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On the distribution of the poles of the scattering matrix for two strictly convex obstacles

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§ 1. Introduction

Let $\mathscr{O}$ be a bounded open set in $\boldsymbol{R}^3$ with sufficiently smooth boundary Γ. We set $\Omega=\boldsymbol{R}^3-\bar{\mathscr{O}}$. Suppose that Ω is connected. Denote by $\mathscr{S}(z)$ the scattering matrix for an acoustic problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\Delta u=0 & \text{in } \Omega\times(-\infty, \infty) \\ u(x, t)=0 & \text{on } \Gamma\times(-\infty, \infty) \end{cases}$$

where $\Delta=\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j{}^2}$. Concerning the definition and the fundamental properties of $\mathscr{S}(z)$, see Lax and Phillips [6, Chapter V]. It is well known that it is holomorphic in $\{z; \operatorname{Im} z\leqslant 0\}$ and meromorphic in the whole complex plane $\boldsymbol{C}$ as $\mathscr{L}(L^2(S^2), L^2(S^2))$[1] valued function (Theorem 5.1 of Chapter V, [6]), and the problem to clarify the relationship between the geometric propoerties of the obstacles and the location of the poles of the scattering matrix is important and has been interested, but we know only a few works about the existence of the poles [1, 3, 8].

Assuming that

$$(1.2)\qquad \begin{cases} \mathscr{O}=\mathscr{O}_1\cup\mathscr{O}_2, \quad \bar{\mathscr{O}}_1\cap\bar{\mathscr{O}}_2=\phi, \\ \text{and the Gaussian curvature of } \Gamma_j \text{ the boundary of } \mathscr{O}_j, \\ j=1, 2 \text{ vanishes nowhere.} \end{cases}$$

Bardos, Guillot and Ralston [1] proved the existence of infinitely many poles in a region $\{z; \operatorname{Im} z\leqslant\varepsilon\log(|z|+1)\}$ for any $\varepsilon>0$.[2] They introduced the notion of pseudo-poles $\alpha_{m,\vec{m}}$ and used it to prove the above result, but it was not considered the problem that the pseudo-poles do approximate the actual poles of $\mathscr{S}(z)$. In [3, 4] we gave a precise information on the location

1) $\mathscr{L}(E, F)$ denotes the set of all linear bounded operator from E to F.

2) Though the condition (1.2) is assumed in [1], they used only a certain condition on the Poincaré mapping at the periodic broken ray. Petkov [8] gives a genelaization of [1].

of the poles in $\operatorname{Im} z \leqslant c_0+c_1$ where c_0 and c_1 are constants determined by $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $0<c_1<c_0/2$, and this result shows that the pseudo-poles $\alpha_{m,\vec{m}}$ for $\vec{m}=0$ approximate the actual poles.

About a question how is the distribution of the poles in the outside of $\{z;\ \operatorname{Im} z \leqslant c_0+c_1\}$ we shall consider the behavior of $\mathcal{S}(z)$ in $\left\{z;\ \operatorname{Im} z \leqslant \frac{3}{2}c_0+c_1\right\}$ posing an additional assumption on $\mathcal{O}$. Namely we like to show the following

THEOREM 1. *Suppose that* (1.2) *holds. Let* $a_j \in \Gamma_j$, $j=1, 2$ *be the points such that*

$$\operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2) = |a_1 - a_2| .$$

If

(1.3) Γ_1 *and* Γ_2 *are umbilical at* a_1 *and* a_2 *respectively,*

the scattering matrix $\mathcal{S}(z)$ *is holomorphic in*

$$\left\{z;\ \operatorname{Im} z \leqslant \frac{3}{2}c_0+c_1\right\} - \bigcup_{j=-\infty}^{\infty}\left\{z;\ |z-z_j| \leqslant C(1+|j|)^{-1/2}\right\}$$
$$- \bigcup_{j=-\infty}^{\infty}\left\{z;\ |z-\tilde{z}_j| \leqslant C(1+|j|)^{-1/2}\right\}$$

where

$$z_j = ic_0 + \frac{\pi}{d}j, \qquad \tilde{z}_j = i\frac{3}{2}c_0 + \frac{\pi}{d}j, \qquad d = \operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2) .$$

Note that $\tilde{z}_j$ are nothing but $\alpha_{j,1}$. Then Theorem 1 means that for $\mathcal{O}$ verifying (1.2) and (1.3) $\mathcal{S}(z)$ is holomorphic in $\left\{z;\ \operatorname{Im} z \leqslant \frac{3}{2}c_0+c_1\right\}$ except small neighborhoods of pseudo-poles. As a matter of course we have to answer a question that $\tilde{z}_j$ do approximate the actual poles. When $\mathcal{O}_j$, $j=1, 2$ are balls, or $\mathcal{O}_j$ are tangent to balls at a_j of a high order, there are no poles near $\tilde{z}_j$. This fact can be verified by checking the orders of the convergences of asymptotic solutions constaucted in this paper. In other cases, though it is not yet proved precisely, according to the form of asymptotic solutions of (1.1) it seems very sure to us that the poles of $\mathcal{S}(z)$ exist near $\tilde{z}_j$.

Let $\operatorname{Re} \mu > 0$ and $g \in C^\infty(\Gamma)$. Denote by $U(\mu)\,g$ the solution of a boundary value problem

$$\begin{cases} (\mu^2-\Delta)\,u=0 & \text{in } \Omega \\ u=g & \text{on } \Gamma \\ u\in \bigcap_{m>0} H^m(\Omega)\,. \end{cases}$$

Then $U(\mu)$ is $\mathscr{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$-valued holomorphic function in $\operatorname{Re}\mu>0$. Mizohata [7] proved that $U(\mu)$ can be prolonged into $\{\mu\,;\ \operatorname{Re}\mu\leqslant 0\}$ as a meromorphic function in the whole complex plane. Theorem 1 follows from the following theorem via Theorem 5. 1 of Chapter V of [6].

THEOREM 2. *Suppose that* (1. 2) *and* (1. 3) *are verified. Then* $U(\mu)$ *is holomorphic in a region*

$$\mathscr{D}_\varepsilon=\left\{\mu\,;\ \operatorname{Re}\mu\geqslant -\frac{3}{2}c_0-c_1+\varepsilon\right\}-\bigcup_{j=-\infty}^{\infty}\left\{\mu\,;\ |\mu-\mu_j|\leqslant C(1+|j|)^{-1/2}\right\}$$
$$-\bigcup_{j=-\infty}^{\infty}\left\{\mu\,;\ |\mu-\tilde{\mu}_j|\leqslant C(1+|j|)^{-1/2}\right\}-\{\mu\,;\ |\mu|\leqslant C_\varepsilon\}$$

where $\mu_j=iz_j$, $\tilde{\mu}_j=i\tilde{z}_j$, ε *is an arbitrary positive constant and* C_ε *is a constant depending on* ε. *Moreover an estimate*

$$\sum_{|\beta|\leq m}\sup_{x\in\Omega_R}\left|D_x^\beta\big(U(\mu)\,g\big)(x)\right|\leqslant C_{R,m,\varepsilon}\sum_{j=0}^{m+7}|\mu|^j\,\|g\|_{H^{m+7-j}(\Gamma)}$$

holds for all $\mu\in\mathscr{D}_\varepsilon$, *where* $\Omega_R=\Omega\cap\{x\,;\ |x|<R\}$.

The plan of the proof of Theorem 2 is essentially same as that of [3]. Namely, we approximate the fundamental solution of (1. 1) by a superposition of asymptotic solutions. Since $U(\mu)$ is nothing but the Laplace transform in t of the fundamental solution, $U(\mu)$ is approximated by a superposition of the Laplace transform of asymptotic solutions. Therefore properties of the Laplace transform of asymptotic solutions play an essential role to consider $U(\mu)$. Roughly speaking the asymptotically periodic property in t of functions produces the poles of their Laplace transform.

