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Indefinite Einstein hypersurfaces with nilpotent shape operators

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§ 1. Introduction

In [4], A. Fialkow classified Einstein hypersurfaces in indefinite space forms if the shape operator is diagonalizable. In [7], it was shown that if the shape operator A is not diagonalizable at each point then there are two possibilities: either $A^2=0$ or $A^2=-b^2I$, where b is a non-zero constant. In this paper those Einstein hypersurfaces with $A^2=0$ and rank A maximal are classified. The main results are the following.

2.2 THEOREM. *If f: $M_n^{2n}\rightarrow N^{2n+1}(c)$ is an isometric immersion of M_n^{2n} into a space form of constant curvature c with $A^2=0$ and rank $A=n$, then the kernel of A is an integrable, totally isotropic and parallel n-dimensional distribution on M. (Here M has signature (n, n). This is a consequence of the conditions on A.)*

2.3 COROLLARY. *If f is as above and $n>1$, then $c=0$.*

In Theorem 4.2, isometric immersions f: $M_n^{2n}\rightarrow \boldsymbol{R}^{2n+1}$ with $A^2=0$ and rank $A=n$ are classified locally.

The Einstein hypersurfaces classified in Theorem 4.2 provide a large family of examples of manifolds which have been studied extensively. A. G. Walker [10, 11, 12] and others (see [13], p. 278 for other references) investigated manifolds with parallel fields of planes. R. Rosca and others ([9], [1], [3]) study manifolds with spin-euclidean connections. In this case the spinor fields can be covariantly differentiated.

If f: $M_1^n\rightarrow N_1^{n+1}(c)$ is an isometric immersion with $A^2=0$ and rank $A=1$, then M_1^n also has constant sectional curvature c. L. Graves [5] classifies such f if $c=0$ and M is complete. In [6], Graves and Nomizu show that for $n\geqslant 4$ there are no umbilic-free isometric imbeddings from $S_1^n(1)$ into $S_1^{n+1}(1)$.

§ 2. Kernel of A is parallel

Let A be a symmetric operator in a vector space V with a non-degenerate inner product (,), so that $(Au, v)=(u, Av)\ \forall u, v\in V$. If $A^2=0$, we can find a basis of V, $\{\hat{L}_1, L_1, \cdots, \hat{L}_n, L_n, E_1, \cdots, E_p\}$, with respect to which

$$A=\begin{bmatrix} 0 & 1 & & & & & & & \\ 0 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 0 & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 0 \end{bmatrix}.$$

Here L_i, $\hat{L}_j$ are lightlike, $(L_i, \hat{L}_j)=-\delta_{ij}$, $(E_k, E_l)=\pm\delta_{kl}$ and all other inner products are 0. If the ratio of the rank of A to the dimension of V is to be as large as possible, then $p=0$, giving a basis $\{\hat{L}_1, L_1, \cdots, \hat{L}_n, L_n\}$ with $A\hat{L}_i=0$ and $AL_i=\hat{L}_i$. In this case V is even dimensional and has signature (n, n) [7].

If $f: M^m\to N^{m+1}$ is a non-degenerate isometric immersion and ξ is a unit normal vector field on M, then the shape operator A of f is defined by

$$\tilde{\nabla}_X\xi=-AX,$$

where $\tilde{\nabla}$ is the indefinite Riemannian connection on N. $A: TM\to TM$ and is symmetric on each T_xM, with respect to the metric on T_xM.

2.1 Lemma. *If $f: M_n^{2n}\to N^{2n+1}(c)$ is an isometric immersion with $A^2=0$ and rank $A=n$, then there are vector fields $\tilde{L}_1, \cdots, \tilde{L}_n$ defined in a neighborhood of any point of M such that $(\tilde{L}_i, \tilde{L}_j)=0=(A\tilde{L}_i, A\tilde{L}_j)$ and $(A\tilde{L}_i, \tilde{L}_j)=-\delta_{ij}$.*

Proof. Because A is symmetric and $A^2=0$, $(AX, AY)=(A^2X, Y)=0$ holds for all tangent vectors X, Y.

