*Hokkaido Mathematical Journal, Vol. 14 (1985), p. 85-96*

# Convexity in Musielak-Orlicz spaces

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(Received March 21, 1984)

**Summary.** Some criterion for uniform convexity of a modular I of Musielak-Orlicz type is given. Moreover, some lower estimates are given for the moduli of convexity in uniformly convex Orlicz spaces.

**0. Introduction.** In paper [4] a criterion is given for uniform convexity under Luxemburg norm of Musielak-Orlicz spaces of vector-valued functions in the case of an atomless measure. In this paper a criterion is given for uniform convexity of a modular I of Musielak-Orlicz type also in the case of an atomless measure. Moreover, in section 2, some lower estimate are given for the moduli of convexity in uniformly convex Orlicz spaces in the case of an atomless as well as a purely atomic measure.

It is well known (see [8] and [13]) that if an Orlicz function satisfies suitable condition $\Delta_2$, then there exist functions $\delta$ and $\eta$ mapping the interval (0, 1) into itself such that the inequalities $I(x)\leq 1-\varepsilon$ and $I(x)\leq\eta(\varepsilon)$ imply $\|x\|_{\Phi}\leq 1-\delta(\varepsilon)$ and $\|x\|_{\Phi}\leq\varepsilon$, respectively (for definitions of functionals I and $\|\ \|_{\Phi}$ see below).

However, for a lower estimate of the modulus of convexity of the Luxemburg norm in Orlicz spaces we need to known some lower estimates of the functions $\delta$ and $\eta$ and of the modulus of convexity $\delta_I$. A lower estimate of the modulus $\delta_I$ follows from results of A. Kamińska [8]. Some estimates of the functions $\delta$ and $\eta$, which yield an estimate of the modulus of convexity of the norm $\|\ \|_{\Phi}$, are main purpose of section 2 of this paper.

R. P. Maleev and S. L. Troyanski [11] gave a best in some sense estimate of the modulus of convexity of the norm $\|\ \|_{\Psi}$, where $\Psi$ is an Orlicz function equivalent to $\Phi$ (two equivalent Orlicz functions $\Phi$ and $\Psi$ yield the same Orlicz space with equivalent norms $\|\ \|_{\Phi}$ and $\|\ \|_{\Psi}$, see [12]). This problem was continuated by T. Figiel [2] (see e. g. Propositions 19, 21 and Lemma 20). However, for a pair of equivalent Orlicz functions $\Phi$ and $\Psi$ an estimate of the modulus of convexity for the norm $\|\ \|_{\Psi}$ need not yield of an estimate of the modulus of convexity for the norm $\|\ \|_{\Phi}$. So, the problem investigated in section 2 of this paper is sensible.

Now, we shall introduce some denotations and definitions. Let $\mathbf{R}$ be the set of real numbers and $\mathbf{N}$ the set of natural numbers. Let $(X, \|\ \|)$ be a real Banach space. By $(T, \Sigma, \mu)$ we denote a space of a non-negative, atomless, complete and $\sigma$-finite measure. By $\Sigma_0$ we denote the class of null sets in $\Sigma$ with respect to measure $\mu$. For every set $A\in\Sigma$, $\chi_A$ denotes the characteristic function of $A$, and $A\cong B$ denotes for $A$, $B\in\Sigma$ that $\mu(A\div B)=0$, where $A\div B=(A\backslash B)\cup(B\backslash A)$. Let $\mathbf{R}_+=[0, +\infty)$ and let $F=F(\mu, T, X)$ denote the space of equivalence classes of strongly $\Sigma$-measurable functions x: $T\rightarrow X$.

By a Musielak-Orlicz function we mean a map $\Phi: T\times X\rightarrow[0, +\infty]$ which is convex, even, vanishing and continuous at zero, not identicaly zero for $\mu$-a. e. $t\in T$ and $\Sigma\times\mathscr{B}$-measurable, where $\mathscr{B}$ denotes the class of Borel subsets of $X$.

We define for any Musielak-Orlicz function $\Phi$ the Musielak-Orlicz space $L^{\Phi}(\mu)$ as the set of all $x\in F$ such that $I(\lambda x)<+\infty$ for some $\lambda>0$ depending on $x$, where

$$I(x)=I_{\Phi}(x)=\int_T\Phi(t,x(t))\mathrm{d}\mu,$$

and its subspace

$$E^{\Phi}(\mu)=\{x\in F: I(\lambda x)<+\infty \text{ for any } \lambda>0\}.$$

The functional $I$ is a convex modular on $F$ (see [12] and [14]). The Luxemburg norm $\|\ \|_{\Phi}$ is defined on $\mathrm{L}^{\Phi}(\mu)$ by

$$\|x\|_{\Phi}=\inf\{u>0: I(x/u)\leq 1\}.$$

A modular I is called uniformly convex if its modulus of convexity $\bar{\delta}_I(\varepsilon)$ defined by

$$\bar{\delta}_I(\varepsilon)=\inf\{1-I\left(\frac{x+y}{2}\right): I(x)=I(y)=1,\ I\left(\frac{x-y}{2}\right)\geq\varepsilon\}$$

is positive for any $\varepsilon>0$ (compare with definitions in [13] and [14]).

We define also an another modulus of convexity of a modular I by

$$\delta_I(\varepsilon)=\inf\{1-I\left(\frac{x+y}{2}\right): I(x)=I(y)=1,\ I(x-y)\geq\varepsilon\}.$$

In the case when $I$ is a norm, we write $\delta_X$ instead of $\delta_{\|\ \|_X}$.