In this paper using the assumption (1. 3) we shall take out more precisely the periodic properties in t of asymptotic solutions than in [3]. This properties are stated in Propositions 2. 1 and 2. 5. Seeing that Theorem 2 can be derived from these propositions by the procedure used in [3] we shall prove only these propositions.

§ 2. Statement of the periodic properties in *t* of asymptotic solutions

We denote by $S_j(\delta)$ the connected component containing a_j of $\{x\,;\ x\in\Gamma_j,\ \operatorname{dis}(x, L)<\delta\}$ where L is a line passing a_1 and a_2. Let $m(x,t\,;\ k)$ be an oscillatory boundary data on $\Gamma_1\times\boldsymbol{R}$ of the form

(2. 1) $$m(x,t;\ k)=e^{ik(\psi(x)-t)}f(x,t)$$

where $f(x,t)\in C_0^\infty(S_1(\delta_0)\times(0,d/2))$ for a sufficiently small $\delta_0>0$ and ψ satisfies the following conditions:

(2. 2) There exists a real valued C^∞ function $\varphi(x)$ defined in a neighborhood $\mathscr{U}$ of $S_1(\delta_0)$ in $\boldsymbol{R}^3$ which satisfies $|\nabla\varphi|=1$ in $\mathscr{U}$ and

$$\varphi(x)=\psi(x),\qquad \frac{\partial\varphi}{\partial n}(x)>0\qquad \text{on}\quad S_1(\delta_0).$$

(2. 3) The principal curvatures at x of $\mathscr{C}_\varphi(x)=\{y;\ \varphi(y)=\varphi(x)\}$ with respect to $-\nabla\varphi(x)$ are positive for all $x\in S_1(\delta_0)$.

We shall construct an asymptotic solutions of the problem

(2. 4) $$\begin{cases}\square u=0 & \text{in}\quad \Omega\times(-\infty,\infty)\\ u=m(x,t;\ k) & \text{on}\quad \Gamma\times(-\infty,\infty)\\ \operatorname{supp} u\subset\Omega\times(0,\infty)\end{cases}$$

following the procedure used in [2] and [3].

As remarked in the introduction we have to take out an asymptotically periodic property in t of them. In order to state this property we introduce same spaces of sequences of functions in addition to spaces $F(p)$, $\mathring{F}(p)$, $K(p)$, $M_r(p)$, $(CH)_s\mathscr{A}_s(p)$ defined in § 6 of [3]. First we consider sequences of functions defined in $\omega\times\boldsymbol{R}$ where ω is a domain surrounded by $S_j(\delta_0)$, $j=1,2$ and $\{x;\ \operatorname{dis}(x,L)=\delta_0\}$. We set

$$\bar{F}(p)=\Big\{\{v_q,\tilde{v}_q\}_{q=0}^\infty\in F(p);\ \sup_{q\geqslant0}\ \sup_{(x,t)\in\omega\times\boldsymbol{R}} e^{3c_0t/2}\Big(\big|D_{x,t}^{r}v_q(x,t)\big|+\big|D_{x,t}^{\beta}\tilde{v}_q(x,t)\big|\Big)<\infty \text{ for all } \beta\Big\}$$

$$\begin{aligned}\bar{K}(p)=\Big\{\{v_q,\tilde{v}_q\}_{q=0}^\infty;\ & v_q(x,t)=(\lambda\tilde{\lambda})^{3q/2}f(x,t-2dq),\\ & \tilde{v}_q(x,t)=(\lambda\tilde{\lambda})^{3q/2}\tilde{f}(x,t-2dq),\ f,\tilde{f}\in C_0^\infty\big(\bar{\omega}\times[0,\infty)\big)\\ & \text{for } q\geqslant p \text{ and } v_q=\tilde{v}_q=0 \text{ fir } q<p\Big\}.\end{aligned}$$

$$\bar{M}_r(p)=\Big\{\{v_q,\tilde{v}_q\}_{q=0}^\infty\in F(p);\ \sup_q\sup_{(x,t)}(1+t)^{-r}e^{(3c_0/2+c_1)t}\Big(\big|D_{x,t}^{\beta}v_q(x,t)\big|+\big|D_{x,t}^{\beta}\tilde{v}_q(x,t)\big|\Big)<\infty \text{ for all } \beta\Big\}$$

and we define for $\boldsymbol{v}=\{v_q,\tilde{v}_q\}_{q=0}^\infty$

$$\|\boldsymbol{v}\|_{\bar{F},m}=\sup_q\sup_{(x,t)}\sum_{|\beta|\leq m}e^{3c_0t/2}\Big(\big|D_{x,t}^{\beta}v_q(x,t)\big|+\big|D_{x,t}^{\beta}\tilde{v}_q(x,t)\big|\Big).$$

$$||\boldsymbol{v}||_{\bar{M}_r,m}=\sup_q \sup_{(x,t)} \sum_{|\beta|\le m} (1+t)^{-r} e^{(3c_0/2+c_1)t} \Big(\Big|D^\beta_{x,t} v_q(x,t)\Big|+\Big|D^\beta_{x,t}\tilde{v}_q(x,t)\Big|\Big).$$

REMARK 1. By the assumption on supports of v_q and $\tilde{v}_q$ $\boldsymbol{v}=\{v_q, \tilde{v}_q\}_{q=0}^{\infty}\in \bar{F}$ is equivalent to

$$\sup_q \sup_{(x,t)} (\lambda\tilde{\lambda})^{-3q/2}\Big(\Big|D^\beta_{x,t} v_q(x,t)\Big|+\Big|D^\beta_{x,t}\tilde{v}_q(x,t)\Big|\Big)<\infty$$

and $\boldsymbol{v}\in\bar{\bar{M}}_r$ is equivalent to

$$\sup_q \sup_{(x,t)} q^{-r}(\lambda\tilde{\lambda})^{-3q/2}\alpha^{-q}\Big(\Big|D^\beta_{x,t} v_q(x,t)\Big|+\Big|D^\beta_{x,t}\tilde{v}_q(x,t)\Big|\Big)<\infty .$$

We have for $\boldsymbol{v}=\{(\lambda\tilde{\lambda})^{3q/2}g(x,t-2dq),\ (\lambda\tilde{\lambda})^{3q/2}\tilde{g}(x,t-2dq)\}_{q\geqslant p}$

$$C^{-1}\Big(|g|_m(\omega\times \boldsymbol{R})+|\tilde{g}|_m(\omega\times \boldsymbol{R})\Big)\leqslant ||\boldsymbol{v}||_{\bar{F},m} \leqslant C\Big(|g|_m(\omega\times \boldsymbol{R})+|\tilde{g}|_m(\omega\times \boldsymbol{R})\Big).$$

Let $(CH)^0_1$ be a subset of $(CH)_1$ of all the elements $\{f^{(j)}\}_{j\in N_+}$ satisfying

$$\sup_{j\in N_+} j^{-1}(\lambda\tilde{\lambda})^{-j/2}\alpha^{-j}\Big|f^{(j)}-\Big(f^{(\infty)}+(\lambda\tilde{\lambda})^{j/2}\bar{f}^{(\infty)}\Big)\Big|_m(\omega\times \boldsymbol{R})<\infty$$

for all m. We set

$$\Big|\{f^{(j)}\}_{j\in N_+}\Big|_{(CH)^1_0,m}=|f^{(\infty)}|_m+|\bar{f}^{(\infty)}|_m + \sup_{0\leqslant j<\infty} j^{-1}(\lambda\tilde{\lambda})^{-j/2}\alpha^{-j}\Big|f^{(j)}-\Big(f^{(\infty)}+(\lambda\tilde{\lambda})^{j/2}\bar{f}^{(\infty)}\Big)\Big|_m(\omega\times \boldsymbol{R}).$$

