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Gap Theorems for Hypersurfaces in R^N

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Introduction

It is an interesting problem in global differential geometry to determine the Riemannian manifolds which are close to some typical space. The Riemannian manifolds which are close to the standard sphere are dealt with in the famous sphere theorem. Recently Greene and Wu [2] proved the following gap theorem for the complete metrics on $\boldsymbol{R}^n$ $(n\geqq 3)$ which are asymptotically flat, i.e., close to the standard metric in some sense.

THE GAP THEOREM OF GREENE AND WU. *A Riemannian manifold M of odd dimension n is isometric to $\boldsymbol{R}^n$ if (and only if) the following conditions hold*:

(i) *There is a point 0 in M that is a pole, i.e.,* $\exp_0: T_0M\rightarrow M$ *is a diffeomorphism.*

(ii) *The curvature of M is either everywhere nonpositive or everywhere nonnegative.*

(iii) $\liminf_{S\rightarrow+\infty} s^2k(s)=0$, *where* $k(s)=\sup\{|$*sectional curvature at* $q|\,; q\in M,\ \mathrm{dist}_M(0, q)=s\}$.

They also showed that the same conclusion holds for even dimension $n\geqq 4$, with some additional assumptions.

The purpose of this paper is to prove gap theorems of this type for asymptotically linear hypersurfaces in a euclidean space $\boldsymbol{R}^N$ $(N\geqq 3)$. Let M be a hypersurface of $\boldsymbol{R}^N$ and $\alpha(p)$ denote the second fundamental form of M at a point p. Put $\tilde{k}(s)=\sup\{\|\alpha(p)\|^2; |p|=s\}$, where $|p|$ stands for the euclidean norm of $p\in\boldsymbol{R}^N$.

THEOREM 1. *Let M be a noncompact properly imbedded convex hypersurface of $\boldsymbol{R}^N$ with* $\liminf_{S\rightarrow+\infty} s^2\tilde{k}(s)=0$. *Then M is a hyperplane.*

We note that the theorem includes the case of 2-dimensional hypersurfaces.

Owing to the convexity condition of Sacksteder [4] combined with a result of Hartman and Nirenberg [3], we also prove

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THEOREM 2. *Let M be a noncompact properly imbedded hypersurface of $\boldsymbol{R}^N$ with $\liminf_{s\to+\infty} s^2\tilde{k}(s)=0$. Suppose that the Ricci curvature of M is everywhere nonnegative. Then M is a hyperplane.*

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§ 1. The asymptotic cone of a convex domain and some preliminary results.

Let D be a noncompact closed convex domain of $\boldsymbol{R}^N$ with smooth boundary $M=\partial D$ and p a point of D. Let V_p denote the union of all half-lines which emanate from p and are contained in D. Then we clearly have

LEMMA 1. 1. *Let p and q be points in D. Then $V_q=V_p+(q-p)$, i. e., V_q is the parallel translate of V_p from p to q.*

DEFINITION 1. 2. We define the asymptotic cone of D at q by V_q. If q is not contained in D, then define $V_q:=V_p+(q-p)$ for some point p of D.

The asymptotic cone has the following properties.

LEMMA 1. 3.

(i) *V_q is a closed convex cone.*

(ii) *Let $\{p_i\}$ be a sequence of points in D such that $\lim_{i\to+\infty}|p_i|=+\infty$ and $\lim_{i\to+\infty}\frac{p_i-q}{|p_i-q|}=v$. Then the half-line $q+\boldsymbol{R}^+v$ is contained in V_q. Conversely, all half-lines in V_q are obtained in this manner.*

Let $S_q(s)=\{p \in \boldsymbol{R}^N\,;\,|p-q|=s\}$ be the sphere of radius s and centered at q. Put $V_q(s)=S_q(s)\cap V_q$ and $D_q(s)=S_q(s)\cap D$.

