$. If μ is i-extremal and $\mu + B_{j+ks} = -aB_{j-1+hs} - (b-1)B_{j+hs}$ or $-(a-1)B_{j+hs} - bB_{j+1+hs}$, $a+b=l$, then $wt^-\mu \geqq a + b + b_i - 2$ by (3.6). By the definition of $h(B)$, $wt^-\mu - b_i \leqq h(B) = l-2$ for $\mu \in B$. Hence $wt^-\mu = a+b+b_i-2$. Therefore by (3.6) 2) $h=k=0$, $\mu = -aB_{j-1} - bB_j$ or $-aB_j - bB_{j+1}$ which is absurd. Therefore there is no extremal μ in B such that $\mu + B_{j+ks} = -aB_{j-1+hs} - (b-1)B_{j+hs}$ or $-(a-1)B_{j+hs} - bB_{j+1+hs}$, $a+b=l$. This implies that the coefficient of $\theta(-(a-1)B_{j-1} - bB_j)$ or $\theta(-aB_j - (b-1)B_{j+1})$ in $(\chi_n(\xi_{ext}), e_j^*)$ is equal to zero if $a+b=l$. This proves $\boldsymbol{E}(h(B)+2)$ by the same argument as in the case where $l \geqq h(B)+3$.

Next we shall show $C(\mu) = D(\mu) = 0$ if μ is i-extremal and if $wt^-\mu \geqq h(B) + b_i$. We notice that this is clear from the definition of $h(B)$ if $wt^-\mu \geqq$

$h(B)+b_i+1$. So let $m_i=h(B)+b_i$, $0\leqq i\leqq s-1$.

Suppose $h(B)>0$. The coefficient of $\theta(-(m_j-1)B_j)$ in $(\chi_n(\xi), e_j^*)$ equals

A7) $C(-m_jB_j)B_j+D(-m_jB_j)B'_j\ (=0)$

by (3.6) 1). The coefficient of $\theta(-h(B)B_{j-f}-B_{j-2})$ in $(\chi_n(\xi), e_j^*)$ equals, by (3.7) 1),

A8) $$\begin{aligned}0&=\sum_{\mu:int}(C(\mu)B_{j+ks}+D(\mu)B'_{j+ks})+\sum_{\mu:ext}(C(\mu)B_{j+ks}+D(\mu)B'_{j+ks})\\&=0+C(-m_{j-1}B_{j-1})B_j+D(-m_{j-1}B_{j-1})B'_j.\end{aligned}$$

Substituting j for $j-1$, we have $C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=0$. From this and A7), we infer $C(-m_jB_j)=D(-m_jB_j)=0$.

Finally we consider the case $h(B)=0$, $m_i=b_i$. First we assume that there is no consecutive subsequence $b_{j+1}, b_{j+2}, \ldots, b_{j+s-3}(=2)$ of $b_k\ (k\in\boldsymbol{Z})$. Then by (3.6) 1)

A9) $C(-m_jB_j)B_j+D(-m_jB_j)B'_j=0$.

By (3.7) 1)

A10) $C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=0$ so $C(-m_jB_j)=D(-m_jB_j)=0$.

If there is a consecutive subsequence $b_{j+1}, \ldots, b_{j+s-3}(=2)$ of $b_k\ (k\in\boldsymbol{Z})$, we may assume that $b_0, b_1, b_2\geqq 3$, $b_j=2\ (3\leqq j\leqq s-1)$. Then by (3.7) 1)-5)

A11) $C(-m_1B_1)B_2+D(-m_1B_1)B'_2+C(-B_2-B_s)B_2+D(-B_2-B_s)B'_2=0$,
A12) $C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=0$ for $j\neq 1$.

Hence by A9) and A12) we have $C(-m_jB_j)=D(-m_jB_j)=0$ for $j\neq 1$. Since $B_2+B_s=B_k+B_{k+1}$ or $2\mathrm{B}_k$ $(3\leqq k\leqq s-1)$ according as $s=2k-1, 2k-2$, we have $C(-B_2-B_s)=D(-B_2-B_s)=0$ by $\boldsymbol{E}(2)$ or the above. Hence by A9) and A11) $C(-m_1B_1)=D(-m_1B_1)=0$. This completes the proof of (4.4) in Case 1.