Let $G(\varepsilon, \Phi)$ denote for every $\varepsilon>0$ and arbitrary Musielak-Orlicz function $\Phi$ the-set af all strongly $\Sigma$-measurable functions $f: T\rightarrow X\backslash\{0\}$ such that $\int_T\Phi(t,f(t))d\mu=\varepsilon$.

A Musielak-Orlicz function $\Phi$ is called uniformly convex if: $\forall\varepsilon\in(0, 1)$ $\exists f\in G(\varepsilon, \Phi)$ $\exists T_0\in\Sigma_0$ $\exists p(\varepsilon)\in(0, 1)$ $\forall x, y\in X$:

$$\Phi(t, f(t))\leq\max(\Phi(t, x), \Phi(t, y))\leq\varepsilon^{-1}\Phi\left(t, \frac{x-y}{2}\right)\Rightarrow$$
$$\Phi\left(\mathrm{t}, \frac{x+y}{2}\right)\leq\frac{1-p}{2}\{\Phi(t, x)+\Phi(t, y)\}$$

(compare with definition in [4] and [13]).

We say that a Musielak-Orlicz function $\Phi$ satisfies the condition $\Delta_2$ if there exist a set $T_1 \in \Sigma_0$, a constant $K>0$ and a non-negative and $\Sigma$-measurable function $h$ with $\int_T h(t)\,d\mu < +\infty$ such that

$$\Phi(t, 2x) \leq K\Phi(t, x) + h(t)$$

for every $t \in T \setminus T_1$ and $x \in X$ (for definition and consequences see e.g. [7] and [12]).

A Musielak-Orlicz function $\Phi$ such that $\Phi(t_1, \cdot) = \Phi(t_2, \cdot)$ for each $t_1, t_2 \in T$ is called an Orlicz function.

We say an Orlicz function $\Phi$ satisfies the condition $\Delta_2$ for all $x$ (at zero) [at infinity] if there exist constants $K, a>0$ such that the inequality $\Phi(2x) \leq K\Phi(x)$ holds for all $x$ (for $x$ satisfying $\Phi(x) \leq a$) [for $x$ satisfying $\Phi(x) \geq a$]

## 1. Results.

LEMMA 1.1. *If $I$ is a convex modular in a linear space $X$ satisfying the following condition*

($\Lambda_2$) $\forall \varepsilon > 0 \; \exists K(\varepsilon) \geq 2 \; \forall x \in X : I(2x) \leq K(\varepsilon) I(x) + \varepsilon$,

*then there exists a function $p : (0, 2) \to (0, 2^{-1})$ such that*

(1.1) $\bar{\delta}_I(p(\varepsilon)) \leq \delta_I(\varepsilon) \leq \bar{\delta}_I(\varepsilon)$.

PROOF. The right-side inequality of (1.1) is obvious. For the proof of the left-side inequality of (1.1) observe that $I(x-y) \geq \varepsilon$ implies

$$\varepsilon \leq I\left(2\frac{x-y}{2}\right) \leq K\left(\frac{\varepsilon}{2}\right) I\left(\frac{x-y}{2}\right) + \frac{\varepsilon}{2}, \text{ i.e. } I\left(\frac{x-y}{2}\right) \geq \varepsilon / 2K\left(\frac{\varepsilon}{2}\right).$$

Thus, the condition (1.1) holds with $p(\varepsilon) = \varepsilon / 2K\left(\frac{\varepsilon}{2}\right)$.

REMARK. If $\Phi$ is a Musielak-Orlicz function satisfying condition $\Delta_2$, then the modular $I = I_\Phi$ satisfies condition ($\Lambda_2$) (see [5]).

LEMMA 1.2. *Let $X$ be a separable Banach space. Then every Musielak-Orlicz function $\Phi$ satisfying the condition*

(*) $\forall\, r > 0 : \tilde{\Phi}(t, r) = \sup \{\Phi(t, x) : \|x\| \leq r\} < +\infty$ *for $\mu - a.e. t \in T$*

*satisfies the following condition (see [9]).*

(*B*) *there exist an ascending sequence $\{T_n\}$ of subsets of $T \cap \Sigma$ of finite measure such that $\bigcup_n T_n \cong T$ and a sequence $\{f_k\}$ of $\Sigma$-measurable and non-negative functions such that $\tilde{\Phi}(t, k) \leq f_k(t)$ for $\mu$-a. e. $t \in T$ and*

$\int_{T_n} f_k(t)\, d\mu < +\infty$ *for any $k, n \in \mathbf{N}$.*

PROOF. The proof of this lemma was shown to me by W. Kurc. Define the sets

$$A_n = \bigcup_{l \leq n} \{t \in T : \tilde{\Phi}(t, l) \leq n\}, \quad n = 1, 2, \cdots.$$

It is obvious that $\{A_n\}$ is an ascending sequence of $\Sigma$-measurable subsets of $T$ such that $\bigcup_n A_n \cong T$. Let $\{B_n\}$ be an ascending sequence of $\Sigma$-measurable subsets of $T$ of finite measure satisfying $\bigcup_n B_n \cong T$. Putting $T_n = A_n \cap B_n$ for any $n \in \mathbf{N}$, we obtain an ascending sequence of sets of finite measure such that $\bigcup_n T_n \cong T$. Denoting $f_k(t) = \tilde{\Phi}(t, k)$, we get

$$\int_{T_n} f_k(t)d\mu = \int_{T_n} \tilde{\Phi}(t,k)d\mu \leq \int_{T_{\max k,n}} \tilde{\Phi}(t,\max(k,n))\,d\mu \leq \max(k,n)\mu(T_{\max(k,n)}) < +\infty.$$

The functions $f_k(t)$ are $\Sigma$-measurable by separability of $X$ and by continuity of $\Phi$, which follows from the condition $(*)$. Thus, the proof is finished.