LEMMA 1. 4. *There is a positive number s_0 such that $S_q(s)$ intersects M transversally for $s>s_0$. Especially, $M\cap S_q(s)$ is a regular submanifold.*

PROOF. Let n_p denote the outward unit normal vector of M at a point p of M. Suppose that there is a sequence $\{p_i\}$ of points of M such that

(i) $\lim_{i\to+\infty}|p_i-q|=+\infty$,

(ii) $n_{p_i}=\pm\frac{p_i-q}{|p_i-q|}$ $(i=1, 2, \cdots)$.

Here we may assume that the sequence of vectors $\{\frac{p_i-q}{|p_i-q|}\}$ converges to a unit vector v. Then Lemma 1. 3 says that the halfline $q+\boldsymbol{R}^+v$ is contained in V_q. Since no points at infinity of V_q are in the n_{p_i}-side of $T_{p_i}M$, we may assume $n_{p_i}=-\frac{p_i-q}{|p_i-q|}$. Then it follows from (i), (ii) and $\lim n_{p_i}=-v$

that $\bigcap_{i=1}^{\infty}$ (*the* $(-n_{p_i})$*-side of* $T_{p_i}M)=\phi$, which contradicts the fact that D is in the $(-n_{p_i})$-side of $T_{p_i}M$ for all i.

LEMMA 1.5. *If D contains no lines, then there is a positive number s_0 such that $D_q(s)$ is contained in an open hemisphere of $S_q(s)$ for $s>s_0$.*

PROOF. Since V_q is a closed convex cone which contains no lines, $V_q(1)$ is contained in an open hemisphere of $S_q(1)$. Put $f(s)=\sup\{\frac{1}{s}\rho(p, V_q(s));$ $p \in D_q(s)\}$, where ρ denotes the spherical distance on $S_q(s)$. Then it follows from (ii) of Lemma 1.3 that $\lim_{s\to+\infty} f(s)=0$. Hence $\frac{1}{s}(D_q(s)-q)+q$ are uniformly contained in an open hemisphere of $S_q(1)$ for large s.

LEMMA 1.6. *Let v be a unit vector which is the initial vector of a half-line in V_q and $\Pi: \boldsymbol{R}^N\to v^\perp$ denote the orthogonal projection. If D contains no lines and if $<v, p-q> > 0$ for any point p of $D_q(s)$, then $\Pi(D_q(s))$ is a convex bounded domain in $v^\perp$.*

PROOF. We assume that v is a vertical vector which points upward and $v^\perp$ is a horizontal hyperplane. Let n_p denote the outward unit normal vector of M at a point p of $M_q(s):=M\cap S_q(s)$. Since no points at infinity of V_q are in the n_p-side of T_pM, we have $<n_p, v>\leqq 0$. Suppose that $<n_p, v><0$. Then $\Pi(S_q(s)\cap T_pM)$ is a convex hypersurface in $v^\perp$ and $\Pi(D_q(s))$ is in the convex side of $\Pi(S_q(s)\cap T_pM)$ about $\Pi(p)$, because p is in the upper hemisphere of $S_q(s)$ and $D_q(s)$ is in the upper side of T_pM. If $<n_p, v>=0$, then D is in one side of the vertical hyperplane T_pM. Therefore $\Pi(D_q(s))$ is in one side of the hyperplane $\Pi(T_p(M_q(s)))$ of $v^\perp$. Hence $\Pi(D_q(s))$ is convex in $v^\perp$.

LEMMA 1.7. *Let M_1 be a compact convex hypersurface of $\boldsymbol{R}^N$ and M_2 a compact hypersurface. If M_1 is inside of M_2, then the volume of M_1 is not greater than that of M_2.*

PROOF. For each point p of M_2, let $h(p)$ denote the point of M_1 which is the nearest to p. Since M_1 is convex, the map $h: M_2\to M_1$ is distance non-increasing, which implies the assertion.

§ 2. Proof of Theorem 1.

Let D be a noncompact closed convex domain of $\boldsymbol{R}^N$ with smooth boundary $M=\partial D$. Let n_p denote the outward unit normal vector of M at a point p.

Case 1: D contains a line l.