Choose $x\in M$. It was noted above that in T_xM there are vectors $(L_1)_x$, $\cdots, (L_n)_x$ such that $(L_i, L_j)_x=0$, $(AL_i)_x\neq 0$ and $(AL_i, L_j)_x=-\delta_{ij}$, $i, j=1, \cdots, n$. Extend the $(L_i)_x$ smoothly in a neighborhood of x so that $(L_i, L_j)=0$. This can be done by extending the appropriate orthonormal frame fields. By continuity, $AL_i\neq 0$ in some, possibly smaller, neighborhood.

Consider the smooth n-dimensional distribution on this neighborhood given by span $\{L_1, \cdots, L_n\}$. We can define an auxiliary negative definite inner product h on this distribution by

$$h(L_i, L_j) = (AL_i, L_j).$$

h is symmetric, bilinear and negative definite near x. Applying the Gram-Schmidt process to $\{L_1, \cdots, L_n\}$ gives $\{\tilde{L}_1, \cdots, \tilde{L}_n\}$ such that

$$h(\tilde{L}_i, \tilde{L}_j) = -\delta_{ij}.$$

These are the desired vector fields. Q. E. D.

2.2 THEOREM. *If $f: M_n^{2n} \to N^{2n+1}(c)$ is an isometric immersion of M_n^{2n} into a space form of constant curvature c with $A^2=0$ and rank $A=n$, then the kernel of A is an integrable, totally isotropic and parallel n-dimensional distribution on M.*

PROOF. In [7], it was proved that kernel A is integrable, totally geodesic and totally isotropic (namely, totally degenerate). A totally geodesic distribution S is one where

$$\nabla_X Y \in S, \quad \text{if} \quad X, Y \in S.$$

To prove that kernel A is parallel we must show that

$$\nabla_U X \in \ker A \quad \text{if} \quad X \in \ker A \quad \text{and} \quad U \in TM$$

or, equivalently, that

$$A(\nabla_U X) = 0 \quad \text{if} \quad AX = 0.$$

In order to do this, let $x \in M$ and choose vector fields in a neighborhood of x, $\{L_1, \cdots, L_n, AL_1, \cdots, AL_n\}$, as in the lemma.

Consider Codazzi's equation with L_i and L_j, $1 \leqslant i, j \leqslant n$:

$$\nabla_{L_i}(AL_j) - A(\nabla_{L_i} L_j) = \nabla_{L_j}(AL_i) - A(\nabla_{L_j} L_i).$$

Taking the inner product of both sides of this equation with AL_k gives

$$(\nabla_{L_i} AL_j, AL_k) = (\nabla_{L_j} AL_i, AL_k) \qquad (\dagger)$$

since $A^2=0$. Denoting AL_j by $L_{j'}$, $j=1, \cdots, n$, and defining Γ_{BC}^D, the Christoffel symbols, as usual, we have

$$\nabla_{L_i} L_{j'} = \sum_{k=1}^{n} \Gamma_{ij'}^k L_k + \Gamma_{ij'}^{k'} L_{k'}.$$

($\dagger$) becomes

$$\Gamma_{ij'}^k = \Gamma_{ji'}^k, \qquad 1 \leqslant i, j, k \leqslant n. \qquad (1)$$

Because the connection in M is metric, $L_i(AL_j, AL_k)=0=(\nabla_{L_i}AL_j, AL_k)+(AL_j, \nabla_{L_i}AL_k)$, so that

$$\Gamma^k_{ij'}+\Gamma^j_{ik'}=0\,, \qquad 1\leqslant i,\ j,\ k\leqslant n\,. \tag{2}$$