THEOREM 1.3. *If $\Phi$ is a uniformly convex Musielak-Orlicz function, then the modular $I = I_\Phi$ is uniformly convex on $L^\Phi(\mu)$. Conversely, if $X$ is separable and $\Phi$ satisfies the condition $(*)$, then uniform convexity of $I$ on the subspace $E^\Phi(\mu)$ implies uniform convexity of $\Phi$.*

PROOF. $\Longrightarrow$ Let $\varepsilon \in (0, 1)$, $\alpha = \varepsilon/4$, $f \in G(\alpha, \Phi)$, $I(x) = I(y) = 1$ and $I\left(\frac{x-y}{2}\right) \geq \varepsilon$. We may assume without loss of generality that the null sets in the definition of $\Phi$ being uniformly convex and satisfying the condition $(*)$ are empty. Define

$$(1.2) \quad B = \left\{ t \in T : \Phi(t, f(t)) \leq \max\{\Phi(t, x(t)), \Phi(t, y(t))\} \leq \varepsilon^{-1}\Phi\left(t, \frac{x(t)-y(t)}{2}\right)\right\}.$$

We have for $t \in B$

$$\Phi(t, 2^{-1}(x(t)+y(t))) \leq 2^{-1}(1-p(\alpha))\{\Phi(t, x(t)) + \Phi(t, y(t))\}.$$

Integrating both-side this inequality over $B$, we get

$$I(2^{-1}(x+y)\chi_B) \leq 2^{-1}(1-p(\alpha))\{I(x\chi_B) + I(y\chi_B)\}.$$

Hence, we obtain

$$(1.3) \quad 1 - I\left(\frac{x+y}{2}\right) = I(x) + I(y) - I\left(\frac{x+y}{2}\right) \geq I(x\chi_B) + I(y\chi_B) - I\left(\frac{x+y}{2}\chi_B\right) \geq 2^{-1}p(\alpha)\{I(x\chi_B) + I(y\chi_B)\}.$$

Denote

$$C = \{t \in T\backslash B : \max(\Phi(t, x(t)), \Phi(t, y(t))) < \Phi(t, f(t))\},$$

$$D = \{t \in T\backslash B : \Phi\left(t, \frac{x(t)-y(t)}{2}\right) < \alpha \max(\Phi(t, x(t)), \Phi(t, y(t)))\}.$$

We have $T\backslash B = C \cup D$ and

$$I\left(\frac{x-y}{2}\chi_C\right) \leq 2^{-1}\{I(x\chi_C) + I(y\chi_C)\} \leq \alpha,$$

$$I\left(\frac{x-y}{2}\chi_D\right) \leq \alpha\{I(x\chi_D) + I(y\chi_D)\} \leq 2\alpha.$$

Hence we get $I\left(\frac{x-y}{2}\chi_{T\setminus B}\right)\leq 3\alpha$ and thus

(1.4) $I\left(\frac{x-y}{2}\chi_B\right)\geq \varepsilon/4.$

So, we have

$$\frac{\varepsilon}{4}\leq I\left(\frac{x-y}{2}\chi_B\right)\leq\frac{1}{2}\{I(x\chi_B)+I(y\chi_B)\},\ \text{i. e.}\ I(x\chi_B)+I(y\chi_B)\geq\frac{\varepsilon}{2}.$$

Hence and by (1.3), we obtain

$$1-I\left(\frac{x+y}{2}\right)\geq \varepsilon p(\alpha)/4,\ \text{ i. e. } \bar{\delta}_I(\varepsilon)\geq \varepsilon p(\alpha)/4,$$

and the proof of sufficiency is finished.

$\Longleftarrow$. Assume that $X$ is separable, $\Phi$ satisfies the condition $(*)$ and $\Phi$ is not uniformly convex. Let $Z=\{z_1, z_2, \cdots\}$ be a countably and dense subset of $X$. Note that we may restrict ourselves in definition of $\Phi$ being uniformly convex to elements $x, y\in Z$ and to rational numbers $\mathrm{p}(\varepsilon)\in(0,1)$ (by continuity of $\Phi(t,\cdot)$ for $\mu$-a. e. $t\in\mathbf{T}$, which follows from the condition $(*)$). Let $\{\delta_k\}$ denote the sequence of all rational numbers from the interval $(0, 1)$. Let $\varepsilon>0, f\in\mathbf{G}(\varepsilon,\Phi)$ be arbitrary and let us consider the set

$$A(\varepsilon,f)=\{t\in T:\ \forall k\in\mathbf{N}\ \exists\, m,\ n\in\mathbf{N}:\Phi(t,f(t))\leq\max(\Phi(t,x(t)),$$
$$\Phi(t,y(t)))\leq\varepsilon^{-1}\Phi\left(t,\frac{z_m-z_n}{2}\right)\text{ and }\Phi\left(t,\frac{z_m-z_n}{2}\right)\geq\frac{1-\delta_k}{2}\{\Phi(t,z_m)+\Phi(t,z_n)\}\}.$$

It is obvious that $A(\varepsilon,f)\in\Sigma$. Since $\Phi$ is not uniformly convex by assumtion, so

$$\exists\,\varepsilon>0\,\forall f\in G(\varepsilon,\Phi):\mu(A(\varepsilon,f))>0.$$

