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An analytical proof of Kodaira's embedding theorem for Hodge manifolds

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Introduction

The main purpose of the present paper is to give a purely analytical proof of a famous theorem due to Kodaira [4] which states that every Hodge manifold X can be holomorphically embedded in a complex projective space $P^N(\boldsymbol{C})$.

Our proof of the theorem is based on Kohn's harmonic theory on compact strongly pseudo-convex manifolds ([2] and [3]), and has been inspired by the proof due to Boutet de Monvel [1] of the fact that every compact strongly pseudo-convex manifold M can be holomorphically embedded in a complex affine space $\boldsymbol{C}^N$, provided dim $M>3$. In this paper the differentiability will always mean that of class C^∞. Given a vector bundle E over a manifold M, $\Gamma(E)$ will denote the space of C^∞ cross sections of E.

1. Let $\tilde{M}$ be an $(n-1)$-dimensional (para-compact) complex manifold, and F a holomorphic line bundle over $\tilde{M}$. Let M' be the holomorphic $\boldsymbol{C}^*$-bundle associated with F, and π' the projection $M'\to\tilde{M}$.

There are an open covering $\{U_\alpha\}$ of $\tilde{M}$ and for each α a holomorphic trivialization

$$\phi_\alpha:\ \pi'^{-1}(U_\alpha)\ni z\longrightarrow\left(\pi'(z),\ f_\alpha(z)\right)\in U_\alpha\times\boldsymbol{C}^*\,.$$

We have

$$f_\alpha(za)=f_\alpha(z)\,a,\ \ z\in\pi'^{-1}(U_\alpha),\ \ a\in\boldsymbol{C}^*\,.$$

Let $\{g_{\alpha\beta}\}$ be the system of holomorphic transition functions associated with the trivializations ϕ_α. Then for any α and β with $U_\alpha\cap U_\beta\neq\phi$ we have

$$f_\alpha(z)=g_{\alpha\beta}\left(\pi'(z)\right)f_\beta(z),\ \ z\in\pi'^{-1}(U_\alpha\cap U_\beta)\,.$$

Let us now consider a $U(1)$-reduction M of the $\boldsymbol{C}^*$-bundle M'. Let π denote the projection $M\to\tilde{M}$. Then there is a unique positive function a_α on U_α such that

$$\pi^{-1}(U_\alpha)=\left\{z\in\pi'^{-1}(U_\alpha)\,\middle|\,|f_\alpha(z)|^2\,a_\alpha\left(\pi'(z)\right)=1\right\}.$$

Clearly we have $a_\alpha|g_{\alpha\beta}|^2=a_\beta$, and hence

$$\gamma=\sqrt{-1}/2\pi\cdot\partial\bar{\partial}\log a_\alpha=\sqrt{-1}/2\pi\cdot\sum_{i,j}\partial^2\log a_\alpha/\partial z_i\partial\bar{z}_j\cdot dz_i\wedge d\bar{z}_j$$

defines a glabal 2-form of type (1, 1) on $\tilde{M}$, where $\{z_1,\cdots,z_{n-1}\}$ denotes any complex coordinate system of $\tilde{M}$ defined on an open set of U_α. The form γ is usually called the Chern form (cf. [5]).

2. M being a real hypersurface of M', it is endowed with a pseudo-complex structure in a natural manner (cf. [6]). Let $T^{(1,0)}(M')$ be the vector bundle of tangent vectors of type (1, 0) to M', and $\boldsymbol{C}T(M)$ the complexification of the tangent bundle $T(M)$ of M. Then the pseudo-complex structure means the subbundle S of $\boldsymbol{C}T(M)$ defined by

$$S_x=\boldsymbol{C}T(M)_x\cap T^{(1,0)}(M')_x\,,\qquad x\in M\,.$$

We have

1) $\dim S_x=n-1$,

2) $S\cap\bar{S}=0$,

3) $[\Gamma(S),\ \Gamma(S)]\subset\Gamma(S)$.