We define $(CH)^0_s$ for $s\geqslant 2$ inductively by the following way: Suppose that $(CH)^0_{s-1}$ is defined. We say that $\{f^{(J_s)}\}_{J_s\in N_+{}^s}\in(CH)_s$ belongs to $(CH)^0_s$ when there exists $\{g^{(J_{s-1})}\}_{J_{s-1}\in N_+{}^{s-1}}\in(CH)^0_{s-1}$ and a linear continuous operator $\boldsymbol{B}^0_s$ from $C^\infty_0(\omega\times \boldsymbol{R})$ into $(CH)^0_1$ such that

$$\{f^{(J_{s-1},j)}\}_{j\in N_+}=\boldsymbol{B}^0_s g^{(J_{s-1})} \qquad \text{for all} \quad J_{s-1}\in N_+{}^{s-1},$$

and we set

$$\Big|\{f^{(J_s)}\}\Big|_{(CH)^0_s,m}=\Big|\{f^{(J_{s-1},\infty)}\}_{J_{s-1}\in N_+{}^{s-1}}\Big|_{(CH)^0_{s-1},m} +\Big|\{\bar{f}^{(J_{s-1},\infty)}\}_{J_{s-1}\in N_+{}^{s-1}}\Big|_{(CH)^0_{s-1},m} +\sup_j j^{-1}(\lambda\tilde{\lambda})^{-j/2}\alpha^{-j}\Big|\Big\{f^{(J_{s-1},j)}-\Big(f^{(J_{s-1},\infty)} +(\lambda\tilde{\lambda})^{j/2}\bar{f}^{(J_{s-1},\infty)}\Big)\Big\}_{J_{s-1}\in N_+{}^{s-1}}\Big|_{(CH)^0_{s-1},m}.$$

We say $W=\{w^{(J_s)}\}_{J_s\in N_+{}^s}\in\mathscr{A}_s(p)$ belongs to $\mathscr{A}_s^0(p)$ when if we set

$$w^{(J_s)}=\Big\{(\lambda\tilde{\lambda})^q f^{(J_s)}(x,t-2dq),\ (\lambda\tilde{\lambda})^q \tilde{f}^{(J_s)}(x,t-2dq)\Big\}_{q\geq|J_s|+p}$$

we have

$$\{f^{(J_s)}\}_{J_s\in N_+{}^s},\ \{\tilde{f}^{(J_s)}\}_{J_s\in N_+{}^s}\in(CH)_s^0,$$

and we set

$$\|W\|_{\mathscr{A}_s^0,m}=\Big|\{f^{(J_s)}\}\Big|_{(CH)_s^0,m}+\Big|\{\tilde{f}^{(J_s)}\}\Big|_{(CH)_s^0,m}.$$

We say $\overline{W}=\{\overline{w}^{(J_s)}\}_{J_s\in N_+{}^s}\in\mathscr{G}_s(p)$ when for each J_s

$$\overline{w}^{(J_s)}=\Big\{(\lambda\tilde{\lambda})^{3q/2} f^{(J_s)}(x,t-2dq),\ (\lambda\tilde{\lambda})^{3q/2} \tilde{f}^{(J_s)}(x,t-2dq)\Big\}_{q\geq|J_s|+p}$$

and we have $\{f^{(J_s)}\}$, $\{\tilde{f}^{(J_s)}\}\in(CH)_s$, and we set

$$\|\overline{W}\|_{\mathscr{G}_s,m}=\Big|\{f^{(J_s)}\}\Big|_{(CH)_s,m}+\Big|\{\tilde{f}^{(J_s)}\}\Big|_{(CH)_{s'},m}.$$

We can define $\mathscr{A}_s^0(p\,;\,\Omega_R)$, $\mathscr{A}_s^0(p\,;\,\Gamma)$, $\mathscr{G}_s(p\,;\,\Omega_R)$ and $\mathscr{G}_s(p\,;\,\Gamma)$ by replacing ω by Ω_R or Γ, and we denote for $W\in\mathscr{A}_s^0(p;\,\Omega_R)$ the seminorm as $\|W\|_{\mathscr{A}_s^0(\Omega_R),m}$.

The result we like to show concerning the asymptotic solution of (2. 4) is the following

Proposition 2. 1. *For an oscillatory boundary data* $m(x,t\,;k)$ *of the form* (2. 1) *verifying* (2. 2) *and* (2. 3) *there exists* $u(x,t\,;k)$ *with the following properties*:

(i) $$u(x,t\,;k)=\sum_{q=0}^{\infty}\Big(u_q(x,t\,;k)-\tilde{u}_q(x,t\,;k)\Big),$$

and $u=\{u_q,\tilde{u}_q\}_{q=0}$ *is decomposed as*

$$u=w_0+\overline{w}_0+z_0+\sum_{r=1}^{N}k^{-r}\Big\{\sum_{J_r\in N_+{}^r}(w^{(J_r)}+\overline{w}^{(J_r)})$$
$$+\sum_{h=0}^{p}\sum_{l=0}^{\infty}\sum_{J_{r-h}\in N_+{}^{r-h}}(w_{r,h,l}^{(J_{r-h})}+\overline{w}_{r,h,l}^{(J_{r-h})})+z_r\Big\}$$

where

$$w_0\in K(0),\ \overline{w}_0\in\bar{K}(0),\ z_0\in\overline{M}(0),$$
$$W_r=\{w^{(J_r)}\}_{J_r\in N_+{}^r}\in\mathscr{A}_r^0(0;\,\Omega_R),\ \overline{W}_r=\{\overline{w}^{(J_r)}\}_{J_r\in N_+{}^r}\in\mathscr{G}_r(0;\,\Omega_R)$$
$$W_{r,h,l}=\{w_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+{}^{r-h}}\in\mathscr{A}_{r-h}^0(l\,;\,\Omega_R)$$
$$\overline{W}_{r,h,l}=\{\overline{w}_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+{}^{r-h}}\in\mathscr{G}_{r-h}(l\,;\,\Omega_R)$$
$$z\in\overline{M}_{2r}(0).$$

They have estimates

$$\|W_r\|_{\mathscr{X}_r^0(\Omega_R),m}+\|\overline{W}_r\|_{\mathscr{G}_r(\Omega_R),m}+\|z_r\|_{\overline{M}_{2r}(\Omega_R),m}\leqslant C_{R,r,m}k^m B_{m+2r}$$

$$\|W_{r,h,l}\|_{\mathscr{X}_{r-h}^0(\Omega_R),m}+\|\overline{W}_{r,h,l}\|_{\mathscr{G}_{r-h}(\Omega_R),m}\leqslant C_{R,r,m}\alpha^l l^{r-h} k^m B_{m+2r}$$

where

$$B_m=|\psi|_m\Big(S_1(\delta_0)\Big)+|f|_m\Big(S_1(\delta_0)\times \boldsymbol{R}\Big)\,.$$

(ii) $$\left(\frac{\partial^2}{\partial t^2}-\Delta\right)u(x,t;k)=0 \qquad \textit{in} \quad \Omega\times(-\infty,\infty)\,,$$