Combining (1) and (2), we see that $\Gamma^k_{ij'}=0$ for all $1\leqslant i,\ j,\ k\leqslant n$. In fact,

$$\Gamma^k_{ij'}\underset{(1)}{=}\Gamma^k_{ji'}\underset{(2)}{=}-\Gamma^i_{jk'}\underset{(1)}{=}-\Gamma^i_{kj'}\underset{(2)}{=}\Gamma^j_{ki'}\underset{(1)}{=}\Gamma^j_{ik'}\underset{(2)}{=}-\Gamma^k_{ij'}\,.$$

The fact that the kernel of A is totally geodesic gives $\Gamma^k_{i'j'}=0$. Thus $\Gamma^k_{Bj'}=0$ for $B=1, \cdots, n, 1', \cdots, n'$, $1\leqslant j,\ k\leqslant n$. This means the kernel of A is parallel. Q. E. D.

2.3 Corollary. *Let $n>1$. If $f: M^{2n}_n \to N^{2n+1}(c)$ is an isometric immersion of M^{2n}_n into a space form of constant curvature c with $A^2=0$ and rank $A=n$, then $c=0$.*

Proof. The Gauss equation of this isometric immersion is

$$R(X, Y)\,Z=c(X\wedge Y)\,Z+(\xi, \xi)\,(AX\wedge AY)\,Z\,,$$

where R is the curvature tensor of M, $X, Y, Z\in T_xM$, and ξ is a unit normal field. Let Z be a vector field in ker A. Expanding the Gauss equation, we have

$$\begin{aligned}&\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z\\ &\quad =c\big((Y, Z)\,X-(X, Z)\,Y\big)\pm\big((AY, Z)\,AX-(AX, Z)\,AY\big)\,.\end{aligned}$$

Since $AZ=0$, this becomes

$$\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z=c\big((Y, Z)\,X-(X, Z)\,Y\big)\,.$$

By Theorem 2.2 the left-hand side of this equation is in ker A. Given dim $M>2$, we can choose X and Y linearly independent with $(X, Z)=0$, $(Y, Z)=1$, and X not in ker A. Then the right-hand side is cX, which is in ker A iff $c=0$. Q. E. D.

L. Graves and K. Nomizu [6] give an example of a Lorentz surface M^2_1 isometrically immersed in S^3_1 with A satisfying $A^2=0$ and rank $A=1$, so the restriction on n cannot be removed.

§ 3. Examples

Before proceeding to the proofs of Theorems 4.1 and 4.2, let us examine a few examples of Einstein hypersurfaces M^{2n}_n with $A^2=0$ and rank $A=n$.

3.1 Example. B-scroll over a null curve in $\boldsymbol{R}^3_1$ [5].

$\boldsymbol{R}_1^3$ is Lorentz 3-space, with signature $(-,+,+)$. Consider a null curve $x(s)$ in $\boldsymbol{R}_1^3$, so that $(\dot{x}(s), \dot{x}(s))=0$. A null curve with a frame $\{A(s), B(s), C(s)\}$ is called a Cartan-framed null curve if the following conditions hold. $A(s)$, $B(s)$ are null; $(C(s), C(s))=1$; $(A(s), B(s))=-1$; all other inner products are zero along $x(s)$; and the Frenet equations of the derivatives of $A(s)$, $B(s)$, $C(s)$ along $x(s)$ have the form:

$$\frac{dx(s)}{ds}=A(s),$$

$$\frac{dA(s)}{ds}=k_2(s)\,C(s),$$

$$\frac{dB(s)}{ds}=k_3(s)\,C(s),$$

$$\frac{dC(s)}{ds}=k_3(s)\,A(s)+k_2(s)\,B(s),$$

The surface $f(u, s)=x(s)+uB(s)$ is called a B-scroll over the null curve $x(s)$. It is Lorentz and is flat iff $k_3(s)=0$. In this case,

$$A=\begin{bmatrix} 0 & -k_2(s) \\ 0 & 0 \end{bmatrix}$$

with respect to $\{\partial/\partial u, \partial/\partial s\}$, where the unit normal $\xi(u, s)=C(s)$. $(\nabla A)=0$ iff $k_2(s)$ is constant. If $k_2\equiv 1$, the surface is given by

$$x(s)+uB(s)=\left(\frac{s^3}{6\sqrt{2}}+\frac{s}{\sqrt{2}},\ \frac{s^3}{6\sqrt{2}}-\frac{s}{\sqrt{2}},\ \frac{s^2}{2}\right)+u\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right).$$

Graves calls this the B-scroll over the null cubic.