Denote $A=A(\varepsilon,f)$, $\varepsilon_1=\int_A\Phi(t,f(t))d\mu$, $\mathscr{X}=X\times X$, and define for arbitrary fixed $k\in\mathbf{N}$ a multifunction $\bar{G}:A\to 2^{\mathscr{X}}$ by

$$\bar{G}(t)=\{(x,y)\in\mathscr{X}:\Phi(t,f(t))\leq\max(\Phi(t,x),\Phi(t,y))\leq\varepsilon^{-1}\ \Phi\left(t,\frac{x-y}{2}\right)$$
$$\text{and }\Phi\left(t,\frac{x+y}{2}\right)\geq\frac{1-\delta_k}{2}\{\Phi(t,x)+\Phi(t,y)\}\}.$$

Obviously, $\bar{G}(t)\neq 0$ for every $t\in A$ and the graph of $\bar{G}$, i. e. the set $\{(t,z)\in T\times\mathscr{X}: z\in\bar{G}(t)\}$ is $\Sigma\times(\mathscr{B}\times\mathscr{B})$-measurable. Hence (see [3]) there exists a $\Sigma$-measurable function (selector) $g:A\to\mathscr{X}$ such that $g(t)\in\bar{G}(t)$ for any $t\in A$. This means that there exist two $\Sigma$-measurable functions $g_1, g_2:A\to X$ such that

(1.5) $\Phi(t,f(t))\leq\max(\Phi(t,g_1(t)),\Phi(t,g_2(t)))\leq\varepsilon^{-1}\Phi\left(t,\frac{g_1(t)-g_2(t)}{2}\right)$

and $\Phi(t,2^{-1}(g_1(t)+g_2(t)))\geq 2^{-1}(1-\delta_k)\{\Phi(t,g_1(t)+\Phi(t,g_2(t))\}$.

Define the sets

$$A_1=\{t\in A:\Phi(t,g_1(t))\geq\Phi(t,g_2(t))\},\ A_2=A\setminus A_1.$$

We have $\int_{A_1}\Phi(t, g_1(t))\,d\mu \geq \varepsilon_1/2$ or $\int_{A_2}\Phi(t, g_2(t))\,d\mu \geq \varepsilon_1/2$. We may assume without loss of generality that $\int_{A_1}\Phi(t, g_1(t))\,d\mu \geq \varepsilon_1/2$. Let us define the sets

$$A^{(n)}=\{t\in A_1 : \max(\|g_1(t)\|, \|g_2(t)\|)\leq n\}, B_n=A^{(n)}\cap T_n,$$

where $\{T_n\}$ is the sequence of set from the condition (B) (see Lemma 1.2). We have $B_n\subset B_{n+1}$ for any $n\in\mathbf{N}$ and $\bigcup_n B_n\cong T$. Thus, there exists $n\in\mathbf{N}$ such that $\int_{B_n}\Phi(t, g_1(t))\,d\mu \geq \varepsilon_1/3$. Next, there exists a set $C\subset B_n$, $C\in\Sigma$, such that $\int_C\Phi(t, g_1(t))\,d\mu = \varepsilon_1/4$. Denote

$$\lambda(t)=\Phi(t, g_1(t))-\Phi(t, g_2(t)).$$

We have for all $t\in C$ : $0\leq\Phi(t, g_1(t))-\Phi(t, g_2(t))<+\infty$ and $\int_C\lambda(t)\,d\mu\leq \int_C\Phi(t, g_1(t))\,d\mu=\varepsilon_1/4$. Next, there exists a set $D\subset C$, $D\in\Sigma$, such that

$$\int_D\lambda(t)\,d\mu=\int_{C\backslash D}\lambda(t)\,d\mu.$$

This is equivalent to the following equality

$$(1.6)\quad \int_D\Phi(t,g_1(t))\,d\mu+\int_{C\backslash D}\Phi(t,g_2(t))\,d\mu=\int_{C\backslash D}\Phi(t,g_1(t))\,d\mu + \int_D\Phi(t,g_2(t))\,d\mu=b,$$

where $b$ is a positive number $\leq\varepsilon_1/4$. Define a multifunction $H : B\backslash C\rightarrow 2^X$ by

$$H(t)=\{x\in X : \Phi(t, x)>0\}.$$

We have $H(t)\neq\emptyset$ for $t\in B\backslash C$ and the graph of $H$ is $\Sigma\times\mathscr{B}$-measurable. Thus, there exists a measurable function $h : B\backslash C\rightarrow X$ such that $h(t)\in H(t)$ for any $t\in B\backslash C$ (see [3]). Let us define

$$D_n=\{t\in B\backslash C : \|h(t)\|\leq n), n\in \boldsymbol{N}.$$

The sequence $\{D_n\}$ is ascending and $\bigcup_n D_n=B\backslash C$. Thus, there exists $n_0\in\boldsymbol{N}$ such that $\int_{D_{n_0}}\Phi(t, h(t))\,d\mu>0$. Since $D_{n_0}\subset B\backslash C$ and $B$ is contained in some set $T_n$ from the condition (B) (see Lemma 1.2), so

$$\forall\lambda>1 : 0<\int_{D_{n_0}}\Phi(t,\lambda h(t))\,d\mu<+\infty.$$

So, there exists a set $E\subset D_{n_0}$, $E\in\Sigma$, and a number $\lambda>1$ such that $\int_E\Phi(t, \lambda h(t))\,d\mu=1-b$. Let us define

$$x=g_1\chi_D+g_2\chi_{C\backslash D}+\lambda h\chi_E : y=g_2\chi_D+g_1\chi_{C\backslash D}+\lambda h\chi_E.$$

Since functions $x$ and $y$ are bounded and vanishing outside some set $T_n$ from the condition (B), so both belong to $E^{\Phi}(\mu)$: Moreover, we have by (1.6): $I(x)=I(y)=1$, and by (1.5) : $I(2^{-1}(x+y))\geq 1-\delta_k$ and $I(2^{-1}(x-y))\geq\varepsilon$. So, by arbitrarity of $k$, this means that the modular $I$ is not uniformly convex, i.e. $\bar{\delta}_I(\varepsilon)=0$ for some $\varepsilon>0$, and the proof is completed.