(4.6) CASE 2. We consider the case $b_0, b_1\geqq 3$, $b_j=2$ for $2\leqq j\leqq s-1$. We shall prove $\boldsymbol{E}(l)$ by induction on $l(\geqq h(B)+3)$. Assume $\boldsymbol{E}(t)$ for $t\geqq l+1$. Let $a, b>0$, $a+b=l$, $m_i=h(B)+b_i$. Then $C(\mu)=D(\mu)=0$ if μ is internal and $wt^-\mu\geqq l+1$. And $C(\mu)=D(\mu)=0$ if μ is i-extremal and if $wt^-\mu\geqq m_i+1$. As is easily seen from (3.6) 2) and 3), the coefficients of $\theta(-aB_{j-1}-(b-1)B_j)$, $\theta(-(a-1)B_j-bB_{j+1})$ in $(\chi_n(\xi), e_j^*)$ are equal to

B1) $$\begin{cases}C(-aB_{j-1}-bB_j)B_j+D(-aB_{j-1}-bB_j)B'_j & (j\neq 1 \text{ or } b>1)\\ C(-aB_0-B_1)B_1+D(-aB_0-B_1)B'_1+C(-B_1-aB_s)B_1+\\ D(-B_1-aB_s)B'_1 & (j=1, b=1)\end{cases}$$

and

$$\mathrm{B}2)\begin{cases} C(-aB_j-bB_{j+1})B_j+D(-aB_j-bB_{j+1})B'_j & (j\neq 0 \text{ or } a>1)\\ C(-B_0-bB_1)B_0+D(-B_0-bB_1)B'_0+C(-bB_1-B_s)B_s+ & \\ D(-bB_1-B_s)B'_s & (j=0,\ a=1)\end{cases}$$

Hence

B3) $C(-aB_{j-1}-bB_j)B_j+D(-aB_{j-1}-bB_j)B'_j=0$

B4) $C(-aB_{j-1}-bB_j)B_{j-1}+D(-aB_{j-1}-bB_j)B'_{j-1}=0$

for $j\neq 1$. Hence $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$ for $j\neq 1 \mod s$. Since $wt(B_1+aB_s)=wt(bB_1+B_s)=l\geqq 3$ and both B_1+aB_s and bB_1+B_s are of the form $\alpha B_{k-1}+\beta B_k$ $(2\leqq k\leqq s)$ or αB_k $(2\leqq k\leqq s-1)$, $C(-B_1-aB_s)=D(-B_1-aB_s)=C(-bB_1-B_s)=D(-bB_1-B_s)=0$. Hence $C(-aB_0-B_1)=D(-aB_0-B_1)=C(-B_0-bB_1)=D(-B_0-bB_1)=0$. By B1) and B2) we have $C(-aB_0-bB_1)=D(-aB_0-bB_1)=0$ for a, $b>1$. This completes the proof of $\boldsymbol{E}(l)$ for $l\geqq h(B)+3$.

Next consider the case $l=h(B)+2$. We shall show $\boldsymbol{E}(h(B)+2)$ and that $\boldsymbol{F}$: $C(-m_iB_i)=D(-m_iB_i)=0$ for any i. Suppose first $h(B)>0$. The coefficient of $\theta(-aB_{j-1}-(b-1)B_j)$ and $\theta(-(a-1)B_{j-1}-bB_j)$ are the same as in B1) and B2) where $l\geqq h(B)+3$. Hence

B5) $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$

if $j\neq 1$ or if $j=1$ and a, $b>1$, $a+b=l$. Next the coefficient of $\theta(-(m_j-1)B_j)$ in $(\chi_n(\xi),\ e^*_j)$ is by (3.6)

B6) $C(-m_jB_j)B_j+D(-m_jB_j)B'_j$.

The coefficients of $\theta(-h(B)B_{j-1}-B_{j-2})$ and $\theta(-h(B)B_{j+1}-B_{j+2})$ in $(\chi_n(\xi),\ e^*_j)$ are equal respectively to

$$\mathrm{B}7)\begin{cases} C(-m_{j-1}B_{j-1})B_j+D(-m_{j-1}B_{j-1})B'_j & (j\neq 1)\\ C(-m_0B_0)B_1+D(-m_0B_0)B'_j+C(-B_1-B_{s-1}-h(B)B_s)B_1+ & \\ D(-B_1-B_{s-1}-h(B)B_s)B'_1 & (j=1)\end{cases}$$

and

$$\mathrm{B}8)\begin{cases} C(-m_{j+1}B_{j+1})B_j+D(-m_{j+1}B_{j+1})B'_j & (j\neq 0)\\ C(-m_1B_1)B_0+D(-m_1B_1)B'_0+C(-h(B)B_1-B_2-B_s)B_s+ & \\ D(-h(B)B_1-B_2-B_s)B'_s & (j=0)\end{cases}$$

hence we have

B9) $C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=0$ $(j\neq 0)$

B10) $C(-m_jB_j)B_{j-1}+D(-m_jB_j)B'_{j-1}=0$ $(j\neq 1)$

B11) $C(-m_jB_j)B_j+D(-m_jB_j)B'_j=0$ (for any j)