3.2. EXAMPLE. Sum of B-scrolls.

For $j=1, \cdots, n$, let $(u_j, s_j)\in I_j\times J_j\subset \boldsymbol{R}\times\boldsymbol{R}$ and suppose $f_j(u_j, s_j)=(a_j(u_j, s_j), b_j(u_j, s_j), c_j(u_j, s_j))$ are n flat B-scrolls in $\boldsymbol{R}_1^3$ which, when written as $x_j(s_j)+u_jB_j$ satisfy the following initial conditions:

$$x_j(0)=0,\ \dot{x}_j(0)=\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right),\qquad B_j=\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

and $C_j(0)=(0, 0, 1)$.

We can define a parametrized hypersurface in $\boldsymbol{R}_n^{2n+1}$ by

$$\begin{aligned} &f(u_1, s_1, \cdots, u_n, s_n) \\ &\quad=\big(a_1(u_1, s_1), \cdots, a_n(u_n, s_n), b_1(u_1, s_1), \\ &\qquad \cdots, b_n(u_n, s_n), c_1(u_1, s_1)+\cdots+c_n(u_n, s_n)\big), \end{aligned}$$

where R_n^{2n+1} has signature $(n, n+1)$. This hypersurface has

$$A=\begin{bmatrix} 0 & -k_1 & & & & \\ 0 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 0 & -k_n \\ & & & & 0 & 0 \end{bmatrix}.$$

If each $k_i(s_i)$ is constant, then $\nabla A=0$.

If rank A is constant but not equal to n, ker A may not be parallel.

3.3 Example. A 4-dimensional scroll with $A^2=0$, rank A constant and ker A not parallel.

According to W. Bonner [2], for every smooth $k(s)$, there is a null curve $x(s)$ in R_1^4 with frame $\{X(s), Y(s), Z(s), C(s)\}$ such that $(X(s), Y(s))=-1$, X and Y are null, Z and C are unit spacelike and whose derivatives are

$$\frac{dx(s)}{ds}=X(s),$$

$$\frac{dX(s)}{ds}=C(s),$$

$$\frac{dY(s)}{ds}=k(s)\,Z(s),$$

$$\frac{dZ(s)}{ds}=k(s)\,X(s),$$

$$\frac{dC(s)}{ds}=+Y(s).$$

Let $x(s)$ be considered as a null curve in R_1^5 by looking at $(x(s), 0)$ with frame $\{(X(s), 0), \cdots, (C(s), 0), W(s)\}$, where $W(s)=W\equiv(0, 0, 0, 0, 1)$.

The Lorentz 4-surface parametrized by $f(u, s, t, v)=x(s)+uY(s)+tZ(s)+vW(s)$ has, with $\xi(u, s, t, v)=(C(s), 0)$, shape operator

$$A=\begin{bmatrix} 0 & -1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

with respect to $\{\partial/\partial u, \partial/\partial s, \partial/\partial t, \partial/\partial v\}$. It is easy to see that the kernel of

A, spanned by $Y(s)$, $Z(s)$, $W(s)$, is not parallel. In fact, $\nabla_{\partial/\partial s}\partial/\partial t=k(s)\,X(s)$ is not in ker A. Thus, if the rank of A is not maximal, then the kernel of A need not be parallel.

§ 4. Local Characterization of M_n^{2n} isometrically immersed in R_n^{2n+1} with $A^2=0$ and rank $A=n$

4.1 THEOREM. *Let $f: M_n^{2n}\to R_n^{2n+1}$ be an isometric immersion with rank $A=n$. Then kernel A is an integrable, totally isotropic, parallel distribution on M_n^{2n} iff $A^2=0$.*

PROOF. If $A^2=0$, the conclusion was obtained in the proof of Theorem 2.2.

