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Regular sequences of ideals in a noncommutative ring

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ABSTRACT: Let R be an associative ring which is not in general commutative. If M is a left R-module we define the notion of an M-regular ideal of R and the notion of an M-regular sequence of ideals of R, generalizing the corresponding notions in commutative ring theory. If M is finitely-generated and if R is left stable and left noetherian, a bound is given for the lengths of M-regular sequences of ideals.

0. Background and notation. Throughout the following, R will denote an associative (but not necessarily commutative) ring with unit element 1. The word "ideal" will mean "proper two-sided ideal" unless modified by an adjective indicating dexterity. The complete brouwerian lattice of all (hereditary) torsion theories defined on the category R-mod of unitary left R-modules will be denoted by R-tors. Notation and terminology regarding such theories will follow [2]. In particular, if M is a left R-module then $\xi(M)$ will denote the smallest torsion theory relative to which M is torsion and $\chi(M)$ will denote the largest torsion theory relative to which M is torsionfree. The unique minimal element of R-tors is $\xi=\xi(0)$ and the unique maximal element of R-tors is $\chi=\chi(0)$. A torsion theory in R-tors is said to be *stable* if and only if its class of torsion modules is closed under taking injective hulls. The ring R is said to be *left stable* if and only if every element of R-tors is stable.

If $\tau\in R$-tors then a nonzero left R-module M is called *τ-cocritical* if and only if it is τ-torsionfree while each of its proper homomorphic images is τ-torsion. A left R-module M is *cocritical* if and only if it is $\chi(M)$-cocritical. A torsion theory in R-tors is said to be *prime* if and only if it is of the form $\chi(M)$ for some cocritical left R-module M. If M is a nonzero left R-module then $\{\chi(N)|N$ is a cocritical submodule of $M\}$ is called the set of *associated primes* of M and is denoted by ass(M).

1. Regular ideals and elements. Let M be a nonzero left R-module. We will say that an ideal I of the ring R is *M-regular* if and only if M is $\xi(R/I)$-torsionfree. An element a of R will be said to be *M-regular* if and only if the ideal RaR generated by a is M-regular. The following is an elementary, and essentially well-known, characterization of regular ideals.

Its proof follows immediately from Proposition 5. 8 of [5] and Corollary 1. 22 of [9].

(1. 1) PROPOSITION: *If M is a nonzero left R-module and if I is an ideal of R then the following conditions are equivalent*:

(1) *I is M-regular*;

(2) *$Im\neq(0)$ for all $0\neq m\in M$*;

(3) *$\mathrm{Hom}_R(N, M)=(0)$ for any $\xi(R/I)$-torsion left R-module N*;

(4) *Im is large in Rm for all $0\neq m\in M$.*

Ideals regular with respect to a given module need not exist, even if the ring R is left noetherian. For example, if R is a simple left noetherian ring then surely no nonzero left R-module can possibly have an associated regular ideal. On the other hand, if R is a fully left bounded left noetherian ring and if M is a left R-module satisfying the condition that $\chi(M)\neq\xi$ then the torsion theory $\chi(M)$ is symmetric (see [5, 8, 10]) and so there exists a proper ideal I of R such that R/I is $\chi(M)$-torsion. Thus M is $\xi(R/I)$-torsionfree and so I is an M-regular ideal of R.

Let us see how this definition relates to the usual notion of regularity for elements of a commutative ring. Indeed, it is immediate that if R is commutative than an element a of R is regular in the above sense if and only if it is regular in the usual sense. If P is a prime ideal of an arbitrary ring R then by Proposition 6. 2 of [5] we see that an ideal I of R is (R/P)-regular if and only if $I\nsubseteq P$. This corresponds to the well-known result that if P is a prime ideal of a commutative ring R then an element a of R is not a zero-divisor on R/P if and only if $a\notin P$. A left R-module M is said to be *definite* if and only if every nonzero homomorphic image of M has a cocritical submodule. (In [2] such modules are called D-modules.) The ring R is *left definite* if and only if every nonzero left R-module is definite. If M is a definite left R-module then $\chi(M)=\wedge\mathrm{ass}(M)$ (see [4] for details) and so an ideal I of R is M-regular if and only if $\xi(R/I)\leq\pi$ for all $\pi\in\mathrm{ass}(M)$. This result corresponds to the well-known result for noetherian modules over commutative rings stating that an element a of R is not a zero-divisor on M if and only if it belongs to none of the prime ideals associated with the zero submodule of M.