Therefore $C(-m_jB_j)=D(-m_jB_j)=0$ for any j. Hence by B5) $C(-B_1-aB_s)=D(-B_1-aB_s)=C(-bB_1-B_s)=D(-bB_1-B_s)=0$. Hence by the coefficients of $\theta(-aB_{j-1})$ and $\theta(-bB_{j+1})$ in $(\chi_n(\xi), e_j^*)$ we have $C(-aB_0-B_1)=D(-aB_0-B_1)=C(-B_0-bB_1)=D(-B_0-bB_1)=0$ for $a=b=l-1$. Hence the proof of $\boldsymbol{E}(h(B)+2)$ and $\boldsymbol{F}$ is complete.

We consider the case $h(B)=0$, $l=2$. Since we need no essential change of the previous proof, we shall give an outline of our proof. First we can show $C(-B_{j-1}-B_j)=D(-B_{j-1}-B_j)=0$ for $j\neq 1$. From the coefficients of $\theta(-B_{j-2})$ and $\theta(-B_{j+2})$ in $(\chi_n(\xi), e_j^*)$ we infer

B12) $C(-m_{j-1}B_{j-1})B_j+D(-m_{j-1}B_{j-1})B'_j=0$ $(j\neq 1, 2)$
B13) $C(-m_{j+1}B_{j+1})B_j+D(-m_{j+1}B_{j+1})B'_j=0$ $(j\neq -1, 0)$, hence
$C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=C(-m_jB_j)B_{j-1}+D(-m_jB_j)B'_{j-1}=0$
for $j\neq 0, 1$.

Hence $C(-m_jB_j)=D(-m_jB_j)=0$ for $j\neq 0, 1$. Since $B_1+B_s=B_{k-1}+B_k$ or $2B_k$ according as $s=2k-2$, $2k-1$, we have $C(-B_1-B_s)=D(-B_1-B_s)=0$. Then by the coefficient of $\theta(-B_{j-1})$ or $\theta(-B_{j+1})$ in $(\chi_n(\xi), e_j^*)$, we have $C(-B_0-B_1)=D(-B_0-B_1)=0$. Then by B6), B7) and B8) we have $C(-m_0B_0)=D(-m_0B_0)=C(-m_1B_1)=D(-m_1B_1)=0$. This completes the proof of $\boldsymbol{E}(2)$ and $\boldsymbol{F}$.

(4.7) CASE 3. Finally we consider the case $b_0\geqq 3$, $b_j=2$ for $1\leqq j\leqq s-1$. We shall prove $\boldsymbol{E}(l)$ for $l\geqq h(B)+3$ by the induction on l. Assume $\boldsymbol{E}(t)$ for $t\geqq l+1$ to prove $\boldsymbol{E}(l)$. Let a, $b>0$, $a+b=l$, $m_i=h(B)+b_i$ $(0\leqq i\leqq s-1)$. We notice that $C(\mu)=D(\mu)=0$ if μ is i-extremal and if $wt^-\mu\geqq m_i+1$. By the coefficients of $\theta(-aB_{j-1}-(b-1)B_j)$ and $\theta(-(a-1)B_j-bB_{j+1})$ we have

C1) $C(-aB_{j-1}-bB_j)B_j+D(-aB_{j-1}-bB_j)B'_j=0$ $(j\neq 1, s$ or $j=1, b>1)$
C2) $C(-aB_{s-1}-bB_s)B_s+D(-aB_{s-1}-bB_s)B'_s+C(-B_0-aB_{s-1}-(b-1)B_s)B_0+D(-B_0-aB_{s-1}-(b-1)B_s)B'_0=0$ $(j=s)$
C3) $C(-aB_0-B_1)B_1+D(-aB_0-B_1)B'_1+C(-B_1-aB_s)B_1+D(-B_1-aB_s)B'_1=0$ $(j=b=1)$

and

C4) $C(-aB_j-bB_{j+1})B_j+D(-aB_j-bB_{j+1})B'_j=0$ $(j\neq 0, s-1$ or $j=s-1, a>1)$
C5) $C(-aB_0-bB_1)B_0+D(-aB_0-bB_1)B'_0+C(-B_s-(a-1)B_0-bB_1)B_s+D(-B_s-(a-1)B_0-bB_1)B'_s=0$ $(j=0)$