EXAMPLE 1.4. If $\Phi_1$, $\Phi_2$ are two Musielak-Orlicz functions on $T\times R$, then the function $\Phi=\Phi_1\cdot\Phi_2$ is a uniformly convex Musielak-Orlicz function.

PROOF. We shall prove only uniform convexity of $\Phi$. Without loss of generality we may assume that $\Phi_1$ and $\Phi_2$ not depends on a parameter $t$, i.e. that $\Phi_1$ and $\Phi_2$ are Orlicz functions. Denote by $\varphi$, $\varphi_1$ and $\varphi_2$ the right-hand derivative of $\Phi, \Phi_1$ and $\Phi_2$, respectively. We have for any $\varepsilon, u>0$:

$$\varphi((1+\varepsilon)u)=\varphi_1((1+\varepsilon)u)\Phi_2((1+\varepsilon)u)+\Phi_1((1+\varepsilon)u)\varphi_2((1+\varepsilon)u)\geq (1+\varepsilon)\{\varphi_1(u)\Phi_2(u)+\Phi_1(u)\varphi_2(u)\}=(1+\varepsilon)\varphi(u).$$

Hence it follows that the function $\Phi$ satisfies the condition

$$(1.7)\quad \forall\, \mathrm{a}\in(0,1)\ \exists\, \mathrm{p(a)}\in(0,1)\ \forall\, \mathrm{u}\geq 0: \Phi\Big(\frac{\mathrm{u}+\mathrm{au}}{2}\Big)\leq 2^{-1}(1-\mathrm{p(a)})\{\Phi(\mathrm{u})+\Phi(\mathrm{au})\},$$

(see [1]), i.e. the function $\Phi$ is uniformly convex on the interval $[0,+\infty)$, because the condition (1.7) implies that (see [5])

$$\forall\, \mathrm{a}\in(0,1)\ \exists\, \mathrm{p(a)}\in(0,1)\ \forall\, x, y\in\mathbf{R}: |x-y|\geq \mathrm{a}\max(|x|,|y|)$$
$$\Longrightarrow \Phi\Big(\frac{x+y}{2}\Big)\leq \frac{1-\mathrm{p(a)}}{2}\{\Phi(x)+\Phi(y)\}.$$

COROLLARY 1.5. *For every Musielak-Orlicz functions $\Phi_1$ and $\Phi_2$ defined on $T\times\boldsymbol{R}$ and for every non-negative measure $\mu$ the modular $I_{\Phi_1\cdot\Phi_2}$ is uniformly convex on the space $L^{\Phi_1\cdot\Phi_2}(\mu)$.*

REMARK 1.6. Let $X$ be arbitrary Banach space and let $\Phi_1, \Phi_2$ be two Musielak-Orlicz functions on $T\times X$. Then the function $\Psi$ complementary to $\Phi=\Phi_1\cdot\Phi_2$ satisfies the condition $\Delta_2$ with $h\equiv 0$.

PROOF. We have for $t\in T, x\in X: \Phi\Big(t,\frac{x}{2}\Big)\leq\frac{1}{4}\Phi(t,x)$ and hence we have for $t\in T$, $x^*\in X^*$:

$$\Psi(t,2x^*)=\sup\{2x^*(x)-\Phi(t,x):x\in X\}=4\sup\Big\{x^*\Big(\frac{x}{2}\Big)-\frac{1}{4}\Phi(t,x)\Big\}\leq$$
$$4\sup\Big\{x^*\Big(\frac{x}{2}\Big)-\Phi\Big(t,\frac{x}{2}\Big)\Big\}=4\Psi(t,x^*).$$

COROLLARY 1.7. *Assume that $\Phi_1$, $\Phi_2$ are Orlicz functions on a Banach space $X$. Then the space $L^{\Phi_1\cdot\Phi_2}(\mu)$ is reflexive iff $X$ is reflexive and $\Phi_1$, $\Phi_2$ satisfy condition $\Delta_2$ for all $x$ (at infinity) [at zero] in the case of $\mu$ being atomless and infinite (atomless and finite) [purely atomic with measure of atoms $b_n$ satisfying $0<\inf_n b_n$].*

PROOF. This follows by applying the theorem on representation of linear continuous functionals over an Orlicz space (see [9] and [15]), by the fact that every reflexive Banach space $X$ has the Radon-Nikodym property, and by the property that an Orlicz function $\Phi_1\cdot\Phi_2$ satisfies suitable condition $\Delta_2$

iff both functions $\Phi_1$ and $\Phi_2$ satisfy suitable condition $\Delta_2$.

## 2. Estimates of the moduli of convexity in Orlicz spaces.

In this section $X=R$ and the uniform convexity of Orlicz function $\Phi$ is defined in an another manner.