(iii) $$u(x,t;\ k)-m(x,t;k)$$
$$=\begin{cases} k^{-N}\sum\limits_{q=0}^{\infty}f_q(x,t;k) & \textit{on} \quad \Gamma_1\times\boldsymbol{R}\\ k^{-N}\sum\limits_{q=0}^{\infty}\tilde{f}_q(x,t;k) & \textit{on} \quad \Gamma_2\times\boldsymbol{R}\end{cases}$$

and $\boldsymbol{f}=\{f_q,\tilde{f}_q\}_{q=0}^{\infty}$ *is decomposed as*

$$\boldsymbol{f}=\sum_{J_N\in N_+{}^N}(\boldsymbol{f}^{(J_N)}+\bar{\boldsymbol{f}}^{(J_N)})+\sum_{h=1}^{N}\sum_{l=0}^{\infty}\sum_{J_{N-h}}(\boldsymbol{f}_{N,h,l}^{(J_{N-h})}+\bar{\boldsymbol{f}}_{N,h,l}^{(J_{N-h})})+\tilde{\boldsymbol{f}}_N\,,$$

where

$$\boldsymbol{F}_N=\{\boldsymbol{f}_N^{(J_N)}\}_{J_N\in N_+{}^N}\in\mathscr{X}_N^0(0;\ \Gamma),\ \overline{\boldsymbol{F}}_N=\{\bar{\boldsymbol{f}}_N^{(J_N)}\}_{J_N\in N_+{}^N}\in\mathscr{G}_N(0;\ \Gamma)\,,$$
$$\boldsymbol{F}_{N,h,l}=\{\boldsymbol{f}_{N,h,l}^{(J_{N-h})}\}_{J_{N-h}\in N_+{}^{N-h}}\in\mathscr{X}_{N-h}^0(l;\ \Gamma)$$
$$\overline{\boldsymbol{F}}_{N,h,l}=\{\bar{\boldsymbol{f}}_{N,h,l}^{(J_{N-h})}\}_{J_{N-h}\in N_+{}^{N-h}}\in\mathscr{G}_{N-h}(l;\ \Gamma)\,,$$
$$\tilde{\boldsymbol{f}}_N\in\overline{M}_{2N}(0;\ \Gamma)$$

and

$$\|\boldsymbol{F}_N\|_{\mathscr{X}_N^0(\Gamma),m}+\|_N\|\overline{\boldsymbol{F}}_{\mathscr{G}_N(\Gamma),m}+\|\tilde{\boldsymbol{f}}_N\|_{\overline{M}_{2N}(\Gamma),m}\leqslant C_{N,m}k^{m+1}B_{m+2(N+N')}$$

$$\|\boldsymbol{F}_{N,h,l}\|_{\mathscr{X}_{N-h}^0(\Gamma),m}+\|\overline{\boldsymbol{F}}_{N,h,l}\|_{\mathscr{G}_{N-h}(\Gamma),m}\leqslant C_{N,m}k^{m+1}\alpha^l l^{N-h}B_{m+2(N+N')}\,.$$

Next we consider the Laplace transform of $u(x,t;k)$ in the above proposition. Let S be an operator from F into $C^\infty(\omega\times\boldsymbol{R})$ defined by

$$(S\boldsymbol{f})(x,t)=\sum_{q=0}^{\infty}\Big(f_q(x,t)-\tilde{f}_q(x,t)\Big)$$

for $\boldsymbol{f}=\{f_q,\tilde{f}_q\}_{q=0}^{\infty}$. We denote the Laplace transform of $w(x,t)$ by $\hat{w}(x,\mu)$, i.e.,

$$\hat{w}(x,\mu)=\int_{-\infty}^{\infty}e^{-\mu t}w(x,t)\,dt\,.$$

We prepare two lemmas whose proof can be achieved by the almost same procedure as in § 8 of [3].

LEMMA 2.2. *Let* $f \in \overline{M}_s(0)$ *and set* $w(x, t) = Sf(x, t)$. *Then* $\hat{w}(x, \mu)$ *is holomorphic in* $\mathscr{D} = \{\mu ; \operatorname{Re} \mu > -3c_0/2 - c_1\}$ *and for any* $\varepsilon > 0$

$$\sup_{\mu \in \mathscr{D}_\varepsilon^\sim} \left| \hat{w}(\cdot, \mu) \right|_m (\omega) \leqq C_{s,m,\varepsilon} \|f\|_{\overline{M}_s, m}$$

where $\mathscr{D}_\varepsilon^\sim = \{\mu ; \operatorname{Re} \mu \geq -3c_0/2 - c_1 + \varepsilon\}$.

LEMMA 2.3. *Let* $W = \{w^{(J_s)}\}_{J_s \in N_+{}^s} \in \mathscr{K}_s^0(p)$ *and set*

$$w(x, t) = \sum_{J_s \in N_+{}^s} SW^{(J_s)} .$$

Then

$$\hat{w}(x, \mu) = (\lambda \tilde{\lambda} e^{-2\mu d})^p \sum_{l+j \leqq s} A(\mu)^{l+1} B(\mu)^j G_{j,l}(x, \mu)$$

where

$$A(\mu) = (1 - \lambda \tilde{\lambda} e^{-2\mu d})^{-1}, \quad B(\mu) = \left(1 - (\lambda \tilde{\lambda})^{3/2} e^{-2\mu d}\right)^{-1}$$

and $G_{j,l}(x, \mu)$, $j + l \leqq s$, *are* $C^\infty(\bar{\omega})$*-valued holomorphic functions in* $\mathscr{D}$ *and have estimates*

$$\sup_{\mu \in \mathscr{D}_\varepsilon^\sim} \left| G_{j,l}(\cdot, \mu) \right|_m (\omega) \leqq C_{s,m,\varepsilon} \|W\|_{\mathscr{K}_s^0, m} \qquad \text{for all } j, l .$$

LEMMA 2.4. *Let* $\overline{W} = \{\overline{w}^{(J_s)}\}_{J_s \in N_+{}^s} \in \mathscr{G}_s(p)$ *and set*

$$\overline{w}(x, t) = \sum_{J_s \in N_+{}^s} S\overline{w}^{(J_s)} .$$

Then

$$\widehat{\overline{w}}(x, \mu) = \left((\lambda \tilde{\lambda})^{3/2} e^{-2\mu d}\right)^p \sum_{j+l \leq s} A(\mu)^j B(\mu)^{l+1} \bar{G}_{j,l}(x, \mu)$$

where $\bar{G}_{j,l}$ *is* $C^\infty(\bar{\omega})$*-valued holomorphic function in* $\mathscr{D}$ *and has estimates*

$$\sup_{\mu \in \mathscr{D}_\varepsilon^\sim} \left| \bar{G}_{j,l}(\cdot, \mu) \right|_m (\omega) \leqq C_{s,m,\varepsilon} \|\overline{W}\|_{\mathscr{G}_s, m} .$$

With the aid of the above lemmas we have from Proposition 2.1 the following

PROPOSITION 2.5. *For* $f(x, t) \in C_0^\infty(S_1(\delta_0) \times (0, d/2))$ *and* $\phi(x)$ *satisfying* (2.2) *and* (2.3), *there exists* $\hat{u}(x, \mu ; k)$ *of the form*

$$\hat{u}(x, \mu ; k) = \sum_{j=0}^{N} k^{-j} \sum_{l+s \leq j+1} A(\mu)^l B(\mu)^s \hat{u}_{j,l,s}(x, \mu ; k)$$

verifying for all $\mu \in \mathscr{D} - \bigcup_{j=-\infty}^{\infty} (\{\mu_j\} \cup \{\tilde{\mu}_j\})$ *an equation*

$$(\mu^2 - \nabla)\,\hat{u}(x, \mu\,;\, k) = 0 \qquad in \quad \Omega\,,$$

$$\hat{u}(x, \mu\,;\, k) - e^{ik\varphi(x)}\hat{f}(x, \mu - ik)$$
$$= k^{-N} \sum_{l+s \leq N+1} A(\mu)^l B(\mu)^s \hat{u}_{N,l,s}(x, \mu\,;\, k) \qquad on \quad \Gamma$$

where $\hat{u}_{j,l,s}(x, \mu\,;\, k)$ *are* $C^\infty(\bar{\Omega})$*-valued holomorphic functions defined in* $\mathscr{D}$ *with estimates*

$$\sup_{\mu \in \tilde{\mathscr{D}}_\varepsilon} \left| \hat{u}_{j,l,s}(\cdot, \mu\,;\, k) \right|_m (\Omega_R) \leqslant C_{R,m,\varepsilon} B_{m+2(N+N')}\,.$$

§ 3. Outline of the proof of Proposition 2.1.