Assume then that ker A is integrable, parallel and totally isotropic. By a motion of R_n^{2n+1} we can assume that ker A is spanned by $B_i=(e_i, e_i, 0)$, $i=1, \cdots, n$, where $e_1, \cdots, e_n$ is the standard basis of R^n. If $(x_1, \cdots, x_{2n})$ is a local coordinate system for M_n^{2n}, then the normal unit vector field ξ must have the following form because it is perpendicular to ker A.

$$\xi_{f(\vec{x})}=\left(\xi_1(\vec{x}), \cdots, \xi_n(\vec{x}), \xi_1(\vec{x}), \cdots, \xi_n(\vec{x}), 1\right).$$

Then,

$$D_{\partial/\partial x_j}\xi=D_{\partial/\partial x_j}\left(\sum_{i=1}^{n}\xi_i(\vec{x})\,B_i+(0, 0, \cdots, 0, 1)\right)$$

which is in ker A. Thus, $-A\,(D_{\partial/\partial x_j}\xi)=0=A^2(\partial/\partial x_j)$. Q. E. D.

4.2 THEOREM. *f: $M_n^{2n}\to R_n^{2n+1}$ is an isometric immersion with $A^2=0$ and rank $A=n$ iff, around each $x\in M$, there is a coordinate system $(t_1, \cdots, t_n, u_1, \cdots, u_n)$ such that f has the following form:*

$$f(\vec{t}, \vec{u})=\left(g_1(\vec{t}), \cdots, g_n(\vec{t}), g_1(\vec{t})+t_1, \cdots, g_n(\vec{t})+t_n, G(\vec{t})\right)+\sum u_j B_j\,.$$

Here $\vec{t}=(t_1, \cdots, t_n)$, $\vec{u}=(u_1, \cdots, u_n)$, B_i, $1\leqslant i\leqslant n$ are as in the proof of Theorem 4.1, $g_1, \cdots, g_n$, $G: U\subset R^n\to R$ are smooth and det $[\partial^2 G/\partial t_i\partial t_j]\neq 0$.

REMARK. Locally, then, each such M_n^{2n} is an n-planed hypersurface.

PROOF. Assume we are given such an isometric immersion. The kernel of A is integrable. Thus, given any x_0 in M, we can find a local coordinate system $(s_1, \cdots, s_n, v_1, \cdots, v_n)$ around x_0 so that ker A is given by $s_1=c_1, \cdots, s_n=c_n$, where the c_i's are constants.

We also can assume, as in Theorem 4.1, that, by a motion of R_n^{2n+1},

$(\ker A)_{f(x)}$ is spanned by $B_1, \cdots, B_n$. Define $g(\vec{s})=f(\vec{s}, 0)$. It is clear then that M_n^{2n} can be locally parametrized near $g(\vec{s})$, with a change of coordinates, by

$$f(\vec{s}, \vec{u}) = g(\vec{s}) + \sum_{j=1}^{n} u_j B_j .$$

The unit normal $\xi(\vec{s}, \vec{u})$ is of the form

$$\xi(\vec{s}, \vec{u}) = \left(\xi_1(\vec{s}), \cdots, \xi_n(\vec{s}), \xi_1(\vec{s}), \cdots, \xi_n(\vec{s}), 1\right).$$

In order for f to have the required properties, several conditions must be satisfied.

i) Rank $A=n$ iff $\{\partial\xi/\partial s_1, \cdots, \partial\xi/\partial s_n\}$ is linearly independent.

ii) M_n^{2n} inherits a non-degenerate metric iff $\det[(\partial g/\partial s_i, B_j)]\neq 0$.

iii) ξ is normal iff $(\partial g/\partial s_i, \xi)=0$ $i=1, \cdots, n$.

