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Structure of Banach quasi-sublattices

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§ 1. Introduction

We begin with the following motivating example. Let D denote the open unit disc in the complex plane and let $\bar{D}$ be its closure. $C(\bar{D})$ means the Banach lattice of all continuous functions on $\bar{D}$ with usual pointwise order and supremum norm. Let H be the subspace of $C(\bar{D})$ consisting of all functions which are harmonic in D. Although H is not a sublattice of $C(\bar{D})$, it enjoys the following properties:

(i) H becomes a Banach lattice with respect to the order and the norm induced by those of $C(\bar{D})$, respectively.

(ii) Let $I:=\{f\in C(\bar{D})\,;\, f=0 \text{ on } \bar{D}\backslash D\}$ and let π denote the canonical surjection from $C(\bar{D})$ onto $C(\bar{D})/I$. Then $\pi_{|H}$ is an isometric lattice isomorphism onto $C(\bar{D})/I$.

(iii) H is the range of a contractive positive projection $P\in\mathscr{L}(C(\bar{D}))$ which is lattice homomorphic as an operator from $C(\bar{D})$ onto the Banach lattice H. ($\mathscr{L}(C(\bar{D}))$ denote the set of all bounded linear operators on $C(\bar{D})$.) In fact, it suffices to define Pf to be the harmonic extension of $f_{|\bar{D}\backslash D}$ to $\bar{D}$ for $f\in C(\bar{D})$.

The purpose of this paper is to investigate the structure of subspaces of a Banach lattice having the same property as the above (i) for H, which we call Banach quasi-sublattices.

In § 2, we give the definition of quasi-sublattices and Banach quasi-sublattices. (The former is introduced to treat the algebraic aspect of the latter separately.) Then we prove the fundamental facts about these spaces, fixing some notations along the way.

In § 3, we show that the analogues of (ii) and (iii) for H is valid for Banach quasi-sublattices of AM-spaces.

§ 2. Quasi-sublattices and Banach quasi-sublattices

DEFINITION 1. *A subspace F of a vector lattice E is called a quasi-sublattice of E if it becomes a vector lattice with respect to the order induced by that of E.*

Let F be a quasi-sublattice of a vector lattice E. Then the positive part, negative part and the absolute value of $x\in F$ in F are denoted by x^{++}, x^{--} and $|x|_F$, respectively. The standard notations x^+, x^- and $|x|$ are used to denote the positive part, negative part and the absolute value of $x\in E$ in E. The supremum and infimum of x, $y\in F$ in F are denoted by $x\Vee y$ and $x\Wedge y$, respectively, while $x\vee y$ and $x\wedge y$ stand for the supremum and infimum of x, $y\in E$ in E, respectively. It follows immediately from the definition that $x\Vee y\geq x\vee y$ and $x\Wedge y\leq x\wedge y$ hold for any x, $y\in F$.

LEMMA 1. *Let F be a quasi-sublattice of a vector lattice E. Suppose two finite families $\{a_{ij}\}_{i\in I,j\in J}$, $\{b_{kl}\}_{k\in K,l\in L}$ of elements of F satisfy*

$$\bigvee_{i\in I}\bigwedge_{j\in J}a_{ij}=\bigvee_{k\in K}\bigwedge_{l\in L}b_{kl}\,.$$

Then

$$\Vee_{i\in I}\Wedge_{j\in J}a_{ij}=\Wedge_{k\in K}\Vee_{l\in L}b_{kl}$$

holds, where $\vee$, $\wedge$ [resp. $\Vee$, $\Wedge$] denote the supremum and the infimum in E [resp. in F], respectively.

PROOF. First we show that $a_{ij}\in F$ and $\bigvee_{i\in I}\bigwedge_{j\in J}a_{ij}\geq 0$ imply $\Vee_{i\in I}\Wedge_{j\in J}a_{ij}\geq 0$. In fact, since the distributive law in E yields

$$\bigwedge_{j\in J}\bigvee_{\sigma\in\Sigma}a_{\sigma(j)j}=\bigvee_{i\in I}\bigwedge_{j\in J}a_{ij}\geq 0\,,$$

where $\Sigma=I^J$, we obtain $\bigvee_{\sigma\in\Sigma}a_{\sigma(j)j}\geq 0$ for any $j\in J$. Hence $\Vee_{\sigma\in\Sigma}a_{\sigma(j)j}\geq 0$ and hence $\Wedge_{j\in J}\Vee_{\sigma\in\Sigma}a_{\sigma(j)j}\geq 0$, which in turn implies $\Vee_{e\in I}\Wedge_{j\in J}a_{ij}\geq 0$.