As it is well known the propagation of solutions of high frequencies of the wave equation $\Box u = 0$ is approximated by the geometric optics. This suggests us that the periodic properties of solutions of (2. 4) are generated by the existemce of a periodic broken ray of the geometric optics in Ω. Under the assumption (1. 2) the periodic broken ray in Ω is unique, namely the broken ray shuttling between a_1 and a_2. Therefore it is essential to construct asymptotic solutions of (2. 4) near the segment $a_1 a_2$ and to examine their behavior. Following § 3 of [2] we choose $0 < \delta_2 < \delta_3 < \delta_0$ so that Lemmas 3. 3 and its corollary of [3] hold. We look for a function $u(x, t\,;\, k)$ satisfying $\Box u = 0$ asymptotically in $\omega \times \boldsymbol{R}$ and $u = m$ on $\Gamma \times \boldsymbol{R}$ in the following form

$$u(x, t\,;\; k) = \sum_{q=0}^{\infty} \Big(u_q(x, t\,;\; k) - \tilde{u}_q(x, t\,;\; k) \Big)\,, \tag{3. 1}$$
$$u_q(x, t\,;\; k) = e^{ik(\varphi_{2q}(x) - t)} \sum_{r=0}^{N} (ik)^{-r} v_{r,q}(x, t)$$
$$\tilde{u}_q(x, t\,;\; k) = e^{ik(\varphi_{2q+1}(x) - t)} \sum_{r=0}^{N} (ik)^{-r} \tilde{v}_{r,q}(x, t)\,.$$

Note that

$$-e^{ik(\varphi_{2q} - t)} \Box u_q$$
$$= \sum_{r=0}^{N} (ik)^{-r+2} \Big\{ (|\nabla \varphi_{2q}|^2 - 1)\, v_{r,q} + T_q v_{r-1,q} - \Box v_{r-2,q} \Big\}$$

holds where we set $v_{-1,q}(x, t) = v_{-2,q}(x, t) = 0$ and

$$T_q = 2\frac{\partial}{\partial t} + 2\nabla \varphi_{2q}(x) \cdot \nabla + \Delta \varphi_q\,.$$

Then, if

$$\begin{cases} |\nabla \varphi_{2q}| = 1 & \text{in} \quad \omega \\ T_q v_{r,q} = \Box v_{r-1,q} & \text{in} \quad \omega \times \boldsymbol{R},\; r = 0, 1, \cdots, N \end{cases} \tag{3. 2}$$

are verified we have

$$\Box u_q = e^{ik(\varphi_{2q}-t)}(ik)^{-N}\Box v_{N,q}\,.$$

Then we require (3.2) so that u_q may satisfy $\Box u=0$ asymptotically when $k\to\infty$. Similarly we require for $\tilde{u}_q$

$$(3.3)\qquad \begin{cases} |\nabla\varphi_{2q+1}|=1 & \text{in } \omega \\ \tilde{T}_q\tilde{v}_{r,q}=\Box\tilde{v}_{r-1,q} & \text{in } \omega\times\boldsymbol{R},\ r=0,1,\cdots,N \end{cases}$$

where $\tilde{T}_q=2\dfrac{\partial}{\partial t}+2\nabla\varphi_{2q+1}\cdot\nabla+\Delta\varphi_{2q+1}$.

In order to hold $u_0(x,t;k)=m(x,t;k)$ on $S_1(\delta_0)\times\boldsymbol{R}$ it suffices to be satisfied

$$(3.4)\qquad \begin{cases} \varphi_0(x)=\psi(x) & \text{on } S_1(\delta_0) \\ v_{0,0}(x,t)=f(x,t),\ v_{r,0}(x,t)=0 & \text{for } r\geqslant 1, \text{ on } S_1(\delta_0)\times\boldsymbol{R}. \end{cases}$$

For $\tilde{u}_q(x,t;k)=u_q(x,t;k)$ on $S_2(\delta_2)\times\boldsymbol{R}$ it suffices to hold

$$(3.5)\qquad \begin{cases} \varphi_{2q+1}(x)=\varphi_{2q}(x) & \text{on } S_2(\delta_2) \\ \tilde{v}_{r,q}(x,t)=v_{r,q}(x,t) & \text{on } S_2(\delta_2)\times\boldsymbol{R} \text{ for } r\geqslant 0. \end{cases}$$

For $u_{q+1}(x,t;k)=\tilde{u}_q(x,t;k)$ on $S_1(\delta_2)\times\boldsymbol{R}$ it suffices to hold

$$(3.6)\qquad \begin{cases} \varphi_{2q+2}(x)=\varphi_{2q+1}(x) & \text{on } S_1(\delta_2) \\ v_{r,q+1}(x,t)=\tilde{v}_{r,q}(x,t) & \text{on } S_1(\delta_2)\times\boldsymbol{R} \text{ for } r\geqslant 0. \end{cases}$$

Thus if we can construct φ_q, $v_{r,q}$, $\tilde{v}_{r,q}$ as (3.2)~(3.5) are satisfied $u(x,t;k)$ defined by (3.1) verifies

$$\begin{cases} \Box u=0(k^{-N}) & \text{in } \omega\times\boldsymbol{R} \\ u=m(x,t;k) & \text{on } S(\delta_2)\times\boldsymbol{R}. \end{cases}$$

Now let us construct φ_q, $q=0,1,2,\cdots$ successively as follows:

$$\begin{cases} |\nabla\varphi_0(x)|=1 & \text{in } \omega \\ \varphi_0(x)=\psi(x),\ \dfrac{\partial\varphi_0}{\partial n}(x)>0 & \text{on } S_1(\delta_0) \end{cases}$$

and for $q\geqslant 1$

$$\begin{cases} |\nabla\varphi_q(x)|=1 & \text{in } \omega \\ \varphi_q(x)=\varphi_{q-1}(x),\ \dfrac{\partial\varphi_q}{\partial n}(x)=-\dfrac{\partial\varphi_{q-1}}{\partial n}(x) & \text{on } S_{\in(q)}(\delta_0) \end{cases}$$

where n denotes the unit outer normal of Γ and

$$\in(q)=\begin{cases}2 & \text{when } q \text{ is odd}\\ 1 & \text{when } q \text{ is even.}\end{cases}$$

We choose as the amplitude function $v_{r,q}$, $\tilde{v}_{r,q}$ the solution of

$$\begin{cases}\boldsymbol{T}\boldsymbol{v}_0=0 & \text{in } \omega\times\boldsymbol{R}\\ \boldsymbol{v}_0=\boldsymbol{f} & \text{on } \Gamma\times\boldsymbol{R}\end{cases}$$

and for $r\geqslant 1$

$$\begin{cases}\boldsymbol{T}\boldsymbol{v}_r=\Box\boldsymbol{v}_{r-1} & \text{in } \omega\times\boldsymbol{R}\\ \boldsymbol{v}_r=0 & \text{on } \Gamma\times\boldsymbol{R}\end{cases}$$

where we use the notation of Definition 6.2 of [3] and we set

$$\boldsymbol{v}_r=\{v_{r,q}, \tilde{v}_{r,q}\}_{q=0}^{\infty}$$

$$\boldsymbol{f}=\{f_q, 0\}_{q=0}^{\infty}, f_0(x,t)=f(x,t), f_0=0 \text{ for } q\geqslant 1\,.$$

After this we look over the behaviors of sequences $\{\varphi_q\}_{q=0}^{\infty}$, $\{v_{r,q}, \tilde{v}_{r,q}\}_{q=0}^{\infty}$. For the sake of economy of pages we shall use freely the notations and results in [3] without explanations.