C6) $C(-B_{s-1}-bB_s)B_{s-1}+D(-B_{s-1}-bB_s)B'_{s-1}+C(-bB_0-B_{s-1})B_{s-1}+D(-bB_0-B_{s-1})B'_{s-1}=0$ $(j=s-1,\ a=1)$

Hence

C7) $C(-aB_{j-1}-bB_j)B_{j-1}+D(-aB_{j-1}-bB_j)B'_{j-1}=0$ $(j\neq 1,\ s,$ or $j=s,\ a>1)$.

Hence $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$ for $j\neq 0,\ 1 \bmod s$. By substituting $b=1$ in C2) we have, from $C(-B_0-aB_{s-1})=D(-B_0-aB_{s-1})=0$,

C8) $C(-aB_{s-1}-B_s)B_s+D(-aB_{s-1}-B_s)B'_s=0$.

By C1) and C4) $C(-aB_{s-1}-B_s)=D(-aB_{s-1}-B_s)=0$ for $a=l-1>1$. We consider $\boldsymbol{E}^*(b')$: $C(-a'B_{s-1}-b'B_s)=D(-a'B_{s-1}-b'B_s)=0$ where $a'+b'=l,\ a',\ b'>0$. We assume $\boldsymbol{E}^*(b')$ for any $b'<b$ and prove $\boldsymbol{E}^*(b)$ for $1\leqq b\leqq l-2$. By C2) and C7) we have

C9) $C(-aB_{s-1}-bB_s)B_s+D(-aB_{s-1}-bB_s)B'_s=0$.

Hence by C4) for $a>1$, $C(-aB_{s-1}-bB_s)=D(-aB_{s-1}-bB_s)=0$. This completes the proof of $\boldsymbol{E}^*(b)$ for $b\leqq l-2$. Similarly

C10) $C(-aB_0-bB_1)=D(-aB_0-bB_1)=0$ for $a+b=l,\ a>0,\ b>1$.

So we infer from C2), C3), C5) and C6)

$$C(-(l-1)B_0-B_1)B_1+D(-(l-1)B_0-B_1)B'_1=0$$
$$C(-(l-1)B_0-B_1)B_0+D(-(l-1)B_0-B_1)B'_0=0$$
$$C(-B_{s-1}-(l-1)B_s)B_s+D(-B_{s-1}-(l-1)B_s)B'_s=0,$$
$$C(-B_{s-1}-(l-1)B_s)B_{s-1}+D(-B_{s-1}-(l-1)B_s)B'_{s-1}=0.$$

Hence $C(\mu)=D(\mu)=0$ for $\mu=-(l-1)B_0-B_1$ or $-B_{s-1}-(l-1)B_s$. This completes the proof of $\boldsymbol{E}(l)$ for $l\geqq h(B)+3$.

Next we consider the case $l=h(B)+2,\ h(B)>0$. We shall prove $\boldsymbol{E}(h(B)+2)$ and $\boldsymbol{F}$: $C(-m_iB_i)=D(-m_iB_i)=0$. In the same way as in Cases 1 and 2 we have

C11) $C(-m_jB_j)B_j+D(-m_jB_j)B'_j=0$ $(j\neq 0)$

C12) $C(-m_0B_0)B_0+D(-m_0B_0)B'_0+C(-(m_0-1)B_0-B_s)B_s+D(-(m_0-1)B_0-B_s)B'_s+C(-(m_0-1)B_s-B_0)B_0+D(-(m_0-1)B_s-B_0)B'_0=0$

C13) $C(-m_{j-1}B_{j-1})B_j+D(-m_{j-1}B_{j-1})B'_j=0$ $(j\neq 1,\ s)$

C14) $C(-m_{s-1}B_{s-1})B_s+D(-m_{s-1}B_{s-1})B'_s+C(-B_0-B_{s-2}-h(B)B_{s-1})B_0+D(-B_0-B_{s-2}-h(B)B_{s-1})B'_0=0$