Returning to $\bigvee_{i\in I}\bigwedge_{j\in J}a_{ij}=\bigvee_{k\in K}\bigwedge_{l\in L}b_{kl}$ and fixing an $i\in I$, we get $\bigvee_{k\in K}\bigwedge_{l\in L}b_{kl}\geq\bigwedge_{j\in J}a_{ij}$. Noting that $\bigwedge_{j\in J}a_{ij}\geq\Wedge_{j\in J}a_{ij}$, we obtain $\bigvee_{k\in K}\bigwedge_{l\in L}(b_{kl}-\Wedge_{j\in J}a_{ij})\geq 0$. Since $b_{kl}-\Wedge_{j\in J}a_{ij}\in F$, the first part of the proof yields $\Vee_{k\in K}\Wedge_{l\in L}(b_{kl}-\Wedge_{j\in J}a_{ij})\geq 0$, and hence $\Vee_{k\in K}\Wedge_{l\in L}b_{kl}\geq\Vee_{i\in I}\Wedge_{j\in J}a_{ij}$. Thus we obtain the desired equality since the converse inequlity is proved similarly.

THEOREM 1. *Let F be a quasi-sublattice of a vector lattice E, and let F_0 be the sublattice of E generated by F. Then there exists a positive (linear) projection P from F_0 onto F, which is lattice homomorphic with respect to the lattice structures of F_0 and F, i.e., P satisfies $P(x\wedge y)=Px\Wedge Py$ and $P(x\vee y)=Px\Vee Py$ for any x, $y\in F_0$.*

PROOF. Since $F_0=\{\bigvee_{i\in I}\bigwedge_{j\in J}a_{ij}\,;\ I,\ J\text{ finite},\ a_{ij}\in F\}$ ([2] p. 74), the mapping

$$P: \bigvee_{i\in I}\bigwedge_{j\in J} a_{ij} \longmapsto \curlyvee_{i\in I}\curlywedge_{j\in J} a_{ij} \qquad (I,\ J \text{ finite},\ a_{ij}\in F)$$

is well defined on F_0. Since $Px\in F$ for $x\in F_0$ and $Px=x$ for $x\in F$, the range of P is F and $P^2=P$. The additivity of P is proved by using the distributive laws in E and F:

Let I, J, K and L be finite sets and a_{ij}, $b_{kl}\in F$ for any $i\in I$, $j\in J$, $k\in K$ and $l\in L$. Then

$$\begin{aligned} P\Big(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij}+\bigvee_{k\in K}\bigwedge_{l\in L} b_{kl}\Big) &= P\Big(\bigvee_{i\in I}\bigwedge_{j\in J}\bigvee_{k\in K}\bigwedge_{l\in L}(a_{ij}+b_{kl})\Big) \\ &= P\Big(\bigvee_{i\in I}\bigvee_{\sigma\in\Sigma}\bigwedge_{j\in J}\bigwedge_{l\in L}(a_{ij}+b_{\sigma(j)l})\Big) \\ &= \curlyvee_{i\in I}\curlyvee_{\sigma\in\Sigma}\curlywedge_{j\in J}\curlywedge_{l\in L}(a_{ij}+b_{\sigma(j)l}) \\ &= \curlyvee_{i\in I}\curlywedge_{j\in J}\curlyvee_{k\in K}\curlywedge_{l\in L}(a_{ij}+b_{kl}) \\ &= P\Big(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij}\Big)+P\Big(\bigvee_{k\in K}\bigwedge_{l\in L} b_{kl}\Big), \end{aligned}$$

where $\Sigma=K^J$. Other assertions are also proved by invoking the distributive law.

The following is a converse to Theorem 1.