3.1. Convergence of φ_q for $q\to\infty$

By checking the considerations in § 3 of [3] we have

LEMMA 3.1. *Suppose that Γ_1 and Γ_2 are umbilical at a_1 and a_2 respectively. Let φ_∞ and $\tilde{\varphi}_\infty$ are the functions defined in § 3 and § 4 of [3]. Then the wave front of φ_∞ passing a_1 and that of $\tilde{\varphi}_\infty$ passing a_2 are umbilical at a_1 and a_2 respectively.*

Applying the implicit function theorem we have the following lemma, which is an improvement of Lemmas 3.1 and 3.2 of [3].

LEMMA 3.2. *Let $i(\sigma)$ and $j(\sigma)$ be S^2-valued C^∞-functions defined on $S_1(\delta_2)$ satisfying Condition C of § 7 of* [2]. *Let σ, $\tilde{\sigma}$ of $S_1(\delta_2)$ be corresponded to $\eta\in S_1(\delta_2)$ by relations*

$$\begin{aligned}z(\eta)&=y(\sigma)+l(\sigma)\,i(\sigma)\\&=y(\tilde{\sigma})+h(\tilde{\sigma})\,j(\tilde{\sigma})\,,\end{aligned}$$

and let $r(\eta)$ and $s(\eta)$ be S^2-valued C^∞-functions defined on $S_2(\delta_2)$ by

$$r(\eta)=i(\sigma)-2\big(i(\sigma)\cdot m(\eta)\big)\,m(\eta)\,,$$

$$s(\eta)=j(\tilde{\sigma})-2\big(j(\tilde{\sigma})\cdot m(\eta)\big)\,m(\eta)\,.$$

Then there exist 2×2 *matrix valued* C^∞ *functions* $E(i;\sigma,\tau)$ *and* $F(i;\sigma,\tau)$ *of* $\sigma\in S_1(\delta_2)$ *and* τ *in a neighborhood of the origin of* $\boldsymbol{R}^2$, *which is determined by* $i(\cdot)$ *only, such that*

(i)
$$\tilde\sigma-\sigma=E\Big(i;\sigma,P\big(j(\tilde\sigma)-i(\tilde\sigma)\big)\Big)P\big(j(\tilde\sigma)-i(\tilde\sigma)\big),$$
$$E(i;\sigma,0)=-l(\sigma)\big[Y(i;\sigma)+l(\sigma)I(\sigma)\big]^{-1},$$

(ii)
$$P\big(r(\eta)-s(\eta)\big)=F\Big(i;\sigma,P\big(j(\tilde\sigma)-i(\tilde\sigma)\big)\Big)P\big(j(\tilde\sigma)-i(\tilde\sigma)\big),$$
$$F(i;0,0)=Y(i;0)\big[Y(i;0)+l(0)I(0)\big]^{-1}.$$

(iii) *There exists an* $\boldsymbol{R}^2$*-valued* C^∞*-function* $p(i;\sigma,\tau)$ *of* $\sigma\in S_1(\delta_2)$ *and* τ *in a neighborhood of the origin of* $\boldsymbol{R}^2$, *which is determined by* $i(\cdot)$ *only, such that*

$$h(\tilde\sigma)=l(\sigma)+p\Big(i;\sigma,P\big(j(\tilde\sigma)-i(\tilde\sigma)\big)\Big)P\big(j(\tilde\sigma)-i(\tilde\sigma)\big)$$
$$p(i;\sigma,0)=\Big(\frac{\partial y}{\partial\sigma_1}(\sigma)\cdot i(\sigma),\ \frac{\partial y}{\partial\sigma_2}(\sigma)\cdot i(\sigma)\Big).$$

COROLLARY. *There exist* $\tilde E(i;\sigma,\tau)$, $\tilde F(i;\sigma,\tau)$ *and* $\tilde p(i;\sigma,\tau)$ *such that*

(i)
$$\tilde\sigma-\sigma=\tilde E\Big(i;\sigma,P\big(j(\sigma)-i(\sigma)\big)\Big)P\big(j(\sigma)-i(\sigma)\big)$$
$$\tilde E(i;\sigma,0)=-l(\sigma)\big[Y(i;\sigma)+l(\sigma)I(\sigma)\big]^{-1},$$

(ii)
$$P\big(r(\eta)-s(\eta)\big)=\tilde F\Big(i;\sigma,P\big(j(\sigma)-i(\sigma)\big)\Big)P\big(j(\sigma)-i(\sigma)\big)$$
$$\tilde F(i;0,0)=Y(i;0)\big[Y(i;0)+l(0)I(0)\big]^{-1},$$

(iii)
$$h(\tilde\sigma)=l(\sigma)+\tilde p\Big(i;\sigma,P\big(j(\sigma)-i(\sigma)\big)\Big)P\big(j(\sigma)-i(\sigma)\big)$$
$$\tilde p(i;\sigma,0)=\Big(\frac{\partial y}{\partial\sigma_1}(\sigma)\cdot i(\sigma),\ \frac{\partial y}{\partial\sigma_2}(\sigma)\cdot i(\sigma)\Big).$$

With the aid of the above lemmas we have

PROPOSITION 3.3. *Let* $\phi(x)$ *be a function satisfying Condition C and let* $\varphi_0, \varphi_1, \cdots, \varphi_q, \cdots$ *be the sequence of phase functions defined for* $\phi(x)$. *Then there exist* $\boldsymbol{R}^3$*-valued* C^∞ *functions* $\xi(x)$ *and* $\tilde\xi(x)$ *defined in* ω *such that*

$$\Big|\nabla\varphi_{2q}-\big(\nabla\varphi_\infty+(\lambda\tilde\lambda)^q\xi\big)\Big|_m(\omega)\leqslant C_m(\lambda\tilde\lambda\alpha)^q,$$

$$\Big|\nabla\varphi_{2q+1}-\big(\nabla\tilde\varphi_\infty+(\lambda\tilde\lambda)^q\tilde\xi\big)\Big|_m(\omega)\leqslant C_m(\lambda\tilde\lambda\alpha)^q$$

hold for all $m=0, 1, 2, \cdots$.

PROOF. Set $\nabla\varphi_{2q}(y(\sigma))=i_q(\sigma)$, $\nabla\varphi_{2q+1}(z(\eta))=r_q(\eta)$, $\nabla\varphi_\infty(y(\sigma))=i_\infty(\sigma)$ and $\nabla\tilde{\varphi}_\infty(z(\eta))=r_\infty(\eta)$. Applying (ii) of Lemma 3.2 and (ii) of its corollary step by step we have

$$\begin{aligned}(i_q-i_\infty)(\sigma) &= F\Big(r_\infty;\ \Psi_{2q,0}(\sigma),\ P(r_q-r_\infty)(\Psi_{2q,0}(\sigma))\Big)\, P(r_q-r_\infty)(\Psi_{2q,0}(\sigma))\\ &= F\Big(r_\infty;\ \Psi_{2q,0}(\sigma),\ P(r_q-r_\infty)(\Psi_{2q,0}(\sigma))\Big)\\ &\qquad \cdot F\Big(i_\infty;\ \Psi_{2q,1}(\sigma),\ P(i_{q-1}-i_\infty)(\Psi_{2q,1}(\sigma))\Big)\, P(i_{q-1}-i_\infty)(\Psi_{2q,1}(\sigma))\\ &= F_0\cdot F_1\cdot\cdots\cdot F_q\cdot \tilde{F}_{q+1}\cdot\cdots\cdot\tilde{F}_{2q}P(i_0-i_\infty)(\Psi_{2q,2q-1}(\sigma))\end{aligned}$$