C15) $C(-m_0B_0)B_1+D(-m_0B_0)B'_1+C(-B_1-B_{s-1}-h(B)B_s)B_1+$

$D(-B_1-B_{s-1}-h(B)B_s)B'_1=0,$

C16) $C(-m_{j+1}B_{j+1})B_j+D(-m_{j+1}B_{j+1})B'_j=0 \qquad (j\neq 1,\ 0)$

C17) $C(-m_1B_1)B_0+D(-m_1B_1)B'_0+C(-h(B)B_1-B_2-B_s)B_s+$
$D(-h(B)B_1-B_2-B_s)B'_s=0$

C18) $C(-m_0B_0)B_{-1}+D(-m_0B_0)B'_{-1}+C(-h(B)B_0-B_1-B_{s-1})B_{s-1}+$
$D(-h(B)B_0-B_1-B_{s-1})B'_{s-1}=0.$

Hence by C11) C13) and C16) $C(-m_jB_j)=D(-m_jB_j)=0$ for $j\neq 0$. In the same way as in Cases 1, 2 we infer also $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$ for $a+b=l$, a, $b>0$, $j\neq 1$, s. So in the same way as in the case $l\geqq h(B)+3$ we have $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$ for $a+b=l$, a, $b>0$, $j=1$, s. By C12) and C15) we have $C(-m_0B_0)=D(-m_0B_0)=0$. Thus the proof of $\boldsymbol{E}(h(B)+2)$ and $\boldsymbol{F}$ is complete if $h(B)>0$.

Finally consider the case $h(B)=0$, $l=2$. It is easy to see $C(-B_{j-1}-B_j)=D(-B_{j-1}-B_j)=0$ $(j\neq 1,\ s)$, $C(-2B_k)=D(-2B_k)=0$ $(k\neq 0,\ 1,\ s-1)$. We prove $C(\mu)=D(\mu)=0$ for $\mu=-2B_{s-1}$, $-b_0B_0$, $-2B_1$, $-B_0-B_1$, $-B_{s-1}-B_s$. We have C2) $a=b=1$, C3) $a=1$, C5) $a=b=1$, C6) $b=1$, C11), C12), C14), C15), C17) and C18) where $m_1=m_{s-1}=2$. By C2), C6) with $a=b=1$ we have $C(-B_{s-1}-B_s)=D(-B_{s-1}-B_s)=0$. By C3), C5) with $a=b=1$ we have $C(-B_0-B_1)=D(-B_0-B_1)=0$. By C11) $j=1$, C17) we have $C(-2B_1)=D(-2B_1)=0$. By C11), C14) we have $C(-2B_{s-1})=D(-2B_{s-1})=0$. Finally by C12), C15) $C(-m_0B_0)=D(-m_0B_0)=0$. Thus we complete the proof of $\boldsymbol{E}(2)$ and $\boldsymbol{F}$.

(4.8) *Proof of Theorem* (4.1). In (4.5)-(4.7) we have proved Lemma (4.4). (4.4) implies that if $h(B)>0$, then ξ is expressed as a linear combination $\sum_{\mu\in B'}(C(\mu)\theta(\mu)\partial_1+D(\mu)\theta(\mu)\partial_2)$ with a finite subset B' of $B(n)$, strictly smaller than B. If $h(B')>0$, then we can apply Lemma (4.4) again. Exentually we obtain B'' with $h(B'')\leqq 0$. If $h(B'')<0$, then by adding $-b_jB_j$ to B'' we may assume that $h(B'')=0$. Then by Lemma (4.4) we have $C(\mu)=D(\mu)=0$ for all internal μ, because any internal μ has $wt^-\mu\geqq 2$. By Lemma (4.4) we also have $C(-mB_j)=D(-mB_j)=0$ for $m\geqq b_j$. So we can write

$$\xi=\sum_j{}'\sum_{m<b_j}(C(-mB_j)\theta(-mB_j)\partial_1+D(-mB_j)\theta(-mB_j)\partial_2).$$

Then by the expression of $\chi_n(\xi_{ext})$ in (4.5) for $l=1$, we derive $C(-mB_j)B_j+D(-mB_j)B'_j=0$ for any j and $m<b_j$. This shows that Ker χ is generated by $\delta_{i,j}$ $(1\leqq i\leqq b_j-1,\ 0\leqq j\leqq s-1)$.

§ 5 The cases $1 \leqq s \leqq 4$.