An Orlicz function $\Phi$ is called uniformly convex on the interval $D\subset \mathbf{R}_+$ if for every $a\in(0,1)$ there exists $p(a, D)\in(0,1)$ such that for any $u\in D$

$$\Phi\left(\frac{u+au}{2}\right)\leq\frac{1-p(a, D)}{2}\{\Phi(u)+\Phi(au)\}$$

(see [1] and [13]). If $\Phi$ is an Orlicz function satisfying condition $\Delta_2$ for all $u$ (at infinity), then uniform convexity of $\Phi$ in the sense of definition in section 1 coincides with uniform convexity in the last sense on the interval $[0, +\infty)$(on every interval $[u, +\infty)$, $u>0$) in the case of an atomless infinite (an atomless finite) measure $\mu$, respectively.

A normed space $(X, \|\ \|)$ is said to be uniformly convex if $\delta_X(\varepsilon)>0$ for every $\varepsilon>0$ (see [10]).

Now, we shall give some auxiliary results.

THEOREM 2.1. (see [8]). *Let $\Phi$ be an Orlicz function. The Orlicz space $L^{\Phi}(\mu)$ is uniformly convex iff $\Phi$ satisfies condition $\Delta_2$ for all $u$ and is uniformly convex on the interval $[0, +\infty)$($\Phi$ is finite, vanishes only at zero, satisfies the condition $\Delta_2$ at infinity and is uniformly convex on every interval $[u, +\infty)$, $u>0$)[$\Phi$ satisfies the condition $\Delta_2$ at zero, there exists a number $u>0$ such that $\Phi(u)=b^{-1}$ and $\Phi$ is uniformly convex on the interval $[0, \Phi^{-1}\left(\frac{1}{2b}\right)]$ in the case of an atomless infinite measure $\mu$ (an atomless finite measure $\mu$) [a purely atomic measure $\mu=\{b_k\}$ such that $0<b=\inf_k b_k=\lim_k\inf b_k$, where $b_k$ denotes the measure of kth atom], respectively.*

LEMMA 2.2. (see [6]). *Let $\Phi$ be an Orlicz function satisfying condition $\Delta_2$ for all $u$ (at infinity) [at zero]. Then $\Phi$ satisfies the condition*

$$\lim_{k\to\infty}\{\Phi(\left(1+\frac{1}{k}\right)u)=\Phi(u)$$

*uniformly with respect to all $u>0$(uniformly on every interval $[c, d]$, where $sup[u>0:\Phi(u)=0]\leq c<d<sup[u>0:\Phi(u)<+\infty]$) [uniformly on every interval $[0, u_0]$, where $u_0<sup[u>0:\Phi(u)<+\infty]$].*

In the following, for arbitrary fixed Orlicz function $\Phi$ and for any $\sigma\in(0, 1)$, we denote by $f_\sigma$ the function from $\mathbf{R}\backslash\{0\}$ into $\mathbf{R}_+$ defined by

$$f_\sigma(u)=\Phi(u/(1-\sigma))/\Phi(u).$$

LEMMA 2.3. *Let $\mu$ be an atomless and infinite measure and let $\Phi$ be an Orlicz function satisfying condition $\Delta_2$ for all $u$. Define the function $\delta:(0,1)\to(0,1)$ by*

$$\delta(\varepsilon)=sup\left\{\sigma\in(0,1):\sup_{u>0} f_\sigma(u)\leq\frac{1}{1-\varepsilon}\right\}.$$

*Then for every $x\in L^\Phi(\mu)$ and $\varepsilon\in(0,1)$ we have $\|x\|_\Phi\leq 1-\delta(\varepsilon)$, whenever $I(x)\leq 1-\varepsilon$.*

PROOF. It follows from Lemma 0.2 that $\delta(\varepsilon)\in(0,1)$ for any $\varepsilon\in(0,1)$. Moreover, by continuity of $\Phi$, which follows from the condition $\Delta_2$ for all $u$, we have $f_{\delta(\varepsilon)}(u)\leq 1/(1-\varepsilon)$ for any $u>0$. Hence it follows immediately that the condition $I(x)\leq 1-\varepsilon$ implies $I(x/(1-\delta(\varepsilon)))\leq 1$, i. e. $\|x\|_\Phi\leq 1-\delta(\varepsilon)$.

LEMMA 2.4. *Let $\mu$ be an atomless and finite measure and let $\Phi$ be a finite Orlicz function satisfying condition $\Delta_2$ at infinity and vanishing only at zero. Define the function $\delta:(0,1)\to(0,\frac{1}{2}]$ by*

$$\delta(\varepsilon)=sup\{\sigma\in(0,\tfrac{1}{2}]:\sup_{u\geq a} f_\sigma(u)\leq 1/(1-\varepsilon/2)\},$$

*where $a=\frac{1}{2}\Phi^{-1}(\varepsilon/(2-\varepsilon)\mu(T))$ and $\Phi^{-1}$ denotes the generalized inverse function of $\Phi$.* (see [13]) *Then for every $x\in L^\Phi(\mu)$ and $\varepsilon\in(0,1)$ the condition $I(x)\leq 1-\varepsilon$ implies $\|x\|_\Phi\leq 1-\delta(\varepsilon)$.*

PROOF. It follows from Lemma 0.1 that $\delta(\varepsilon)\in(0,\frac{1}{2}]$ for any $\varepsilon>0$. Suppose that $I(x)\leq 1-\varepsilon$ and define the set $A=\{t\in T:|x(t)|\leq a\}$. We have $f_{\delta(\varepsilon)}(u)\leq 1/(1-\varepsilon/2)$ for all $u\geq a$ and hence

$$I(x/(1-\delta(\varepsilon)))=I(x\chi_A/(1-\delta(\varepsilon)))+I(x\chi_{T\setminus A}/(1-\delta(\varepsilon)))\leq I(2x\chi_A)$$
$$+(1/(1-\varepsilon/2))I(x\chi_{T\setminus A})\leq\Phi(2a)\mu(T)+(1-\varepsilon)/(1-\varepsilon/2)=1,$$ i. e. $\|x\|_\Phi\leq 1-\delta(\varepsilon)$.