We remark that the differential π_* of π maps S onto $T^{(1,0)}(\tilde{M})$, the bundle of tangent vectors of type (1, 0) to $\tilde{M}$. We also remark that S is invariant under the action of $U(1)$ on M. More precisely, for each $a\in U(1)$ let R_a denote the right translation $M\ni x\rightarrow xa\in M$. Then we have $(R_a)_*S=S$ or in other words, R_a is an automorphism of the pseudo-complex manifold M.

For any integer k we denote by $\mathscr{C}^k$ the space of cross sections of $\Lambda^k\bar{S}^*$, and define an operator $\bar{\partial}:\mathscr{C}^k\rightarrow\mathscr{C}^{k+1}$ by

$$(\bar{\partial}\varphi)(\bar{X}_1\wedge\cdots\wedge\bar{X}_{k+1})=\sum_i(-1)^{i+1}\bar{X}_i\varphi(\bar{X}_1\wedge\cdots\wedge\hat{\bar{X}}_i\wedge\cdots\wedge\bar{X}_{k+1})$$
$$+\sum_{i<j}(-1)^{i+j}\varphi\Big([\bar{X}_i,\bar{X}_j]\wedge\bar{X}_1\wedge\cdots\wedge\hat{\bar{X}}_i\wedge\cdots\wedge\hat{\bar{X}}_j\wedge\cdots\wedge\bar{X}_{k+1}\Big)\,,$$

where $\varphi\in\mathscr{C}^k$ and $X_i\in\Gamma(S)$. Then we have $\bar{\partial}^2=0$, and hence the system $\{\mathscr{C}^k,\bar{\partial}\}$ gives a complex (cf. [6]).

A function $\varphi\in\mathscr{C}^0$ is said to be holomorphic if it satisfies the (tangential Cauchy-Riemann) equation $\bar{\partial}\varphi=0$.

For any integer m we define a subspace $\mathscr{C}^0_{(m)}$ of $\mathscr{C}^0$ by

$$\mathscr{C}^0_{(m)}=\Big\{\varphi\in\mathscr{C}^0\Big|R_a^*\varphi=a^{-m}\varphi\ \text{ for all }\ a\in U(1)\Big\}\,.$$

Let $\varphi\in\mathscr{C}^0$. Then it is clear that φ is in $\mathscr{C}^0_{(0)}$ if and only if there is a (unique) function $\tilde{\varphi}$ on $\tilde{M}$ with $\varphi=\pi^*\tilde{\varphi}$. Since $\pi_*S=T^{(1,0)}(\tilde{M})$, we see that a function $\varphi\in\mathscr{C}^0_{(0)}$ is holomorphic if and only if $\tilde{\varphi}$ is holomorphic. In general consider the m-th power F^m of the line bundle F. Then it can be shown that there is a natural isomorphism of $\mathscr{C}^0_{(m)}$ onto $\Gamma(F^m)$, say $\varphi\rightarrow\tilde{\varphi}$, and that φ is holomorphic if and only if $\tilde{\varphi}$ is holomorphic (cf. [6]).

3. Assume that $\tilde{M}$ is compact. As is well known, the line bundle F is negative if and only if there is a $U(1)$-reduction M of M' such that the hermitian matrix $(\partial^2 \log a_\alpha/\partial z_i\partial\bar{z}_j)$ is positive definite at each point of $\tilde{M}$ (cf. [5]).

Hereafter we assume that $\tilde{M}$ is compact and that F is negative with respect to a $U(1)$-reduction M of M'. Since M is locally defined by the equations $|f_\alpha|^2\pi'^*a_\alpha=1$ or equivalently

$$\log f_\alpha+\overline{\log f_\alpha}+\pi'^*(\log a_\alpha)=0\,,$$

we see that M is a (compact) strongly pseudo-convex real hypersurface of M' (cf. [6]).

Let $d(p,q)\,(p,q\in\tilde{M})$ be a distance function on $\tilde{M}$ associated with a Riemannian metric on $\tilde{M}$. Fix a point p_0 of $\tilde{M}$ and define a function ρ on $\tilde{M}$ by

$$\rho(p)=d(p_0,p)^2\,,\qquad p\in\tilde{M}\,,$$

which can be confused with a function on M, i.e., the function $\pi^*\rho$. (Analogous confusions will be made frequently.)