where

$$F_j=\begin{cases} F\Big(r_\infty;\ \Psi_{2q,j}(\sigma),\ P(r_{q-l}-r_\infty)(\Psi_{2q,j}(\sigma))\Big) & \text{for } j=2l\\ F\Big(i_\infty;\ \Psi_{2q,j}(\sigma),\ P(i_{q-l}-i_\infty)(\Psi_{2q,j}(\sigma))\Big) & \text{for } j=2l+1,\end{cases}$$

and

$$\tilde{F}_{q+j}=\begin{cases} \tilde{F}\Big(i_{q-l};\ \Psi_{2q,q+j}(\sigma),\ P(i_{q-l}-i_\infty)(\Psi_{2q,q+j}(\sigma))\Big) & \text{for } q+j=2l+1\\ \tilde{F}\Big(r_{q-l};\ \Psi_{2q,q+j}(\sigma),\ P(r_{q-l}-r_\infty)(\Psi_{2q,q+j}(\sigma))\Big) & \text{for } q+j=2l.\end{cases}$$

Recall that we have

$$\left|\Psi_{2q,j}(\sigma)-\Psi_{\infty,j}(\sigma)\right|\leqslant C\alpha^q \qquad \text{for } 0\leqslant j\leqslant q$$

and

$$\left|P(i_{q-l}-i_\infty)\right|\leqslant C\alpha^q,\ \left|P(r_{q-l}-r_\infty)\right|\leqslant C\alpha^q \qquad \text{for } 2l\leqslant q.$$

Then it holds that

$$\left|F_j-F\Big(r_\infty;\ \Psi_{\infty,j}(\sigma),0\Big)\right|\leqslant C\alpha^q \qquad \text{for } j \text{ even}$$

$$\left|F_j-F\Big(i_\infty;\ \Psi_{\infty,j}(\sigma),0\Big)\right|\leqslant C\alpha^q \qquad \text{for } j \text{ odd.}$$

Note that

$$\left|F\Big(r_\infty;\ \Psi_{\infty,j}(\sigma),0\Big)-\tilde{\lambda}I\right|\leqslant C\alpha^j$$

$$\left|F\Big(i_\infty;\ \Psi_{\infty,j}(\sigma),0\Big)-\lambda I\right|\leqslant C\alpha^j$$

follow from $F(i_\infty;\ 0,0)=\lambda I$, $F(r_\infty;\ 0,0)=\tilde{\lambda}I$ and $|\Psi_{\infty,j}(\sigma)|\leqslant C\alpha^j$. Therefore a sequence of matrix valued function

$$\xi_j^+(\sigma)=\frac{F(r_\infty;\Psi_{\infty,0}(\sigma),0)}{\tilde{\lambda}}\frac{F(i_\infty;\Psi_{\infty,1}(\sigma),0)}{\lambda}\cdots\frac{F(r_\infty;\Psi_{\infty,2j}(\sigma),0)}{\tilde{\lambda}}$$

converges to $\xi^+(\sigma)$ as $j\to\infty$ and we have

$$\left|\frac{F_0}{\tilde{\lambda}}\frac{F_1}{\lambda}\cdots\frac{F_q}{\tilde{\lambda}}-\xi^+(\sigma)\right|\leqslant C\alpha^q.$$

Similarly by using

$$\left|X_{-(q+j)}(x,\nabla\varphi_{2q})-A_{q-j}\right|\leqslant C\alpha^{q+j}\qquad\text{for}\quad 0\leqslant j\leqslant q$$

we can find a 2×2 constant matrix ξ^- such that

$$\left|\frac{F_{q+1}}{\lambda}\frac{F_{q+2}}{\tilde{\lambda}}\cdots\frac{F_{2q}}{\lambda}-\xi^-\right|\leqslant C\alpha^q\qquad\text{for all }\sigma.$$

Thus we have

$$P(i_q-i_\infty)(\sigma)=(\lambda\tilde{\lambda})^q\left(\xi^+(\sigma)\,\xi^-+O(\alpha^q)\right),$$

which implies that the required estimate holds for $m=0$. For the other cases we can prove it by the same process.

3.2. Convergence of broken rays

By using (i) of Lemma 3.2 and Proposition 3.3 we have a sharp form of Proposition 4.6 of [3].

PROPOSITION 3.4. *For* $0\leqslant j\leqslant q$

$$\left|PX_{-j}(\cdot,\nabla\varphi_{2q})-\left(\lambda_0^j E_\infty(\cdot)+\lambda_0^{2q-j}E_0\right)P\right|_m\left(S_1(\delta_2)\right)\leqslant C_m(\lambda_0\alpha)^q$$

$$\left|PX_{-j}(\cdot,\nabla\varphi_{2q+1})-\left(\lambda_0^j\tilde{E}_\infty(\cdot)+\lambda_0^{2q-j}\tilde{E}_0\right)P\right|_m\left(S_2(\delta_2)\right)\leqslant C_m(\lambda_0\alpha)^q$$

hold, where $\lambda_0=(\lambda\tilde{\lambda})^{1/2}$, $E_\infty(x)$, $\tilde{E}_\infty(x)$ *are* 2×2 *matrix valued* C^∞ *functions defined on* $S_1(\delta_2)$ *and* $S_2(\delta_2)$ *respectively, and* E_0, $\tilde{E}_0$ *are* 2×2 *constant matrices.*

PROPOSITION 3.5. *For* $0\leqslant j\leqslant q$

$$\left|PX_{-2q+j}(\cdot,\nabla\varphi_{2q})-\left(PA_j+\lambda_0^{2q-j}F_\infty(\cdot)\right)\right|_m\left(S_1(\delta_2)\right)\leqslant C_m(\lambda_0\alpha)^q$$

$$\left|PX_{-2q-1+j}(\cdot,\nabla\varphi_{2q})-\left(PA_j+\lambda_0^{2q-j}\tilde{F}_\infty(\cdot)\right)\right|_m\left(S_2(\delta_2)\right)\leqslant C_m(\lambda_0\alpha)^q$$

hold where F_∞ *and* $\tilde{F}_\infty$ *are* 2×2 *matrix valued* C^∞ *functions defined on* $S_1(\delta_2)$ *and* $S_2(\delta_2)$ *respectively.*

3.3. On solutions of the transport equations

Let $\boldsymbol{v}=\{v_q, \tilde{v}_q\}_{q=0}^{\infty}$ be the solution of

$$(3.6)\qquad \begin{cases} \boldsymbol{T}\boldsymbol{v}=0 & \text{in } \omega\times\boldsymbol{R} \\ \boldsymbol{v}=\boldsymbol{f}^{\boldsymbol{\cdot}} & \text{on } S(\delta_2)\times\boldsymbol{R} \end{cases}$$

for $\boldsymbol{f}^{\boldsymbol{\cdot}}=\{f_q, 0\}_{q=0}^{\infty}$ such that $f_j(x, t)=f(x, t)\in C_0^{\infty}(S_1(\delta_2)\times(0, d/2))$ and $f_q=0$ for all $q\neq j$. Taking account of the difference of Definitions 5.1 and 6.2 of [3] we have from Lemma 5.1 of

$$v_q(x, t)=\Lambda_{2q}(x)\,\Lambda_{2q-1}\Big(X_{-1}(\mathrm{x}, \nabla\varphi_{2q})\Big)\cdots\Lambda_{2j}\Big(X_{-2q+2j}(x, \nabla\varphi_{2q})\Big)$$
$$\cdot f\Big(X_{-2q+2j-1}(x, \nabla\varphi_{2q}),\ t-h_{2q,2j}(x)\Big),$$
$$\tilde{v}_q(x, t)=\Lambda_{2q+1}(x)\,\Lambda_{2q}\Big(X_{-1}(x, \nabla\varphi_{2q+1})\Big)\cdots\Lambda_{2j}\Big(X_{-2q+2j-1}(x, \nabla\varphi_{2q+1})\Big)$$
$$\cdot f\Big(X_{-2q+2j-2}(x, \nabla\varphi_{2q}),\ t-h_{2q+1,2j}(x)\Big).$$