From Proposition 1. 1 we note that if M is a nonzero left R-module and if I is an M-regular ideal of R then IM must be large in M. If the ideal I is idempotent and satisfies the condition that R/I is flat as a right R-module then the converse is also true. That is to say, if M is a nonzero left R-module then I is an M-regular ideal of R if and only if IM is large in M [1]. A related result is Proposition 22.12 of [2], which states that if I is an idempotent ideal of a ring R then the following conditions are equivalent:

(1) R/I is projective as a left R-module;

(2) If M is a left R-module such that I is an M-regular ideal of R then $M=IM$.

(1.2) PROPOSITION: *If M is a nonzero left R-module then the set of all M-regular ideals of R, together with R itself, forms a multiplicatively-closed filter.*

PROOF: By definition, this set is nonempty. If $H\subseteq I$ are ideals of R and if H is M-regular then $\xi(R/I)\leq\xi(R/H)\leq\chi(M)$ by [1, Proposition 8.6] and so I is also M-regular. If I and H are both M-regular ideals of R then by [2, Proposition 5.9] we have $\xi(R/IH)=\xi(R/[I\cap H])=\xi(R/I)\vee\xi(R/H)$ and so $\xi(R/[I\cap H])\leq\chi(M)$ and $\xi(R/IH)\leq\chi(M)$. This shows that $I\cap H$ and IH are both M-regular.

(1.3) COROLLARY: *Let R be a left noetherian ring and let M be a nonzero left R-module. Then there exists a torsion theory $\kappa(M)\in R$-tors defined by the condition*

(*) *A left R-module N is $\kappa(M)$-torsion if and only if every element of N is annihilated by an M-regular ideal of R.*

PROOF: We must show that $\{H|H$ is a left ideal of R containing an M-regular ideal of $R\}$ is an idempotent filter, and this is a direct consequence of Proposition 1.2 and [10, Proposition 1.2.3].

Note that in the above situation we clearly have $\kappa(M)\leq\chi(M)$.

(1.4) PROPOSITION: *If $0\rightarrow M'\rightarrow M\rightarrow M''\rightarrow 0$ is an exact sequence of nonzero left R-modules then*

(1) *Any M-regular ideal of R is also M'-regular.*

(2) *Any ideal of R which is both M'-regular and M''-regular is also M-regular.*

PROOF: By [2, Proposition 8.6] we see that the hypothesis implies that $\chi(M')\geq\chi(M)\geq\chi(M')\wedge\chi(M'')$. The result follows immediately from this and from the definition of regularity.

The following Proposition is based on a private communication to the author by Jacques Raynaud.

(1.5) PROPOSITION: *If R is a left noetherian ring then the following conditions on a nonzero finitely-generated left R-module M and a nonzero ideal I of R are equivalent:*

(1) *I contains an M-regular element*;

(2) *I is M-regular.*

PROOF: It is clear from Proposition 1.1 that (1) implies (2), so we need only prove the converse. Indeed, assume (2). Since R is left noetherian and since M is finitely-generated, we known from Propositions 21.21 and 21.22 of [2] that ass (M) is finite and nonempty, say ass $(M)=\{\pi_1, \cdots, \pi_n\}$. Then, in particular, $\xi(R/I)\leq\pi_i$ for all $1\leq i\leq n$. For each such i, let N_i be a π_i-cocritical left R-module and let $H_i=\sum\{(0: N')|N'$ is a nonzero submodule of $N_i\}$. Then H_i is a prime ideal of R. (Indeed, H_i is just the *tertiary radical* of N_i in the sense of [7] or of [4, Chapter 3].) Moreover, by Theorem 1.2 of [7] we know that H_i is the unique maximal element of the set of all ideals of R which are not π_i-dense in R. Thus, in particular, $I\not\subseteq H_i$ for all $1\leq i\leq n$.