PROPOSITION 1. *Let E be a vector lattice and let P be a positive linear projection in E. Then the range PE of P is a quasi-sublattice of E and* $P(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij})=\curlyvee_{i\in I}\curlywedge_{j\in J} a_{ij}$ *holds for any finite family* $\{a_{ij}\}_{i\in I, j\in J}$ *of elements of PE.*

PROOF. PE is indeed a quasi-sublattice of E and $x\curlyvee y=P(x\vee y)$, $x\curlywedge y=P(x\wedge y)$ hold for any x, $y\in PE$ ([6] p. 214). Since

$$P\Big(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij}\Big)\geq P\Big(\bigwedge_{j\in J} a_{ij}\Big)\geq P\Big(\curlywedge_{j\in J} a_{ij}\Big)=\curlywedge_{j\in J} a_{ij}$$

hold for any fixed $i\in I$, $P(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij})\geq\curlyvee_{i\in I}\curlywedge_{j\in J} a_{ij}$. On the other hand, $\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij}=\bigwedge_{\sigma\in\Sigma}\bigvee_{i\in I} a_{i\sigma(i)}$ $(\Sigma=J^I)$ implies

$$P\Big(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij}\Big)\leq P\Big(\bigvee_{i\in I} a_{i\sigma(i)}\Big)\leq P\Big(\curlyvee_{i\in I} a_{i\sigma(i)}\Big)=\curlyvee_{i\in I} a_{i\sigma(i)}$$

for any fixed $\sigma\in\Sigma$. Hence $P(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij})\leq\curlywedge_{\sigma\in\Sigma}\curlyvee_{i\in I} a_{i\sigma(i)}=\curlyvee_{i\in I}\curlywedge_{j\in J} a_{ij}$, and hence $P(\bigvee_{i\in I}\bigwedge_{j\in J} a_{ij})=\curlyvee_{i\in I}\curlywedge_{j\in J} a_{ij}$.

Now we proceed to the study of Banach quasi-sublattices.

DEFINITION. *A closed subspace F of a Banach lattice E is called a Banach quasi-sublattice of E if F becomes a Banach lattice with respect to the order and the norm induced by those of E, respectively.*

Note that a closed subspace F of a Banach lattice E is a Banach quasi-sublattice of E if and only it is a quasi-sublattice of E and $\| |x|_F\| = \|x\|$ holds for any $x \in F$. By the definition, a closed sublattice of a Banach lattice E is a Banach quasisublattice of E. An important non-trivial example of a Banach quasi-sublattice is the range of a contractive positive projection. In fact let P be a contractive positive projection in a Banach lattice E and let F be the range of P. Then F is a quasi-sublattice of E and $|x|_F = P|x|$ for $x \in F$ ([6] p. 214). Hence $\| |x|_F\| = \|P|x|\| \leq \|x\|$. This implies $\| |x|_F\| = \|x\|$ since it always holds that $|x|_F \geq |x|$ for $x \in F$. The space H described in the introduction is a concrete example of such Banach quasi-sublattices.

On the other hand, certain Banach lattices admit no Banach quasi-sublattices other than closed sublattices.

PROPOSITION 2. *Suppose a Banach lattice E has a strictly monotone norm, i.e., $x, y \in E$ $0 \leq x \leq y$ and $\|x\| = \|y\|$ imply $x = y$. Then any Banach quasi-sublattice of E is a sublattice of E.*

PROOF. Let F be a Banach quasi-sublattice of E and $x \in F$. Then $0 \leq |x| \leq |x|_F$ and $\|x\| = \| |x| \| \leq \| |x|_F\| = \|x\|$. By the assumption this implies $|x| = |x|_F$. The identity

$$x \veebar y = \frac{1}{2}(x + y + |x - y|_F)$$

for $x, y \in F$ shows that F is a sublattice of E.

Concerning the analogues for general Banach quasi-sublattices of the properties (ii) and (iii) in § 1, we have the following result. The (b)$\Rightarrow$(a) part of the proof is due to Professor T. Ando.

THEOREM 2. *Let F be a Banach quasi-sublattice of a Banach lattice E and let $\tilde{F}$ be the closed sublattice of E generated by F. Then the following are equivalent.*

(a) *There exists a closed ideal I of E for which the restriction $\pi|_F$ of the canonical map $\pi: E \to E/I$ is isometric and lattice homomorphic with respect to the lattice structures of F and E/I.*

(b) *There exists a positive contractive projection $P \in \mathscr{L}(\tilde{F})$ with range F.*

PROOF. (a)$\Rightarrow$(b): Suppose a closed ideal I of E meet the condition in (a) and let $\pi: E \to E/I$ be the natural map. Let $\{a_{ij}\}_{i \in I, j \in J}$ be a finite family of elements of F. Then

$$\|\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij}\| \geq \left\|\pi(\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij})\right\| = \left\|\bigwedge_{i \in I} \bigvee_{j \in J} \pi(a_{ij})\right\|$$
$$= \left\|\pi(\veebar_{i \in I} \barwedge_{j \in J} a_{ij})\right\| = \left\|\veebar_{i \in I} \barwedge_{j \in J} a_{ij}\right\|$$

holds since $\pi|_F$ is isometric and lattice homomorphic. This shows that the mapping P in Theorem 1 can be uniquely extended to a contractive positive projection from $\tilde{F}$ to F, hence (b) holds.