LEMMA 1. *There are a function h on M and a neighborhood V of p_0 having the following properties*:

1) *h is in $\mathscr{C}^0_{(-1)}$,*
2) *h is holomorphic on $\pi^{-1}(V)$,*
3) *$|h(x)|\leqq e^{-K_1\rho(x)}$, $x\in M$, where K_1 is a positive constant,*
4) *$|h(x)|\geqq e^{-K_2\rho(x)}$, $x\in\pi^{-1}(V)$, where K_2 is a positive constant.*

PROOF. Fix an α with $p_0\in U_\alpha$, and denote by u the restriction of f_α to $\pi^{-1}(U_\alpha)$. Then u is holomorphic, and we have:

$$R_a^*u=ua\,,\qquad a\in U(1)\,,$$

$$|u|^2a_\alpha=1 \text{ on } \pi^{-1}(U_\alpha)\,.$$

Let $\{z_1,\cdots,z_{n-1}\}$ be a complex coordinate system around p_0 with $z_i(p_0)=0$. Then the function $b=\log a_\alpha$ can be expanded as follows:

$$b=b(p_0)+2\mathrm{Re}\sum_i b_i(p_0)\,z_i+\mathrm{Re}\sum_{i,j}b_{ij}(p_0)\,z_iz_j$$
$$+\sum_{i,j}b_{i\bar{j}}(p_0)\,z_i\bar{z}_j+O(|z|^3)\,,$$

where $b_i=\partial b/\partial z_i$, $b_{ij}=\partial^2 b/\partial z_i\partial z_j$, $b_{i\bar{j}}=\partial^2 b/\partial z_i\partial\bar{z}_j$, and $|z|^2=\sum_i|z_i|^2$. We define a function t on U_α by

$$t = 1/2 \cdot b(p_0) + \sum_i b_i(p_0)\, z_i + 1/2 \cdot \sum_{i,j} b_{ij}(p_0)\, z_i z_j$$

and a function h' on $\pi^{-1}(U_\alpha)$ by

$$h' = u \cdot e^t .$$

Since $|h'|^2 = |u|^2 \cdot e^{2\mathrm{Re}\, t}$, it follows that

$$\begin{aligned} \log |h'|^2 &= \log |u|^2 + 2\mathrm{Re}\, t \\ &= -b + 2\mathrm{Re}\, t \\ &= -\sum_{i,j} b_{i\bar{j}}(p_0)\, z_i \bar{z}_j + O(|z|^3) . \end{aligned}$$

Since the hermitian matrix $(b_{i\bar{j}}(p_0))$ is positive definite, we can find a neighborhood $V'(\subset U_\alpha)$ of p_0 and positive constants K_1 and K_2 such that

$$-K_2 \rho(x) \leqq \log |h'(x)| \leqq -K_1 \rho(x) , \qquad x \in \pi^{-1}(V') .$$

Now take a neighborhood V of p_0 with $V \subset\subset V'$ and a function η on $\widetilde{M}$ having the following properties: 1) $0 \leqq \eta \leqq 1$, 2) Supp $\eta \subset V'$, and 3) $\eta = 1$ on V. And define a function h on M by $h(x) = 0$ if $x \notin \pi^{-1}(V')$ and $h(x) = \eta(x)\, h'(x)$ if $x \in \pi^{-1}(V')$. Then it is easy to see that h and V, thus obtained, have the desired properties.

4. Let g be a Riemannian metric on M such that $g(X, Y) = 0$ for all $X, Y \in S_x$ and $x \in M$. Since S is $U(1)$-invariant, we may assume that g is $U(1)$-invariant, i.e., $R_a^* g = g$, $a \in U(1)$. Let ω denote the volume element associated with g, which is also $U(1)$-invariant.