If $g(\vec{s})=(g_1(\vec{s}), \cdots, g_{2n+1}(\vec{s}))$, let $h_i(\vec{s})=g_{n+i}(\vec{s})-g_i(\vec{s})$ $i=1, \cdots, n$. Condition ii can be rewritten as

ii′) $\det[\partial h_i/\partial s_j] \neq 0$,

while iii becomes

iii′) $\sum_{i=1}^{n} \xi_i(\partial h_i/\partial s_j) + \partial g_{2n+1}(\vec{s})/\partial s_j = 0 \qquad j=1, \cdots, n$.

Finally, in order to insure that A_ξ is symmetric and that the mixed partials of g_{2n+1} be equal, we need

iv) $\sum_{k=1}^{n} (\partial\xi_k/\partial s_i)(\partial h_k/\partial s_j) = \sum_{k=1}^{n} (\partial\xi_k/\partial s_j)(\partial h_k/\partial s_i)$.

By ii′, we can change coordinates from $(s_1, \cdots, s_n, u_1, \cdots, u_n)$ to $(h_1, \cdots, h_n, u_1, \cdots, u_n)$ which we rename $(t_1, \cdots, t_n, u_1, \cdots, u_n)$. With this new coordinate system, ii′ is automatically fulfilled, while iii′ becomes

iii″) $\xi_j + \partial g_{2n+1}(\vec{t})/\partial t_j = 0$,

and iv becomes

iv′) $\partial\xi_j/\partial t_i = \partial\xi_i/\partial t_j \qquad i, j=1, \cdots, n$.

Summarizing, we see that after the changes of coordinates, we must have

i) $\{\partial\xi/\partial t_1, \cdots, \partial\xi/\partial t_n\}$ linearly independent;

iii″) $\xi_j = -\partial g_{2n+1}(\vec{t})/\partial t_j$; and

iv′) $\partial\xi_j/\partial t_i = \partial\xi_i/\partial t_j$.

Given the immersion f, let $G(\vec{t})=g_{2n+1}(\vec{t})$ which is smooth. Then by iii″,

$\xi_j = -\partial G/\partial t_j$, and we have $\partial\xi_j/\partial t_i = -\partial^2 G/\partial t_i \partial t_j = -\partial^2 G/\partial t_j \partial t_i = \partial\xi_i/\partial t_j$ so that iv′ is satisfied. The only condition we must impose on $G(\vec{t})$ is det $[\partial^2 G/\partial t_i \partial t_j] \neq 0$, so that i holds. Thus, given any such f, we have transformed it into the desired form. It is easy to check that any f in this form has $A^2=0$ and rank $A=n$. Q. E. D.

We show that sums of B-scrolls in 3. 2 Example do not, even locally, exhaust M_n^{2n} as in 4. 2 Theorem.

$T_x(M_n^{2n})$ can be given the structure of a commutative algebra using the covariant derivative of A.

$$X \cdot Y := \nabla_X(AY) - A(\nabla_X Y)$$

For any 4-dimensional sum of two B-scrolls with $\nabla A \neq 0$ we can find a basis $\{e_1, e_2, u_1, u_2\}$ of T_x with the following products.

$e_1 \cdot e_1 = u_1$ and $e_2 \cdot e_2 = u_2$, while all others are zero. (If $\nabla A = 0$, $X \cdot Y = 0$ everywhere.)

Use the classification theorem to define an M_2^4 in $\boldsymbol{R}_2^5$ by setting $g_1 = 0 = g_2$ and $G(t_1, t_2) = t_2^3 + t_1^2 t_2 + t_1 + t_2$. Then there is a basis $\{f_1, f_2, v_1, v_2\}$ of $T_0 M$ so that the non-zero products are

$$f_1 \cdot f_1 = 2v_2,$$
$$f_2 \cdot f_2 = -6v_2,$$
$$f_1 \cdot f_2 = 2v_1.$$

These two 4-dimensional algebras are not isomorphic. Thus, the second hypersurface is not a sum of B-scrolls.

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