(5.1) In the cases $1 \leqq s \leqq 4$, the cusp singularity T is a complete intersection singularity of embedding dimension 3 ($s \leqq 3$) or 4 ($s=4$). Their defining equations are known explicitly;

$$T_{p,q,r}: x^p+y^q+z^r-xyz=0 \quad (s \leqq 3)$$
$$\Pi_{p,q,r,t}: x^p+z^r-yw=y^q+w^t-xz=0 \quad (s=4).$$

We notice that by [6], T is isomorphic to $\Pi_{p,q,r,t}$ iff $(p, q, r, t)=(b_0, b_1, b_2, b_3)$. By a well known formula [9] $\dim \boldsymbol{T}^1 = \dim \mathrm{Ext}^1(\Omega^1_T, \mathcal{O}_T)=p+q+r-2$, $p+q+r+t-2$ respectively. Hence in order to describe $\boldsymbol{T}^1$, we suffice to find $(p+q+r-2)$ or $(p+q+r+t-2)$ linearly independent elements in Ker χ. This is an easier way to describe $\boldsymbol{T}^1$ than doing in the same way as in § 4.

THEOREM (5.2) *Suppose* $s=4$. *Then* $\dim \boldsymbol{T}^1$ *is equal to* $\sum_{j=0}^{3}(b_j-1)+2$. *A basis of* $\boldsymbol{T}^1$ *is given by* $\delta_{i,j}:=\theta(-iB_j)\delta_j$ $(1 \leqq i \leqq b_j-1,\ 0 \leqq j \leqq 3)$ *and* $\theta(-b_0B_0)\delta_0+\theta(-b_2B_2)\delta_2$, $\theta(-b_1B_1)\delta_1+\theta(-b_3B_3)\delta_3$ *where* $\delta_j=B'_j\partial_1-B_j\partial_2$.

PROOF. Let $\xi_1=\theta(-b_0B_0)\delta_0+\theta(-b_2B_2)\delta_2$, $\xi_2=\theta(-b_1B_1)\delta_1+\theta(-b_3B_3)\delta_3$. Then we have by (3.8),

$$\chi(\xi_1)=\theta(-b_0B_0+B_1)[B_0, B_1]+\theta(-b_2B_2+B_1)[B_2, B_1])e_1$$
$$+(\theta(-b_0B_0+B_{-1})[B_0, B_{-1}]+\theta(-b_2B_2+B_3)[B_2, B_3])e_3$$
$$\chi(\xi_2)=(\theta(-b_1B_1+B_2)[B_1, B_2]+\theta(-b_3B_3+B_2)[B_3, B_2])e_2$$
$$+(\theta(-b_1B_1+B_0)[B_1, B_0])+\theta(-b_3B_3+B_4)[B_3, B_4])e_0$$

where $[A, B]=A'B-AB'$. Since $[B_2, B_1]=[-B_0+b_1B_1, B_1]=-[B_0, B_1]$, $[B_k, B_{k+1}]=[B_0, B_1]$ for any k, $[A, B]=-[B, A]$, $\theta(-B_{-1})=\theta(-B_3)$. $\theta(-B_0)=\theta(-B_4)$ in $H^1(V, H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(n\mathcal{C}))$ for $n \geqq 1$, we have $\chi(\xi_1)=\chi(\xi_2)=0$. It is clear from (1.10) that ξ_1, ξ_2 and $\delta_{i,j}$ are linearly indepe-ndent.

q. e. d.

THEOREM (5.3) *Suppose that* $s=3$. *Then* $\dim \boldsymbol{T}^1=b_0+b_1+b_2+1$. *A basis of* $\boldsymbol{T}^1$ *is given by* $\delta_{i,j}:=\theta(-iB_j)\delta_j$ $(1 \leqq i \leqq b_j-1)$,

$$\xi:=\sum_{j=0}^{2}\theta(-(b_j+1)B_j)\delta_j \quad (\textit{if } b_0, b_1 \geqq 3 \textit{ or } b_0 \geqq 4);$$
$$\sum_{j=0}^{2}\theta(-(b_j+1)B_j)\delta_j-2\theta(-B_1-B_2)(\delta_1+\delta_2) \quad (\textit{if } (b_0, b_1, b_2)=(3, 2, 2))$$
$$\xi_j:=\theta(-b_jB_j)\delta_j+\theta(-B_{j+1}-B_{j+2})(\delta_{j+1}+\delta_{j+2}) \quad (j=0, 1, 2)$$

where $\delta_j=B'_j\partial_1-B_j\partial_2$.