For any $\varphi, \psi \in \mathscr{C}^k$ we define a function $\langle \varphi, \psi \rangle$ on M in the following manner: Let $x \in M$ and let $\{e_1, \cdots, e_{n-1}\}$ be any basis of S_x with $g(e_i, \bar{e}_j) = \delta_{ij}$. Then

$$\langle \varphi, \psi \rangle (x) = 1/k! \cdot \sum_{i_1, \cdots, i_k} \varphi(\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k})\, \overline{\psi(\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_k})} .$$

We now define an inner product (,) in $\mathscr{C}^k$ by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle\, \omega .$$

Let $\varphi \in \mathscr{C}^k$ and $a \in U(1)$. Since S is $U(1)$-invariant, $R_a^* \varphi$ can be naturally defined to give an element of $\mathscr{C}^k$. In this way the group $U(1)$ acts on the space $\mathscr{C}^k$, and we see that the inner product (,) is $U(1)$-invariant, i.e., $(R_a^* \varphi, R_a^* \psi) = (\varphi, \psi)$, $a \in U(1)$.

We denote by ϑ the formal adjoint operator of the operator $\bar{\partial}$ with respect to the inner product (,). The operator $\square = \vartheta \bar{\partial} + \bar{\partial} \vartheta$ is usually called the Laplacian.

Now it is known that, for every $1 \leqq k \leqq n-2$, there are unique operators $H, G: \mathscr{C}^k \to \mathscr{C}^k$ such that

$$\square H = HG = 0, \quad \text{and} \quad \square G + H = 1.$$

(See [2], [3] and [6].) The operator G is usually called the Green operator.

Here we notice that the operators $\bar{\partial}$, ϑ, $\square$, H and G are all compatible with the $U(1)$-action: For any $a \in U(1)$ and $\varphi \in \mathscr{C}^k$ we have $R_a^*(\bar{\partial}\varphi) = \bar{\partial}(R_a^*\varphi)$, $R_a^*(\vartheta\varphi) = \vartheta(R_a^*\varphi)$, etc.

In the following we assume that $n \geqq 3$. Then we define an operator $H: \mathscr{C}^0 \to \mathscr{C}^0$ by

$$H\varphi = \varphi - \vartheta G \bar{\partial}\varphi, \qquad \varphi \in \mathscr{C}^0.$$

It is easy to see that $H\varphi$ is holomorphic and that the operator $H: \mathscr{C}^0 \to \mathscr{C}^0$ is compatible with $U(1)$-action. In particular we have $H\mathscr{C}^0_{(m)} \subset \mathscr{C}^0_{(m)}$.

5. Let p_0 be any point of $\tilde{M}$. We take a function h on M and a neighborhood V of p_0 having the properties in Lemma 1. Let φ be a function on $\tilde{M}$ that is holomorphic on a neighborhood $O(\subset V)$ of p_0. For any positive integer m let us consider the function $h^m\varphi$ on M, which is clearly in $\mathscr{C}^0_{(-m)}$. Accordingly the function

$$H(h^m\varphi) = h^m\varphi - \vartheta G \bar{\partial}(h^m\varphi)$$

is holomorphic and is in $\mathscr{C}^0_{(-m)}$.

We denote by $\| \ \|_{(s)}$ (resp. by $| \ |_s$) a Sobolev norm (resp. a C^s-norm) in $\mathscr{C}^k$ corresponding to any non-negative integer s (cf. [2]). Putting

$$a = \operatorname*{Min}_{p \in \tilde{M} - O} \rho(p)\ (>0) \quad \text{and} \quad A = e^{-K_1 a}$$

we see that

$$|h(x)| \leqq e^{-K_1\rho(x)} \leqq A \ \text{ if } \ x \in \pi^{-1}(\tilde{M} - O).$$

LEMMA 2. *For every non-negative integer s there is a positive constant C_s such that*

$$\|\bar{\partial}(h^m\varphi)\|_{(s)} \leqq C_s m^{s+1} A^m, \qquad m > 0.$$

PROOF. Let $\{x_1, \cdots, x_l\}$ $(l = 2n-1)$ be a coordinate system of M defined on an open set W of M. Let X be a cross section of S supported in W. Then we have

$$\bar{X}(h^m\varphi) = mh^{m-1}\bar{X}h \cdot \varphi + h^m \cdot \bar{X}\varphi.$$