(b)$\Rightarrow$(a): Suppose P: $\tilde{F}\to F$ satisfy the condition in (b). Then P is lattice homomorphic with respect to the lattice structure of $\tilde{F}$ and F, respectively. In fact, if we denote by F_0 the sublattice of E generated by F, Lemma 1 and Proposition 1 imply that $P|_{F_0}$ is lattice homomorphic with respect to the corresponding lattice structures, hence P is also lattice homomorphic. Therefore, Ker P is a closed sublattice of $\tilde{F}$ containing $x^{++}-x^{+}$ and $|x|_F-|x|$ for any $x\in F$.

Let I be the closed ideal of E generated by Ker P. Then the above observation implies that the natural map π: $E\to E/I$ satisfies $\pi(x)^{+}=\pi(x^{+})=\pi(x^{++})$ for any $x\in F$. Hence $\pi|_F$ is lattice homomorphic with respect to the lattice structure of F and E/I, respectively.

To see that $\pi|_F$ is isometric, it suffices to show $||\pi(x)||\geq||x||$ for positive $x\in F$, since for general $x\in F$

$$\left\|\pi(x)\right\|=\left\||\pi(x)|\right\|=\left\|\pi(|x|)\right\|=\left\|\pi(|x|_F)\right\|$$

and $||\,|x|_F||=||x||$ hold. So let $x\in F$ be positive and $u\in I$. Then there exists two sequences $\{u_n\}_{n\in N}$ and $\{v_n\}_{n\in N}$ satisfying $u_n\in E$, $v_n\in$Ker P and $|u_n|\leq v_n$ for any $n\in\boldsymbol{N}$, and $\lim_{n\to\infty}u_n=u$. Since $x+u_n\geq x-v_n$, $(x+u_n)^{+}\geq(x-v_n)^{+}$ holds for any $n\in\boldsymbol{N}$. By the remark in the first paragraph of this part of proof, the above inequality and the fact $(x-v_n)^{+}\in\tilde{F}$ imply $||(x+u_n)^{+}||\geq||(x-v_n)^{+}||\geq||P((x-v_n)^{+})||=||(P(x-v_n))^{++}||=||x||$. Thus we obtain $||x+u||\geq||(x+u)^{+}||=\lim_{n\to\infty}||(x+u_n)^{+}||\geq||x||$, which implies $||\pi(x)||\geq||x||$.

Next we turn to the problem of positive extension of linear functionals. First we prepare the following

LEMMA 2. *Let F be a Banach quasi-sublattice of a Banach lattice E. Then $||x^{++}||=||x^{+}||$ holds for any $x\in F$.*

PROOF. For any $x\in F$ and non-negative integer n, put $x_n:=nx^{++}+x\in F$. Since $x_n=(n+1)x^{++}-x^{--}=nx^{++}+x^{+}-x^{-}$, $|x_n|_F=(n+1)x^{++}+x^{--}\geq(n+1)x^{++}+x^{-}\geq nx^{++}+x^{+}+x^{-}$ and $|x_n|\leq nx^{++}+x^{+}+x^{-}$. Therefore

$$\left\||x_n|_F\right\|\geq\left\|(n+1)x^{++}+x^{-}\right\|\geq\left\|nx^{++}+x^{+}+x^{-}\right\|\geq\left\||x_n|\right\|=\left\||x_n|_F\right\|,$$

hence

$$\left\|nx^{++}+x^{+}+x^{-}\right\|=\left\|(n+1)x^{++}+x^{-}\right\|.$$

Using this equality, we can inductively prove the following inequality for any non-neagative integer n:

$$||nx^{++}+x^{-}|| \leq n||x^{+}||+||x^{-}||\,.$$

Dividing the above inequality by n and letting $n\to\infty$, we obtain $||x^{++}||\leq ||x^{+}||$, hence $||x^{++}||=||x^{+}||$.