PROOF. By (3.9) $\delta_{i,j}$ is contained in Ker χ. Let $p=b_0+1$, $q=b_1+1$, $r=b_2+1$. Then if $(p, q, r)=(4,3,3)$, by (3.9)

$$\begin{aligned}\chi(\xi)=&\theta(-4B_0+B_2)[B_0, B_2]e_2+\theta(-4B_0+B_{-2})[B_0, B_{-2}]e_1\\&+\theta(-4B_0+B_1)[B_0, B_1]e_1+\theta(-4B_0+B_{-1})[B_0, B_{-1}]e_2\\&+\theta(-3B_1+B_2)[B_1, B_2]e_2+\theta(-3B_1+B_0)[B_1, B_0]e_0\\&+\theta(-3B_2+B_3)[B_2, B_3]e_0+\theta(-3B_2+B_1)[B_2, B_1]e_1\\&+\theta(-3B_1+B_3)[B_1, B_3]e_0+\theta(-3B_2+B_0)[B_2, B_0]e_0\\&-2\theta(-B_1)[B_1, B_2]e_2-2\theta(-B_2)[B_2, B_1]e_1=0\end{aligned}$$

If $p, q\geqq 4$ or $p\geqq 5$, then

$$\begin{aligned}\chi(\xi)=&\theta(-pB_0+B_1)[B_0, B_1]e_1+\theta(-pB_0+B_{-1})[B_0, B_{-1}]e_1\\&+\theta(-qB_1+B_2)[B_1, B_2]e_2+\theta(-qB_1+B_0)[B_1, B_0]e_0\\&+\theta(-rB_2+B_3)[B_2, B_3]e_0+\theta(-rB+B_1)[B_2, B_1]e_1=0\end{aligned}$$

Similarly by (3.9) $\chi(\xi_j)=0$. q. e. d.

(5.4) In the cases $s=1$ and 2, we define

$s=1)$ $C_{3k}=2B_{k-1}+B_k$, $C_{3k+1}=B_{k-1}+B_k$, $C_{3k+2}=B_k$ $(k\in\mathbf{Z})$
$s=2)$ $C_{3k}=B_{2k-1}+B_{2k}$, $C_{3k+1}=B_{2k}$, $C_{3k+2}=B_{2k+1}$ $(k\in\mathbf{Z})$.

We also define a homomorphism χ of $H^1(V, H^0(\mathscr{D}, \tilde{\Theta}_{\mathscr{D}}(n\mathscr{C})))$ into $H^1(V, H^0(\mathscr{D}, \mathscr{O}_{\mathscr{D}}(n\mathscr{C})))^3$ by

$$\chi(C(\mu)\theta(\mu)\partial_1+D(\mu)\theta(\mu)\partial_2)=\sum_{j=0}^{2}\ \sum_k{}'\ (C(\mu)C_{j+3k}+D(\mu)C'_{j+3k})$$
$$\theta(\mu+C_{j+3k})e_j$$

where $\mu\in B(n)$, $\sum_k{}'$ is the summation for all k such that $\mu+C_{j+3k}\in-(M^*)^+\cup\{0\}$. By [1,7] the space $\boldsymbol{T}^1$ is Ker χ.

THEOREM (5.5) *Suppose* $s=2$. *Then* $\dim\boldsymbol{T}^1=b_0+b_1+4$. *A basis of* $\boldsymbol{T}^1$ *is given by*

$$\begin{aligned}\delta_{i,j}\ &:\ =\theta(-iB_j)\delta_j\ \ j=0,\ 1,\ 1\leqq i\leqq b_j\\ \xi\ &:\ =\theta(-(b_0+2)B_0)\delta_0+\theta(-(b_1+2)B_1)\delta_1+\theta(-2B_{-1}-2B_0)\\&\qquad(\delta_{-1}+\delta_0)+\theta(-2B_0-2B_1)(\delta_0+\delta_1)\\ \xi_0\ &:\ =\theta(-B_{-1}-B_0)(\delta_0+\delta_{-1})+\theta(-B_0-B_1)(\delta_0+\delta_1)\\ \xi_1\ &:\ =\theta(-(b_0+1)B_0)\delta_0+\theta(-2B_1-B_2)(2\delta_1+\delta_2)+\\&\qquad\theta(-B_0-2B_1)(\delta_0+2\delta_1)\\ \xi_2\ &:\ =\theta(-(b_1+1)B_1)\delta_1+\theta(-B_1-2B_2)(\delta_1+2\delta_2)+\\&\qquad\theta(-2B_0-B_1)(2\delta_0+\delta_1)\end{aligned}$$

where $\delta_j = B'_j\partial_1 - B_j\partial_2$.