We claim that $I\not\subseteq\cup_{i=1}^n H_i$. Indeed, by discarding some of the H_i if necessary we can assume without loss of generality that $H_i\not\subseteq H_j$ whenever $i\neq j$. Since I is not contained in any of the H_i, we can select an element a_i of $[I\cap(\cap_{j\neq i}H_j)]\backslash H_i$ for each $1\leq i\leq n$. Then $a=\sum_{i=1}^n a_i$ is an element of I which clearly does not belong to $\cup_{i=1}^n H_i$.

If a is any element of $I\backslash\cup_{i=1}^n H_i$ then surely $RaR\not\subseteq H_i$ for each i and so, by the selection of the H_i, this means that RaR is π_i-dense in R for all i. Thus $\xi(R/RaR)\leq\chi(M)=\wedge_{i=1}^n\pi_i$. This, together with Proposition 1.1, proves (1).

The notion of the torsion-theoretic Krull dimension of a left R-module is explored in [4]. We denote this dimension of a left R-module M by TTK-dim (M). If the ring R is left stable and left noetherian, then TTK-dim (M) just coincides with the Gabriel dimension of M for any left R-module M. See [4, Proposition 13.3].

(1.6) PROPOSITION: *Let R be a left stable left noetherian ring and let M be a finitely-generated left R-module. If I is an M-regular ideal in R then* TTK-dim $(M)>TTK$-dim (M/IM).

PROOF: Set $\bar{M}=M/IM$. If TTK-dim $(\bar{M})=k$ then there exists a chain of prime torsion theories in R-mod of the form

$$\pi_k<\cdots<\pi_0=\pi',$$

where $\pi'\in$ass (M') for some homomorphic image M' of $\bar{M}$. Since $\bar{M}$ is $\xi(R/I)$-torsion, so is M' and so $\xi(R/I)\not\leq\pi'$. On the other hand, by [3, Proposition 2] there exists an element π of ass (M) satisfying $\pi\geq\pi'$. Since M is $\xi(R/I)$-torsionfree, this implies that $\pi\geq\xi(R/I)$ and so $\pi>\pi'$. Therefore TTK-dim $(M)>TTK$-dim $(\bar{M})$.

2. Regular sequences of ideals. Let M be a nonzero left R-module and let K be an ideal of R. A (finite or infinite) sequence $I=\langle I_1, I_2, \cdots\rangle$ of ideals of R contained in K will be called an *M-regular sequence in K* if and only if

(1) I_1 is an M-regular ideal of R;

(2) If $t>1$ then $\sum_{j=1}^{t-1} I_j M \neq M$ and I_t is an $(M/[\sum_{j=1}^{t-1} I_j M])$-regular ideal of R.

It is immediate from this definition that if the ring R is commutative then a sequence $\langle a_1, \cdots, a_n\rangle$ of elements of K is M-regular in the usual sense of commutative ring theory if and only if $\langle(a_1), \cdots, (a_n)\rangle$ is an M-regular sequence of ideals of R in the above sense. In [6, Chapter 8] Lubkin considers a generalization of this situation. In particular, he considers sequences $\langle a_1, \cdots, a_n\rangle$ of elements of a (not-necessarily commutative) ring R satisfying the following conditions:

(1) If $1\leq h\leq n$ then $Ra_h M\subseteq \sum_{i=1}^{h} a_i M$;

(2) If $1\leq h\leq n$ then the function α_h from $M/[\sum_{i=1}^{h-1} a_i M]$ to itself defined by $\bar{x}\mapsto a_h\bar{x}$ is an R-monomorphism.

If each of the modules $M/[\sum_{i=1}^{h-1} a_i M]$ is nonzero then, in such a situation, it is clear that $\langle Ra_1R, \cdots, Ra_nR\rangle$ is an M-regular sequence of ideals of R in the sense defined above.