Let l_p denote the line which passes a point p of D and is parallel to l. Since D is convex and closed, l_p is also contained in D. Hence there is a closed convex domain D' of the orthogonal complement to l such that $D=D'\times l$. Thus $M=\partial D'\times l$. Since lim inf $s^2\tilde{k}(s)=0$, $\partial D'$ is a linear subspace and so is M.

Case 2 : D contains no lines.

It follows from Lemma 1. 5 that there are positive numbers s_0, $\theta(0<\theta<\pi/2)$ and a unit vector v in the asymptotic cone V_0 at the origin 0 such that $\measuredangle(v, q)<\theta$ for $q\in D_0(s)$, $s>s_0$, where $\measuredangle(v, q)$ denotes the angle which v and q make. We assume the vector v is vertical and points upward. Let Π: $\boldsymbol{R}^N\rightarrow v^{\perp}$ denote the orthogonal projection. Let p' be a point of M such that $<p', v>=\min_{p\in M}<p, v>$. Here we may assume that $<p', v><<q, v>$ for $q\in D_0(s)$, $s>s_0$. It follows from Lemma 1. 6 that there are two points $\bar{p}_0$ and $\bar{p}_1$ in $\Pi(M_0(s))$ such that outward unit normal vectors $\bar{n}_0$ and $\bar{n}_1$ of $\Pi(M_0(s))$ at $\bar{p}_0$ and $\bar{p}_1$ are in opposite directions, where $M_0(s):=M\cap S_0(s)$. Let p_i be points in $M_0(s)$ such that $\Pi(p_i)=\bar{p}_i$ $(i=0,1)$. Put $n_i'=(<p_i, v>n_{p_i}-<n_{p_i}, v>p_i)/s$, then $n_i'\in\boldsymbol{R}\bar{n}_i$ and $|n_i'|\leqq 2$. Since D is convex, we have $<p_i-p', n_{p_i}>\geqq 0$ which implies $<p_i, n_{p_i}>\geqq <p', n_{p_i}>\geqq -|p'|$. Hence it follows from $0\leqq-<n_{p_i}, v>\leqq 1$ that $<n_i'/|n_i'|, n_{p_i}>\geqq (s\cos\theta-|p'|)/2s$. Therefore $\bar{n}_i=n_i'/|n_i'|$ and $<\bar{n}_i, n_{p_i}>$ is uniformly positive for, large s, i. e., we may assume that there is a positive constant C such that $<\bar{n}_i, n_{p_i}>>C$, $s>s_0$. Hence we get

$$|n_{p_1}-n_{p_0}|\geqq|<n_{p_1}-n_{p_0}, \bar{n}_1>|=|<n_{p_1}, \bar{n}_1>+<n_{p_0}, \bar{n}_0>|>2C.$$

On the other hand, since $\Pi(M_0(s))$ is a convex hypersurface, there is a 2-plane β in $v^{\perp}$ which passes $\bar{p}_0$ and $\bar{p}_1$ and intersects $\Pi(M_0(s))$ transversally. Then the intersection $\beta\cap\Pi(M_0(s))$ is an oval in β which is inside of a circle of radius $s\sin\theta$. It follows from Lemma 1. 7 that the length of $\beta\cap\Pi(M_0(s))$ is not greater than $2\pi s\sin\theta$. Hence there is a curve $\bar{c}(t)$ $(0\leqq t\leqq 1)$ in $\Pi(M_0(s))$ from $\bar{p}_0$ to $\bar{p}_1$ such that length$(\bar{c})\leqq \pi s\sin\theta$. Let c be the lift of $\bar{c}$ to $M_0(s)$. Since the norm of the differential of the map $\Pi|M_0(s)$ is not less than $\cos\theta$, we get length$(c)\leqq \pi s\tan\theta$. Then we derive

$$0<2C<|n_{p_1}-n_{p_0}|\leqq\int_0^1|\frac{d}{dt}n_{c(t)}|\,dt$$

$$\leqq\int_0^1\|\alpha(c(t))\|\,|\frac{d}{dt}c(t)|\,dt\leqq\sqrt{\tilde{k}(s)}\,\text{length}(c)\leqq\sqrt{\tilde{k}(s)}\,\pi s\tan\theta,$$

which contradicts our assumption lim inf $s^2\tilde{k}(s)=0$.