Lemma 3.6. *We have for* $j\leqslant q$

$$\Big|\Lambda_{2q}(\cdot)\,\Lambda_{2q-1}\Big(X_{-1}(\cdot, \nabla\varphi_{2q})\Big)\cdots\Lambda_{2j}\Big(X_{-2q+2j}(\cdot, \nabla\varphi_{2q})\Big)$$
$$-\lambda_0^{2q-2j}\Big(a_{\infty}(\cdot)\,b_j+\lambda_0^{q-j}\,a_{\infty,1}(\cdot)\,c_j\Big)\Big|_m\Big(S_1(\delta_2)\Big)\leqslant C(\lambda_0^3\alpha)^{q-j}$$

$$\Big|\Lambda_{2q+1}(\cdot)\,\Lambda_{2q}\Big(X_{-1}(\cdot, \nabla\varphi_{2q+1})\Big)\cdots\Lambda_{2j}\Big(X_{-2q+2j-1}(\cdot, \nabla\varphi_{2q+1})\Big)$$
$$-\lambda_0^{2q-2j}\Big(\tilde{a}_{\infty}(\cdot)\,b_j+\lambda_0^{q-j}\,\tilde{a}_{\infty,1}(\cdot)\,\tilde{c}_j\Big|_m\Big(S_2(\delta_2)\Big)\leqslant C(\lambda_0^3\alpha)^{q-j}$$

where $a_{\infty}(x)$, $\tilde{a}_{\infty}(x)$, $a_{\infty,1}(x)$, $\tilde{a}_{\infty,1}(x)$ *are* C^{∞} *functions and* b_j, c_j *and* $\tilde{c}_j$ *are constants such that*

$$|b_j-1|\leqslant C\alpha^{2j},\quad |c_j-c_{\infty}|\leqslant C\alpha^{2j},\quad |\tilde{c}_j-\tilde{c}_{\infty}|\leqslant C\alpha^{2j}.$$ [3)]

Lemma 3.7. *It holds that for all* $j\leqslant q$

$$\Big|h_{2q,2j}(\cdot)-\Big(2(q-j)\,d+j_{\infty}(\cdot)\Big)\Big|_m\Big(S_1(\delta_2)\Big)\leqslant C_m(\lambda_0\alpha)^{q-j}$$

$$\Big|h_{2q+1,2j}(\cdot)-\Big((2q+1-2j)\,d+\tilde{j}_{\infty}(\cdot)\Big)\Big|_m\Big(S_2(\delta_2)\Big)\leqslant C_m(\lambda_0\alpha)^{q-j}$$

where $j_{\infty}(x)$ *and* $\tilde{j}_{\infty}(x)$ *are* C^{∞} *functions.*

Then we have

Proposition 3.8. *The solution* $\boldsymbol{v}$ *of the equation* (3.6) *is decomposed as*

3) When $\mathscr{O}_j$, $j=1, 2$ are tangent to a ball at a_j of a high order $a_{\infty,1}(x)$ and $\tilde{a}_{\infty,1}(x)$ vanishes identiaclly. This implies that the poles of $\mathscr{S}(z)$ do not exist near $\tilde{z}_j$.

$$v=w+\bar{w}+z$$

where $w\in K(j)$, $\bar{w}\in\bar{K}(j)$ *and* $z\in\bar{M}_1(j)$, *and the correspondances from* $f(x,t)$ *to* w, $\bar{w}$, z *are continuous for* $C_0^\infty(S_1(\delta_2)\times(0,d/2))$ *into* $K(j)$, $\bar{K}(j)$, $\bar{M}_1(j)$ *respectively.*

COROLLARY. *The solution of the transport equation* $v_0=\{v_{0,q}, \tilde{v}_{0,q}\}_{q=0}^\infty$ *is decomposed as*

$$v_0=w_0+\bar{w}_0+z_0,\ w_0\in K(0),\ \bar{w}_0\in\bar{K}(0),\ z_0\in\bar{M}_1(0)\,.$$

Next consider the solutions of higher order transport equation, that is, $v_r=\{v_{r,q}, \tilde{v}_{r,q}\}_{q=0}^\infty$, $r=1,2,\cdots,N$.

PROPOSITION 3.9. *We have for* $r\geqslant 1$

$$v_r=\sum_{J_r\in N_+{}^r}(w_r^{(J_r)}+\bar{w}_r^{(J_r)})+z_r$$
$$+\sum_{h=1}^{r}\sum_{l=0}^{\infty}\sum_{J_{r-h}\in N_+{}^{r-h}}(w_{r,h,l}^{(J_{r-h})}+\bar{w}_{r,h,l}^{(J_{r-h})})\,,$$

where

$$W_r=\{w_r^{(J_r)}\}_{J_r\in N_+{}^r}\in\mathscr{K}_r^0(0),\ \ \bar{W}_r=\{\bar{w}_r^{(J_r)}\}_{J_r}\in_{N+^r}\in\mathscr{G}_r(0)\,,$$
$$W_{r,h,l}=\{w_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+{}^{r-h}}\in\mathscr{K}_{r-h}^0(l)$$
$$\bar{W}_{r,h,l}=\{\bar{w}_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+{}^{r-h}}\in\mathscr{G}_{r-h}(l)\,,$$
$$z_r\in\bar{M}_{2r}(0)$$

and we have estimates

$$||W_r||_{\mathscr{K}_r^0,m},\ ||\bar{W}_r||_{\mathscr{G}_r,m},\ |z|_{M_{2r},m}\leqslant C_{r,m}B_{m+2r}\,,$$
$$||W_{r,h,l}||_{\mathscr{K}_{r-h}^0,m},\ ||\bar{W}_{r,h,l}||_{\mathscr{G}_{r-h},m}\leqslant C_{r,m}\alpha^l l^{r-h}B_{m+2r}\,.$$

In order to prove the above proposition the following two lemmas are fundamental.

LEMMA 3.10. *Let* $g\in K(p)$. *The solution* v *of*

$$\begin{cases} T_\infty v=g & in\quad \omega\times\boldsymbol{R}\\ \quad v=0 & on\quad S(\delta_2)\times\boldsymbol{R}\end{cases}$$

is decomposed as

$$v=\sum_{j=0}^{\infty}w^{(j)}\,,\qquad w^{(j)}\in K(p+j)$$

and $W=\{w^{(j)}\}_{j\in N_+}\in\mathscr{K}_1^0(p)$. *Moreover the correspondance from* g *to* W *is continuous.*

LEMMA 3.11. *Let* $g\in\bar{K}(p)$. *The solution* v *of*

$$\begin{cases} T_\infty v=g & \text{in} \quad \omega\times R \\ v=0 & \text{on} \quad S(\delta_2)\times R \end{cases}$$

is decomposed as

$$v=\sum_{j=0}^{\infty}\bar{w}^{(j)},\quad \bar{w}^{(j)}\in\bar{K}(p+j),$$

and

$$\bar{W}=\{\bar{w}^{(j)}\}_{j\in N_+}\in\mathscr{G}_1(p).$$

The correspondance from g *to* $\bar{W}$ *is continuous.*

Using these lemmas we can derive Proposition 3.9 by an almost same procedure as in § 6 of [3].

Proposition 2.1 is derived from Propositions 3.3 and 3.9 by the same reasoning as in § 7 of [3].

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