Since both h and φ are holomorphic on $\pi^{-1}(O)$, we have $\bar{X}(h^m\varphi) = 0$ on $\pi^{-1}(O)$. Therefore it follows that

$$|\bar{X}(h^m\varphi)|_0 \leqq C_0 m A^m .$$

Applying the operator $\partial_i = \partial/\partial x_i$ to the equality above for $\bar{X}(h^m\varphi)$, we obtain

$$\partial_i\big(\bar{X}(h^m\varphi)\big) = m(m-1)\, h^{m-2}\partial_i h\cdot\bar{X}h\cdot\varphi + mh^{m-1}\partial_i(\bar{X}h\cdot\varphi)$$
$$+ mh^{m-1}\partial_i h\cdot\bar{X}\varphi + h^m\cdot\partial_i(\bar{X}\varphi) .$$

As above it follows that

$$|\partial_i\big(\bar{X}(h^m\varphi)\big)|_0 \leqq C_1 m^2 A^m .$$

In general consider the operators $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_l^{\alpha_l}$ where $\alpha = (\alpha_1, \cdots, \alpha_l)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l \leqq s$. Then we have

$$|D^\alpha\big(\bar{X}(h^m\varphi)\big)|_0 \leqq C_s m^{s+1} A^m ,$$

from which follows easily the lemma.

LEMMA 3. *There is a positive constant C such that*

$$|H(h^m\varphi) - h^m\varphi|_1 \leqq Cm^{n+2}A^m , \qquad m > 0 .$$

PROOF. Using the Sobolev lemma, we obtain

$$|\vartheta G\bar{\partial}(h^m\varphi)|_1 \leqq C_1 \|\vartheta G\bar{\partial}(h^m\varphi)\|_{(n+1)} \leqq C_2 \|G\bar{\partial}(h^m\varphi)\|_{(n+2)} .$$

By Folland-Kohn [2] we know that

$$\|G\psi\|_{(n+2)} \leqq C_3 \|\psi\|_{(n+1)} , \qquad \psi \in \mathscr{C}^1 .$$

Therefore it follows from Lemma 2 that

$$|H(h^m\varphi) - h^m\varphi|_1 \leqq C_4 \|\bar{\partial}(h^m\varphi)\|_{(n+1)} \leqq Cm^{n+2}A^m .$$

6. By using Lemmas 1 and 3 we shall show that the complex manifold $\widetilde{M}$ can be holomorphically embedded in a complex projective space.

Let $p_0 \in \widetilde{M}$, and let $\varphi_1, \cdots, \varphi_{n-1}$ be functions on $\widetilde{M}$ having the following properties:

1) Each function φ_i is holomorphic on a common neighborhood $O(\subset V)$ of p_0,

2) $\{\varphi_1, \cdots, \varphi_{n-1}\}$ gives a coordinate system on O.

Putting $\varphi_n = 1$, we define functions $f_1^{(m)}, \cdots, f_n^{(m)}$ on M by

$$f_j^{(m)} = H(h^m\varphi_j) , \qquad 1 \leqq j \leqq n .$$

Then $f_j^{(m)}$ are holomorphic and are in $\mathscr{C}^0_{(-m)}$. Furthermore by Lemma 3 we have

$$(*) \qquad |f_j^{(m)} - h^m\varphi_j|_1 \leqq Cm^{n+2}A^m , \qquad m > 0 .$$

Let us define functions $\psi_1^{(m)}, \cdots, \psi_n^{(m)}$ on $\pi^{-1}(V)$ by

$$\psi_j^{(m)} = f_j^{(m)}/h^m , \qquad 1 \leqq j \leqq n .$$

Let ε be a positive number with $K_1 a - K_2 \varepsilon > 0$, and let $O'(\subset O)$ be a neighborhood of p_0 such that $\rho(p) \leqq \varepsilon$ for all $p \in O'$. Putting $B = e^{-(K_1 a - K_2 \varepsilon)}$, we see that if $x \in \pi^{-1}(O')$,

$$A|h(x)|^{-1} \leqq A e^{K_2 \rho(x)} \leqq e^{-K_1 a + K_2 \rho(x)} \leqq B .$$

For every cotangent vector α we denote by $|\alpha|$ the norm of α with respect to a fixed Riemannian metric on M.