LEMMA 2.5. *Let $\mu$ be a purely atomic measure as in Theorem 0.2 and let $\Phi$ be an Orlicz function satisfying condition $\Delta_2$ at zero and such that there exists a number $u_0>0$ such that $\Phi(u_0)=\frac{1}{b}$. Let us define the function $\delta:(0,1)\to(0,1)$ by*

$$\delta(\varepsilon)=sup\left\{\sigma\in(0,1):sup[f_\sigma(u):u\in(0,(1-\sigma)\Phi^{-1}(\tfrac{1}{b})]\cap[0,\Phi^{-1}(\tfrac{1-\varepsilon}{b})]\leq\frac{1}{1-\varepsilon}\right\}.$$

*Then for every $x\in L^\Phi(\mu)$ and $\varepsilon>0$ the condition $I(x)\leq 1-\varepsilon$ implies $\|x\|_\Phi\leq 1-\delta(\varepsilon)$.*

PROOF. It follows from Lemma 0.1 that $\delta(\varepsilon)\in(0,1)$ for every $\varepsilon\in(0,1)$.

Now, assume that $\varepsilon\in(0,1)$ and $I(x)\leq 1-\varepsilon$, i. e. $\sum_{k=1}^{\infty}\Phi(x_k)b_k\leq 1-\varepsilon$. Hence it follows that $\Phi(x_k)\leq(1-\varepsilon)/b$ for $k=1,2,\cdots$, i. e. $|x_k|\in[0,\Phi^{-1}\left(\frac{1-\varepsilon}{b}\right)]$. We have by continuity of $\Phi$ on the interval $[0,\Phi^{-1}\left(\frac{1}{b}\right)]$ that $f_{\delta(\varepsilon)}(u)\leq\frac{1}{1-\varepsilon}$ for every $u\in(0,\Phi^{-1}\left(\frac{1-\varepsilon}{b}\right)]$. Hence it follows that $\Phi(x_k/(1-\delta(\varepsilon)))\leq\Phi(x_k)/(1-\varepsilon)$ for $k=1,2,\cdots$. Thus, we get

$$I(x/1-\delta(\varepsilon)))=\sum_{k=1}^{\infty}\Phi(x_k/(1-\delta(\varepsilon)))b_k\leq(\sum_{k=1}^{\infty}\Phi(x_k)b_k)/(1-\varepsilon)\leq 1,\text{ i. e.}$$

$\|x\|_\Phi\leq 1-\delta(\varepsilon)$.

In the following $\lambda(u)$ is defined as $\log_2 u^{-1}$ if $\log_2 u^{-1}$ is a positive integer and as $E(\log_2 u^{-1})+1$ in an opposite case, where $E(u)$ denotes the integer part of $u$.

LEMMA 2.6. *Let $\Phi$ be an Orlicz function satisfying condition $\Delta_2$ for all $u$. Let $K=sup[\Phi(2u)/\Phi(u):u>0]$ and $\eta(\varepsilon)=K^{-\lambda(\varepsilon)}$. Then for every $x\in L^\Phi(\mu)$ and $\varepsilon\in(0,1)$ the condition $I(x)\leq\eta(\varepsilon)$ implies $\|x\|_\Phi\leq\varepsilon$.*

PROOF. Let $\varepsilon\in(0,1)$ and $I(x)\leq\eta(\varepsilon)$. We have

$$I(x/\varepsilon)\leq I(2^{\lambda(\varepsilon)}x)\leq K^{\lambda(\varepsilon)}I(x)\leq 1,\text{ i. e.}\|x\|_\Phi\leq\varepsilon.$$

LEMMA 2.7. *Let $\mu$ be an atomless and finite measure and let $\Phi$ be a finite Orlicz function satisfying condition $\Delta_2$ at infinity and vanishing only at zero. Let $a=\varepsilon\Phi^{-1}(1/2\mu(T))$, $K=sup[\Phi(2u)/\Phi(u):u\geq a]$, and $\eta(\varepsilon)=K^{-\lambda(\varepsilon)}/2$. Then for every $x\in L^\Phi(\mu)$ and $\varepsilon\in(0,1)$ the condition $I(x)\leq\eta(\varepsilon)$ implies $\|x\|_\Phi\leq\varepsilon$.*

PROOF. Take arbitrary $\varepsilon\in(0,1)$ and $x\in L^\Phi(\mu)$ satisfying $I(x)\leq\eta(\varepsilon)$ and define the set $A=\{t\in T:|x(t)|\leq a\}$. We have

$$I(x)/\varepsilon)=I(x\chi_A/\varepsilon)+I(x\chi_{T\setminus A}/\varepsilon)\leq 2^{-1}+I(2^{\lambda(\varepsilon)}x\chi_{T\setminus A})\leq$$
$$\leq 2^{-1}+K^{\lambda(\varepsilon)}I(x)\leq 1,\ i.\ e.\ \|x\|_\Phi\leq\varepsilon.$$