Proposition 3. *Let F be a Banach quasi-sublattice of a Banach lattice E. Then any positive linear functional ϕ on F has a norm preserving extension to a positive linear functional on E.*

Proof. Let $p(x):=||\phi||\,||x^{+}||$ for $x\in E$. Then p is a sub-additive positively homogeneous function on E, and $\phi(x)\leq p(x)$ holds for any $x\in F$, since ϕ is positive and $||x^{++}||=||x^{+}||$ by Lemma 2. It readily follows that any Hahn-Banach extension $\tilde{\phi}$ of ϕ dominated by p meets the requirement of the proposition.

§ 3. Quasi-sublattices of AM-spaces

Proposition 4. *Let F be a quasi-sublattice of an* AM-*space E. Then F is also an* AM-*space.*

Proof. It suffices to show $||x+y||=\max\{||x||, ||y||\}$ for any x, $y\in F$ satisfying $x\curlywedge y=0$ ([4] p. 22). Let x, y be such elements. Then $|x-y|_F= x+y$, hence

$$||x+y|| = \big\| |x-y|_F \big\| = ||x-y||\,.$$

But $||x-y||\leq\max\{||x||, ||y||\}$ since E is an AM-space. Therefore $||x+y||\leq \max\{||x||, ||y||\}$, which implies $||x+y||=\max\{||x||, ||y||\}$ since the converse inequality is always valid for x, $y\geq 0$.

Theorem 3. *Let F be a Banach quasi-sublattice of an* AM-*space E. Then there exists a closed ideal I of E for which the restriction of the canonical map $\pi: E\to E/I$ to F is isometric and lattice homomorphic with respect to the lattice structures of F and E/I.*

Proof. Let $X:=\{f\in E'\,;\, f\geq 0,\ ||f||\leq 1\}$ and $Y:=\{\phi\in F'\,;\,\phi\geq 0,\ ||\phi||\leq 1\}$, where E' and F' denote the Banach space dual of E and F, respectively, which are also Banach lattices ([6] p. 85). Then X [resp. Y] is compact with respect to the relative w^*-topology, and the set X_0 [resp. Y_0] of the non-zero extreme points of X [resp. Y] consists of lattice homomorphic linear functionals on E [resp. F] ([4] p. 59). Proposition 3 shows that the

$$r:\begin{cases} X\longrightarrow Y \\ f\longmapsto f|_F \end{cases}$$

mapping is surjective. Since $r^{-1}(\phi)$ is a closed face of X for any $\phi\in Y_0$, $r^{-1}(\phi)\cap X_0$ is non-void ([3] p. 133).

Put $X_1:=r^{-1}(Y_0)\cap X_0$ and $I:=\{x\in E; f(|x|)=0$ for any $f\in X_1\}$. Then I is clearly a closed ideal of E which meets the requirement of Theorem, as we shall see below.

First we verify the equality $\|\pi(x)\|=\|x\|$ for $x\in F$, where $\pi: E\rightarrow E/I$ is the natural map. This follows from the following two observations:

(i) For any $x\in F$, $y\in I$ and $f\in X_1$, $\|x+y\|\geq|f(x+y)|=|f(x)|$ hold since $f(y)=0$.

(ii) For any $x\in F$ $\|x\|=\|\,|x|_F\|=\sup\{\phi(|x|_F);\ \phi\in Y_0\}=\sup\{|\phi(x)|;\ \phi\in Y_0\}=\sup\{|f(x)|;\ f\in X_1\}$ hold, where the third equality is due to the fact that $\phi\in Y_0$ is lattice homomorphic on F, and the last equality holds since $r(X_1)=Y_0$.

To see that $\pi|_F$ is lattice homomorphic with respect to the lattice structure of F and E/I, it suffices to note that for any x, $y\in F$ and $f\in X_1$,

$$\begin{aligned} f(x\veebar y-x\vee y)&=r(f)\,(x\veebar y)-f(x\vee y)=r(f)\,(x)\vee r(f)\,(y)-f(x)\vee f(y)\\ &=0 \end{aligned}$$

hold and hence $x\veebar y-x\vee y\in I$.

COROLLARY 1. *Let F be a closed subspace of an* AM*-space E. Then F is a Banach quasi-sublattice of E if and only if there exists a closed sublattice $\tilde{F}$ of E and a contractive positive projection $P\in\mathcal{L}(F)$ with $F=P\tilde{F}$.*

PROOF. The "if part" readily follows from the remark after the definition of Banach quasi-sublattices. On the other hand let F be a Banach quasi-sublattice of E and let $\tilde{F}$ be the closed sublattice of E generated by F. Then Theorem 2 and Theorem 3 imply that there exists a contractive positive projection $P\in\mathcal{L}(\tilde{F})$ with $F=P\tilde{F}$.