LEMMA 4. *There is a positive constant C' such that*

$$|\psi_j^{(m)}(x) - \varphi_j(x)| + |d\psi_{jx}^{(m)} - d\varphi_{jx}| \leqq C' m^{n+3} B^m ,$$
$$m > 0 , \qquad x \in \pi^{-1}(O') , \qquad 1 \leqq j \leqq n .$$

PROOF. By (*) we have the inequalities:

$$|f_j^{(m)}(x) - h(x)^m \varphi_j(x)| + |df_{jx}^{(m)} - d(h^m \varphi_j)_x| \leqq C m^{n+2} A^m ,$$
$$m > 0 , \qquad x \in M , \qquad 1 \leqq j < n .$$

For every $x \in \pi^{-1}(O')$ we have

$$\psi_j^{(m)}(x) - \varphi_j(x) = h(x)^{-m}\left(f_j^{(m)}(x) - h(x)^m \varphi_j(x)\right) ,$$

$$d\psi_{jx}^{(m)} - d\varphi_{jx} = h(x)^{-m}\left(df_{jx}^{(m)} - d(h^m \varphi_j)_x\right)$$
$$- m\left(\psi_j^{(m)}(x) - \varphi_j(x)\right) h(x)^{-1} dh_x .$$

From these facts follows easily the lemma.

LEMMA 5. *There are neighborhoods O_1 and O_2 with $O_2 \subset O_1 \subset O$, and a positive integer μ such that for all $m \geqq \mu$ the following hold:*

1) *$f_n^{(m)}(x) \neq 0$ for all $x \in \pi^{-1}(O_1)$,*

2) *The functions $f_i^{(m)}/f_n^{(m)}$ $(1 \leqq i \leqq n-1)$ on $\pi^{-1}(O_1)$ are holomorphic, and are reduced to holomorphic functions on O_1,*

3) *The functions $f_i^{(m)}/f_n^{(m)}$, regarded as holomorphic functions on O_1, give a coordinate system on O_1,*

4) $|f_n^{(m)}(y)|/|f_n^{(m)}(x)| < 1/2$, $x \in \pi^{-1}(O_2)$, $y \in \pi^{-1}(\tilde{M} - O_1)$.

PROOF. By Lemma 4 we see that $\lim_{m \to \infty} |\psi_n^{(m)}(x) - 1| = 0$ uniformly for $x \in \pi^{-1}(O')$. Hence there is a positive integer μ such that $\psi_n^{(m)}(x) \neq 0$ and hence $f_n^{(m)}(x) \neq 0$ for all $m \geqq \mu$ and $x \in \pi^{-1}(O')$. For any $1 \leqq i \leqq n-1$ and $m \geqq \mu$, the function $\varphi_i^{(m)} = f_i^{(m)}/f_n^{(m)}$ on $\pi^{-1}(O')$ is holomorphic, and is reduced to a holomorphic function on O', because $\varphi_i^{(m)}(xa)=\varphi_i^{(m)}(x)$, $x\in\pi^{-1}(O')$ and $a\in U(1)$. Clearly we have $\varphi_i^{(m)}=\psi_i^{(m)}/\psi_n^{(m)}$. Therefore we see from Lemma 4 that

$$\lim_{m\to\infty}\Big(|\varphi_i^{(m)}\big((x)-\varphi_i(x)\big)|+|d\varphi_{ix}^{(m)}-d\varphi_{ix}|\Big)=0$$

uniformly for $x\in\pi^{-1}(O')$. Let O_1 be a neighborhood of p_0 with $O_1\subset\subset O'$. Since $\{\varphi_1,\cdots,\varphi_{n-1}\}$ gives a coordinate system on O, it follows that if we choose a sufficiently large μ, $\{\varphi_1^{(m)},\cdots,\varphi_{n-1}^{(m)}\}$ gives a coordinate system on O_1 for every $m\geqq\mu$.