THEOREM (5.6) *Suppose* $s=1$. *Then* $\dim \boldsymbol{T}^1 = b_0 + 7$. *Let* $b = b_0$, $\delta_j = B'_j\partial_1 - B_j\partial_2$ *and suppose* $b \geqq 4$. *Then a basis of* $\boldsymbol{T}^1$ *is given by*

$$\begin{aligned}
\delta_{1,1} &: = \theta(-B_{-1} - B_0)(\delta_{-1} + \delta_0), \\
\delta_{i,2} &: = \theta(-iB_0)\delta_0 \quad (1 \leqq i \leqq b+1) \\
&\quad\ \theta(-(b+2)B_0)\delta_0 + \theta(-2B_{-1} - 2B_0)(\delta_{-1} + \delta_0) \quad (i = b+2) \\
\zeta_0 &: = \theta(-2B_{-1} - B_0)(2\delta_{-1} + \delta_0) + \theta(-B_{-1} - 2B_0)(\delta_{-1} + 2\delta_0) \\
\zeta_1 &: = \theta(-2B_{-1} - 2B_0)(\delta_{-1} + \delta_0) + \theta(-3B_0 - B_1)(3\delta_0 + \delta_1) + \\
&\quad\ \theta(-B_0 - 3B_1)(\delta_0 + 3\delta_1) \\
\zeta_2 &: = \theta(-(b+3)B_0)\delta_0 + \theta(-3B_0 - 2B_1)(3\delta_0 + 2\delta_1) + \\
&\quad\ \theta(-2B_0 - 3B_1)(2\delta_0 + 3\delta_1) + 2\theta(-4B_0 - B_1)(4\delta_0 + \delta_1) + \\
&\quad\ 2\theta(-B_0 - 4B_1)(\delta_0 + 4\delta_1) \\
\zeta &: = \theta(-(b+4)B_0)\delta_0 + \theta(-3B_0 - 3B_1)(\delta_0 + \delta_1) + \\
&\quad\ \theta(-4B_0 - 2B_1)(2\delta_0 + \delta_1) + \theta(-2B_0 - 4B_1)(\delta_0 + 2\delta_1) + \\
&\quad\ \theta(-5B_0 - B_1)(5\delta_0 + \delta_1) + \theta(-B_0 - 5B_1)(\delta_0 + 5\delta_1) + \\
&\quad\ 9\delta_{b,4}\theta(-6B_0)\delta_0
\end{aligned}$$

where $\delta_{b,4} = 0$ $(b \neq 4)$, $1 (b=4)$.

Theorems (5.5) and (5.6) are proved in the same manner as (4.1), (5.2) and (5.3). We notice that $\theta(-aB_{j+s} - bB_{k+s})\delta_{l+s} = \theta(-aB_j - bB_k)\delta_l$ for any j, k, l. We were unable to find a basis of $\boldsymbol{T}^1$ in the case $s=1$, $b_0 = b = 3$. In this case $\delta_{1,1}$, $\delta_{i,2}$, $(1 \leqq i \leqq b+2)$ ζ_0, ζ_1 and $\zeta_2 + 14\theta(-5B_0)\delta_0$ are contained in Ker χ and they are linearly independent. It remains to find ξ such that $\chi(\zeta + \xi) = 0$.

CONJECTURE. There is a natural way to relate the bases in (5.2) and (5.3) and the deformations of $\Pi_{p,q,r,t}$ and $T_{p,q,r}$ given by

$$\begin{aligned}
&x^p + z^r + \sum_{j=1}^{p-1} u_{0,j}x^j + \sum_{j=1}^{r-1} u_{2,j}z^j - yw + u_1 = 0, \\
&y^q + w^t + \sum_{j=1}^{q-1} u_{1,j}y^j + \sum_{j=1}^{t-1} u_{3,j}w^j - xz + u_2 = 0
\end{aligned}$$

and

$$\begin{aligned}
&x^p + y^q + z^r + \sum_{j=1}^{p-2} v_{0,j}x^j + \sum_{j=1}^{q-2} v_{1,j}y^j + \sum_{j=1}^{r-2} v_{2,j}z^j \\
&+ v_0(x^{p-1} + yz) + v_1(y^{q-1} + wx) + v_2(z^{r-1} + xy) - xyz + v = 0
\end{aligned}$$

where $T_{4,3,3}$ is perhaps to be excluded. (Compare [7, 8].)

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