Let K be an ideal of the ring R. If M is a nonzero left R-module and if $\langle I_1, I_2, \cdots\rangle$ is an M-regular sequence in K then for each $t\geq 1$ it is surely true that the truncated sequence $\langle I_1, I_2, \cdots, I_t\rangle$ is also M-regular. Moreover, if $H=\sum_{j=1}^{t} I_j$ then $\langle(I_{t+1}+H)/H, (I_{t+2}+H)/H, \cdots\rangle$ is an (M/HM)-regular sequence in K. If the ring R is left noetherian and if I is an ideal of R then a nonzero left R-module M is $\xi(R/I)$-torsion if and only if every nonzero element of M is annihilated by some power of I. (See, for example, Proposition 5. 6 of [5].) As an immediate consequence of this we note that if I and H are ideals of a left noetherian ring R then $\xi(R/I)=\xi(R/H)$ if and only if there exist positive integers p and q such that $I^p\subseteq H$ and $H^q\subseteq I$. (Ideals having this property are said to be *radically equivalent.*) In particular $\xi(R/I)=\xi(R/I^k)$ for each ideal I of R and each positive integer k.

If $I=\langle I_1, I_2, \cdots\rangle$ is a (finite or infinite) sequence of ideals of a ring R then we can define a descending chain of torsion theories

$$\rho_1(I)\geq\rho_2(I)\geq\cdots$$

in R-tors by setting $\rho_t(I)=\xi(R/[\sum_{j=1}^{t} I_j])=\wedge_{j=1}^{t}\xi(R/I_j)$ for all $t\geq 1$. See [9] for further characterizations of such torsion theories. In particular, we note that by [9, Proposition 1. 20] a left R-module M is $\rho_t(I)$-torsionfree if and only if $\mathrm{Hom}_R(R/I_j, M)=0$ for all $1\leq j\leq t$. For any such sequence I and for any nonzero left R-module M we can also define another descending chain of torsion thories

$$\chi_0(I, M)\geq\chi_1(I, M)\geq\cdots$$

in R-tors by setting $\chi_0(I, M)=\chi(M)$ and $\chi_t(I, M)=\chi_{t-1}(I, M)\wedge\chi(M/[\sum_{j=1}^{t}I_jM])$ for all $t\geq 1$.

(2.1) PROPOSITION: *Let M be a nonzero left R-module and let $I=\langle I_1, I_2, \cdots\rangle$ be a sequence of nonzero ideals of R contained in an ideal K of R. Then*

(1) *I is an M-regular sequence in K*

implies

(2) (a) $\chi>\rho_1(I)>\rho_2(I)>\cdots$;

(b) $\chi_0(I, M)>\chi_1(I, M)>\cdots$;

(c) $\rho_h(I)\leq\chi_{h-1}(I, M)$ *and* $\rho_h(I)\not\leq\chi_h(I, M)$ *for all* $h\geq 1$.

Moreover, the converse holds if the ring R is left definite.

PROOF: (1)$\Rightarrow$(2): In order to prove (2a) we must show that equality cannot occur at any stage of the sequence. By definition, M is a nonzero $\rho_1(I)$-torsionfree left R-module and so $\chi>\rho_1(I)$. If $h>1$ and if $N=\sum_{j=1}^{h-1}I_jM$ then M/N is a nonzero $\rho_h(I)$-torsionfree left R-module which is $\rho_{h-1}(I)$-torsion and so $\rho_h(I)\neq\rho_{h-1}(I)$.

Since I is assumed to be an M-regular sequence we know that, in particular, $\sum_{j=1}^{h-1}I_jM\neq M$ for all $h>1$. Moreover, $\xi(R/I_h)\leq\chi(M/[\sum_{j=1}^{h-1}I_jM])$ for all such h. In particular, this implies that $\rho_h(I)\leq\chi_{h-1}(I, M)$ for all $h\geq 1$. We further note that $M/[\sum_{j=1}^{h}I_jM]=M/[\sum_{j=1}^{h}I_j]M$ for all $h\geq 1$. This module is $\rho_h(I)$-torsion and so surely cannot be $\rho_h(I)$-torsionfree. Therefore $\rho_h(I)\not\leq\chi_h(I, M)$ for all such h, proving (2c).

Finally, to establish (2b) we note that if $\chi_h(I, M)=\chi_{h-1}(I, M)$ then $\rho_{h-1}(I)\leq\chi_{h-1}(I, M)$ which, as we have already seen, cannot happen.