§ 3. The convexity and curvature.

The following condition for a hypersurface of $\boldsymbol{R}^N$ to be convex is known.

THEOREM 3. 1. (Sacksteder [4]). *Let M be a properly imbedded hypersurface of $\boldsymbol{R}^N$. If its sectional curvature is nonnegative and not* *identically zero, then M is convex, i. e., M is the boundary of a convex domain.*

We note that the condition for sectional curvature can be replaced by the condition for Ricci curvature.

LEMMA 3. 2. *Let M be a hypersurface of* $\boldsymbol{R}^N$ *and p a point of M. Then the following (i), (ii) and (iii) are equivalent.*

(i) *The sectional curvature of M is nonnegative at p.*

(ii) *The Ricci curvature of M is nonnegative at p.*

(iii) *The second fundamental form of M is semi-definite at p.*

PROOF. It follows from Gauss equation that (i) and (iii) are equivalent. And (i) clearly implies (ii). Therefore it suffices to show that (ii) implies (iii). Let $\lambda_1, \cdots, \lambda_{N-1}$ $(\lambda_1 \leqq \cdots \leqq \lambda_{N-1})$ be the eigen values of the second fundamental form at p and $v_1, \cdots, v_{N-1} \in T_p M$ the corresponding eigen vectors. It follows from (ii) that Ricci $(v_1) = \lambda_1(\lambda_2 + \cdots + \lambda_{N-1}) \geqq 0$ and Ricci $(v_{N-1}) = (\lambda_1 + \cdots + \lambda_{N-(?)})\lambda_{N-1} \geqq 0$. Suppose that the second fundamental form is not semi-definite at p, i. e., $\lambda_1 < 0 < \lambda_{N-1}$. Then we get $0 \leqq \lambda_1 + \cdots + \lambda_{N-2}$ and $\lambda_2 + \cdots + \lambda_{N-1} \leqq 0$, which together with $\lambda_1 \leqq \cdots \leqq \lambda_{N-1}$ derive $\lambda_1 = \cdots = \lambda_{N-1} = 0$. It contradicts $\lambda_1 < 0 < \lambda_{N-1}$. Hence (iii) follows from (ii).

REMARK 3. 3. By a similar argument, we can show that the sectional curvature of M is zero at p if and only if the Ricci curvature of M is zero at p.

Owing to Lemma 3. 2, we get

COROLLARY 3. 4. *Let M be a properly imbedded hypersurface of* $\boldsymbol{R}^N$. *If its Ricci curvature is nonnegative and not dientically zero, then M is convex.*

§ 4. Proof of Theorem 2.

Case 1 : Ricci $\equiv 0$.

It follows from Remark 3. 3 that the sectional curvature is identically zero. Then the following theorem of Hartman and Nirenberg [3] says that M is a hypercylinder, i. e., $M = \boldsymbol{R}^{N-2} \times$ *a plane curve.*

THEOREM 4. 1. (Hartman and Nirenberg) *Let M be a properly imbedded hypersurface of* $\boldsymbol{R}^N$. *If its sectional curvature is identically zero, then M is a hypercylinder.*

Since lim inf $s^2 \tilde{k}(s) = 0$, the plane curve must be a line and M is a hyperplane.

Case 2 : Ricci $\geqq 0$, $\not\equiv 0$.

In this case Corollary 3. 4 implies that M is convex. Hence it follows from Theorem 1 that M is a hyperplane.

References

[1] A. KASUE: Gap theorems for minimal submanifolds of euclidean space, preprint.
[2] R. E. GREENE and H. WU: Gap theorems for noncompact Riemannian manifolds, Duke Math. J. 49 (1982), 731-756.
[3] P. HARTMAN and L. NIRENBERG: On spherical image maps whose Jacobians do not change sign, Amer. J. Math. 81 (1959), 901-920.
[4] R. SACKSTEDER: On hypersurfaces with no negative sectional curvatures, Amer. J. Math. 82 (1960), 609-630.

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