LEMMA 2.8. *Let $\mu$ be a purely atomic measure as in Theorem 0.2 and let $\Phi$ be an Orlicz function satisfying condition $\Delta_2$ at zero, i. e. $K=sup[\Phi(2u)/\Phi(u):0<u\leq a]<+\infty$ for some $a>0$ such that $\Phi(a)>0$. Define the function $\eta:(0,1)\to(0,+\infty)$ by*

$$\eta(\varepsilon)=min(\eta_1(\varepsilon),\eta_2(\varepsilon)),$$

*where $\eta_1(\varepsilon)=b\Phi(a/2^{\lambda(\varepsilon)})$ and $\eta_2(\varepsilon)=K^{-\lambda(\varepsilon)}$. Then for every $x\in L^\Phi(\mu)$ and $\varepsilon\in(0,1)$, we have $\|x\|_\Phi\leq\varepsilon$ whenever $I(x)\leq\eta(\varepsilon)$.*

PROOF. Take arbitrary $\varepsilon\in(0,1)$ and $x\in L^\Phi(\mu)$ satisfying $I(x)\leq\eta(\varepsilon)$. Then $I(x)\leq\eta_1(\varepsilon)$ and hence $2^{\lambda(\varepsilon)}|x_k|\leq a$ for any $k\in N$. Hence it follows by the condition $\Delta_2$ for the interval $[0,a]$ that

$$I(x/\varepsilon)\leq I(2^{\lambda(\varepsilon)}x)\leq K^{\lambda(\varepsilon)}I(x)\leq K^{\lambda(\varepsilon)}\eta_2(\varepsilon)\leq 1, \text{ i. e. } \|x\|_\Phi\leq\varepsilon.$$

THEOREM 2.9. *If $L^\Phi(\mu)$ is a uniformly convex Orlicz space, then its modulus of convexity $\delta_L^\Phi$ satisfies for every $\varepsilon\in(0,1)$ the following inequality*

$$\delta_L^\Phi(\varepsilon)\leq\delta(\delta_I(\eta(\varepsilon))),$$

*where $\delta$ is the function from Lemmas* 2.3, 2.4 *and* 2.5 (*respectively*), *$\delta_I$ is the modulus of convexity of the modular I, and $\eta(\varepsilon)$ is the function from Lemmas* 2.6, 2.7 *and* 2.8 (*respectively*).

PROOF. Let $\varepsilon\in(0,1)$, $\|x\|_\Phi=\|y\|_\Phi=1$, $\|x-y\|_\Phi\geq\varepsilon$. Then, by the respective condition $\Delta_2$ (see Theorem 2.1) we have $I(x)=I(y)=1$. Moreover, by Lemmas 2.6, 2.7 and 2.8 (respectively), we have $I(x-y)\geq\eta(\varepsilon)$. So, by uniform convexity of the modular $I$ (see [8]), we have $I((x+y)/2)\leq 1-\delta_I(\eta(\varepsilon))$. *Applying Lemmas* 1.3, 1.4 *and* 1.5 *in suitable cases, we obtain* $\|(x+y)/2\|_\Phi\leq 1-\delta(\delta_I(\eta(\varepsilon)))$, and the proof is finished.

REMARK 2.10. Since the functions $\delta_I$(see [8]) and $\delta$, $\eta$ are estimated from below, so the function $\delta_I\Phi$ is estimated from below also.

REMARK 2.11. In the case of an atomless and infinite measure $\mu$, and $\log_2\varepsilon^{-1}$ being integer, our estimate for the function $\eta$ is a best estimate. For the same measure $\mu$ and for any $\varepsilon\in(0,1)$ our estimate for the function $\delta$ is a best estimate.

## References


[1] AKIMOVIC, V., On uniformly convex and uniformly smooth Orlicz spaces, Teoria Funkcii Funk. Anal i Pril. 15 (1970), pp. 114–120 (in Russian).

[2] FIGIEL, T., On the moduli of convexity and smoothness, Studia Math. 56 (1976), pp. 121–155.

[3] HIMMELBERG, C. J., Measurable relations, Fundamenta Math. 87 (1975), pp. 53–72.

[4] HUDZIK, H., A criterion of uniform convexity of Orlicz-Musielak spaces with Luxemburg norm, Bull. Acad. Polon. Sci., to appear.

[5] HUDZIK, H., Uniformly convex Musielak-Orlicz spaces with Luxemburg's norm, Commentationes Math. 23.1 (1983), pp. 21–32.

[6] HUDZIK, H., Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm, Studia Math., to appear.

[7] HUDZIK, H. and KAMIŃSKA, A., Some remarks on convergence in Orlicz spaces, Commentationes Math. 23.1 (1980), pp. 21–32.

[8] KAMIŃSKA, A., On uniformly convex Orlicz spaces, Indagationes Math. 44. 1 (1982), pp. 27–36.

[9] KOZEK, A., Convex integral functionals in Orlicz spaces, Commentationes Math. 21.1 (1980), pp. 109–135.

[10] LINDENSTRAUSS, J. and TZAFRIRI, L., Classical Banach spaces II, Function spaces, Springer-Verlag, Berlin Heidelberg New York (1979).

[11] MALEEV, R. P. and Troyanski, S. L., On the moduli of convexity and smoothness in Orlicz spaces, Studia Math. 54 (1975), pp. 131–141.

[12] MUSIELAK, J., Orlicz spaces and modular spaces, Springer-Verlag Berlin Heidelberg

New York Tokyo 1034 (1983).
[13] LUXEMBURG, W. A. J., Banach function spaces, Thesis, Delft (1955).
[14] NAKANO, H., Topology and linear-topological spaces, Tokyo (1951).
[15] TURETT, B., Fenchel-Orlicz spaces, Dissertationes Math. 181 (1980), pp. 1–60.

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