(2)$\Rightarrow$(1): We now assume that R is left definite and that (2) holds. If $h\geq 2$ then by (2b) we have $\chi>\chi(M)>\chi_{h-1}(I, M)$ and so, in particular, $\sum_{j=1}^{h-1}I_jM\neq M$. Moreover, by (2c) we see that $\chi(M)=\chi_0(I, M)\geq\rho_1(I)=\xi(R/I_1)$ and so $\langle I_1\rangle$ is an M-regular sequence in K. Now assume inductively that $h\geq 1$ and that we have already established that $\langle I_1, \cdots, I_h\rangle$ is an M-regular sequence in K. For notational convenience, set $\sigma=\chi(M/[\sum_{j=1}^{h}I_jM])$. Then $\rho_h(I)\wedge\xi(R/I_{h+1})=\rho_{h+1}(I)\leq\chi_h(I, M)=\chi_{h-1}(I, M)\wedge\sigma\leq\sigma$.
If $\pi\in\mathrm{ass}\,(M/[\sum_{j=1}^{h}I_jM])$ then $\pi=\chi(N)$,where N is a cocritical submodule of $M/[\sum_{j=1}^{h}I_jM]$. This implies, in particular, that N is $\rho_h(I)$-torsion and so $\rho_h(I)\not\leq\pi$. But π is prime and so $\xi(R/I_{h+1})\leq\pi$. Since R is left definite,

by [4, Proposition 0.5] we see that $\xi(R/I_{h+1})\leq\sigma$ and so $\langle I_1, \cdots, I_{h+1}\rangle$ is an M-regular sequence in K. This proves (1).

(2.2) COROLLARY: *If R is a left stable left noetherian ring and if M is a nonzero left R-module then there does not exist an infinite M-regular sequence of ideals of R.*

PROOF: Let $I=\langle I_1, I_2, \cdots\rangle$ be an infinite M-regular sequence of ideals of R. Then by Proposition 2, 1 we have an infinite descending chain in R-tors:

$$\rho_1(I)>\rho_2(I)>\cdots$$

By [5, Proposition 4.12], each one of the torsion theories $\rho_i(I)$ is compact, yielding a contradiction by [5, Proposition 4.10].

(2.3) PROPOSITION: *Let R be a left stable left noetherian ring and let M be a finitely-generated left R-module having an M-regular sequence of ideals $\langle I_1, \cdots, I_n\rangle$ in R. Then $n\leq TTK\text{-dim}(M)+1$.*

PROOF: We will proceed by induction on n. The case $n=1$ is trivial and so assume that $n>1$ and that the result has been shown true for all finitely-generated left R-modules having associated regular sequences of length $n-1$. Set $\bar{M}=M/I_1M$. Then $\langle[I_2+I_1]/I_1, \cdots, [I_n+I_1]/I_1\rangle$ is an $\bar{M}$-regular sequence and so, by the induction hypothesis, $TTK\text{-dim}(\bar{M})\geq n-2$. By Proposition 1.6, this implies that $TTK\text{-dim}(M)\geq n-1$, i.e., $n\leq TTK\text{-dim}(M)+1$.

References

[1] G. BACCELLA: *On flat factor rings and fully right idempotent rings*, preprint, 1981.

[2] J. GOLAN: "Localization of noncommutative rings", Marcel Dekker, New York, 1975.

[3] J. GOLAN: *On the maximal support of a module*, Comm. Algebra 4 (1976), 219–231.

[4] J. GOLAN: Decomposition and dimension in module categories", Marcel Dekker, NewYork, 1977.

[5] J. GOLAN: "Structure sheaves over a noncommutative ring", Marcel Dekker, New York, 1980.

[6] S. LUBKIN: "Cohomology of completions", North-Holland, Amsterdam, 1980.

[7] J. RAYNAUD: *Quelques résultats sur les localisations*, C. R. Acad. Sc. Paris 277 (1973), A 299–A 302.

[8] S. K. SIM: *On rings of which every prime kernel functor is symmetric*, Nanta Math. 7 (1974), 69–70.

[9] J. THOMAS: *On filters generated by sequences of ideals,* Comm. Algebra 6 (1978), 551–581.

[10] F. VAN OYSTAEYEN: "Prime spectra in noncommutative algebra", Lecture Notes in Mathematics #444, Springer-Verlag, Berlin, 1975.

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