In case E is realized as a closed sublattice of the Banach lattice $C(K)$ (K: a compact Hausdorff space), we have the following

COROLLARY 2. *Let K be a compact Hausdorff space and let E be a closed sublattice of $C(K)$. Then for any Banach quasi-sublattice F of E, there exists a closed subset K_0 of K such that $F|_{K_0}:=\{x|_{K_0};\ x\in F\}$ is a sublattice of $C(K_0)$ and $\|x\|=\|x|_{K_0}\|$ holds for any $x\in F$. Moreover if F contains the constant functions, there exist a compact Hausdorff space K_1, a continuous surjection $p: K_0\rightarrow K_1$ and a continuous mapping $\mu: K\rightarrow\mathcal{M}_1^+(M_1)$*

($\mathscr{M}_1^+(K_1)$ *denotes the space of probability Radon measures on* K_1 *endowed with the relative* w^**-topology) satisfying the following conditions*:

(i) *The mapping* $p^*: g \mapsto g \circ p$ *gives an isometric lattice isomorphism from* $C(K_1)$ *onto* $F|_{K_0}$.

(ii) *For any* $x \in F$ *and* $s \in K$,

$$x(s) = \int p^{*-1}(x|_{K_0})\, d\mu_s$$

holds, where μ_s *denotes the value of* μ *at* s.

PROOF. Let X, Y, X_0 and Y_0 be defined as in the proof of Theorem 3, and let $r: X \to Y$ be the restriction map, i.e., $r(\varphi) := \varphi|_F$ for $\varphi \in X$. Then as noted in the proof of Theorem 3, $r(X_0) \supset Y_0$. On the other hand consider the evaluation mapping $\varepsilon: K \to X$ which maps $s \in K$ to the functional $E \ni x \mapsto x(s)$. Then $\varepsilon(K) \supset X_0$ since $\varepsilon(K) \cup \{0\}$ is compact and its closed convex hull is X. Therefore the closed subset $K_0 := \overline{(r \circ \varepsilon)^{-1}(Y_0)}$ of K satisfies $r \circ \varepsilon(K_0) \supset Y_0$. This implies that $F|_{K_0}$ is a sublattice of $C(K_0)$ and that $\|x\| = \|x|_F\|$ holds for any $x \in F$, which in turn implies that $F|_{K_0}$ is closed in $C(K_0)$. This proves the first part of the Corollary.

Assume now F contains the constant functions. Let the equivalence relation $\sim$ on K_0 be defined by $s \sim t$ if and only if $x(s) = x(t)$ holds for any $x \in F$. Let $K_1 := K_0/\sim$ be the quotient space and let $p: K_0 \to K_1$ be the canonical surjection. Then K_1 is a compact Hausdorff space ([5] pp. 125–126) and the Stone-Weierstrass theorem implies that the mapping $p^*: g \mapsto g \circ p$ gives an isometric lattice isomorphism from $C(K_1)$ onto $F|_{K_0}$.

On the other hand, the first part of the proof shows that the mapping $\tau: x \mapsto x|_F$ is an isometric lattice isomorphism from F onto $F|_K$. Hence $\tau^{-1} \circ p^*$ is an isometric lattice isomorphism from $C(K_1)$ onto F. It is easy to see that for any $s \in K$ there exists a unique probability Radon measure μ_s on X_1 which satisfies

$$\int g d\mu_s = (\tau^{-1} \circ p^*)(g)(s)$$

for any $g \in C(K_1)$. That the mapping $\mu: s \in K \mapsto \mu_s \in \mathscr{M}_1^+(K_1)$ is continuous and that the assertion (ii) in the Corollary holds are clear from the construction of μ_s.

REMARK. In an unpublished note [1], Professor T. Ando studied the structure of certain subspaces of a Banach lattice. Among his results, the following is closely related to our results in § 3:

If a closed linear subspace F of a Banach lattice E satisfies the following conditions (i), (ii) and (iii), then F is the range of a positive projection.

(i) F is a quasi-sublattice of E;

(ii) The sublattice generated by F is dense in E;

(iii) $E=F-E_+$.

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