Now from (*) we obtain

$$|f_n^{(m)}(z)-h(z)^m|\leqq Cm^{n+2}A^m\,,\qquad z\in M,\qquad m>0\,.$$

Therefore if $x\in\pi^{-1}(O_1)$, we have

$$|f_n^{(m)}(x)|\geqq|h(x)|^m-Cm^{n+2}A^m\geqq e^{-mK_2\rho(x)}-Cm^{n+2}A^m\,,$$

and if $y\in\pi^{-1}(\tilde{M}-O_1)$, we have

$$|f_n^{(m)}(y)|\leqq|h(y)|^m+Cm^{n+2}A^m\leqq e^{-mK_1\rho(y)}+Cm^{n+2}A^m\,.$$

Put $b=\operatorname*{Min}_{p\in\tilde{M}-O_1}\rho(p)$ (>0), and let δ be a positive number such that $K_1b-K_2\delta>0$ and hence $K_1a-K_2\delta>0$. Let O_2 $(\subset O_1)$ be a neighborhood of p_0 such that $\rho(p)\leqq\delta$ for all $p\in O_2$. Then it follows that if $x\in\pi^{-1}(O_2)$ and $y\in\pi^{-1}(\tilde{M}-O_1)$, then

$$\begin{aligned}|f_n^{(m)}(y)|/|f_n^{(m)}(x)|&\leqq(e^{-mK_1\rho(y)}+Cm^{n+2}A^m)/(e^{-mK_2\rho(x)}-Cm^{n+2}A^m)\\&\leqq(e^{-mK_1b}+Cm^{n+2}A^m)/(e^{-mK_2\delta}-Cm^{n+2}A^m)\\&=(B_2^m+Cm^{n+2}B_1^m)/(1-Cm^{n+2}B_1^m)\,,\end{aligned}$$

(provided $Cm^{n+2}B_1^m<1$), where $B_1=Ae^{K_2\delta}=e^{-(K_1a-K_2\delta)}$ and $B_2=e^{-(K_1b-K_2\delta)}$. Therefore if we again choose a sufficiently large μ, we know that $|f_n^{(m)}(y)|/|f_n^{(m)}(x)|<1/2$ for all $x\in\pi^{-1}(O_2)$, $y\in\pi^{-1}(\tilde{M}-O_1)$ and $m\geqq\mu$. We have thus proved Lemma 5.

7. The functions $f_j^{(m)}$, the neighborhoods O_1, O_2, and the integer μ in Lemma 5 are all dependent on the arbitrarily given point $p=p_0$. Thus we write these things respectively as follows: $f_{j,p}^{(m)}$, $O_1(p)$, $O_2(p)$, and $\mu(p)$. Since $\tilde{M}$ is compact, we can find a finite number of points $p_1,\cdots,p_k$ of $\tilde{M}$ such that $\tilde{M}=\bigcup_\lambda O_2(p_\lambda)$. Let $\mu_0=\operatorname*{Max}_\lambda\mu(p_\lambda)$. Then for every $m\geqq\mu_0$ we define a map $\boldsymbol{f}$: $M\to C^{nk}$ by

$$\boldsymbol{f}=(f_{1,p_1}^{(m)},\cdots,f_{n,p_1}^{(m)},\cdots,f_{1,p_k}^{(m)},\cdots,f_{n,p_k}^{(m)})\,.$$

We have $R_a^*\boldsymbol{f}=a^m\boldsymbol{f}$, $a\in U(1)$, and by Lemma 5 we have $\boldsymbol{f}(x)\neq0$ for all

$x \in M$. Hence we see that f induces a map $\tilde{f}$ of $\tilde{M}$ into the $(nk-1)$-dimensional complex projective space $P^{nk-1}(C)$. By virtue of Lemma 5 we can easily show that $\tilde{f}$ is a holomorphic embedding.

As is well known, a compact complex manifold is a Hodge manifold if and only if it admits a negative line bundle (cf. [5]). Therefore we have shown that every Hodge manifold $\tilde{M}$ of dimension $\geqq 2$ can be holomorphically embedded in a complex projective space. Finally we note that a compact Riemann surface R, being a Hodge manifold, can be holomorphically embedded in a complex projective space, because the product $R \times R$ is a 2-dimensional Hodge manifold.

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