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On rings for which various types of nonzero preradicals are cofaithful

(Dedicated to Professor Kentaro Murata for the celebration of his sixtieth birthday)

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In [11] we have studied those rings with trivial preradical ideals. Dually in this paper, we shall investigate rings for which every nonzero (idempotent, left exact or cohereditary, etc.) preradical or radical is cofaithful in the sense of [10]. Among other things we prove the following theorems.

THEOREM I. *The following properties are equivalent for a ring* R:

(1) *Every nonzero left exact radical for* R-mod *is cofaithful.*
(2) *There exist only two left exact radicals for* R-mod.
(3) *Every nonzero injective left* R*-module is a cogenerator for* R-mod.
(4) *Every nonzero* QF-$3'$ *left* R*-module is a cogenerator for* R-mod.
(5) ([8]) R *is left ChC.*
(6) R *is left CTF and left Kasch.*

THEOREM II. *The following properties are equivalent for a ring* R:

(1) *Every nonzero cohereditary radical for* R-mod *is cofaithful.*
(2) *Every nonzero left ideal of* R *is cofaithful.*
(3) *If* $a(\neq 0)\in R$ *and* ${}_RQ$ *is injective, then* $RaQ=Q$.
(4) *If* σ *is a preradical for* R-mod, *then either* $\sigma(E(R))=0$ *or* $\sigma(E(R))=E(R)$.
(5) *If* σ *is a radical for* R-mod, *then either* $\sigma(E(R))=0$ *or* $\sigma(E(R))=E(R)$.
(6) *If* σ *is a left exact preradical for* R-mod, *then either* $\sigma(E(R))=0$ *or* $\sigma(E(R))=E(R)$.
(7) R *is left SP.*
(8) R *is left SSP and left CTF.*

THEOREM III. *The following properties are equivalent for a ring* R:

(1) *Every nonzero preradical for* R*-mod is cofaithful.*
(2) *Every nonzero radical for* R-mod *is cofaithful.*

(3) *Every nonzero idempotent preradical for* R-mod *is cofaithful.*

(4) *Every nonzero left exact preradical for* R-mod *is cofaithful.*

(5) *Every nonzero factor module of an injective left R-module is a cogenerator for* R-mod.

(6) *Every nonzero factor module of* $E(R)$ *is a cogenerator for* R-mod.

(7) R *is left SP and left ChC.*

(8) R *is left SP and left Kasch.*

(9) R *is left SP and has nonzero socle.*

(10) R *is left ChC and left weakly regular.*

(11) R *is simple artinian.*

(12) *There exist only two preradicals for* R-mod.

(13) *There exist only two radicals for* R-mod.

(14) *There exist only two idempotent preradicals for* R-mod.

(15) ([4]) *There exist only two left exact preradicals for* R-mod.

1. Preliminaries

Let R be a ring with identity and R-mod the category of all unital left R-modules. We denote by $E(M)$ the injective hull of ${}_RM$. We refer for the definitions and basic properties concerning preradicals and torsion theories to [10], [11], [12] and [14], but we shall briefly summarize them. To a preradical σ for R-mod, one associate the pair $(\boldsymbol{T}(\sigma), \boldsymbol{F}(\sigma))$ of classes of modules in R-mod given by

$$\boldsymbol{T}(\sigma)=\left\{{}_RX \mid \sigma(X)=X\right\} \text{ and } \boldsymbol{F}(\sigma)=\left\{{}_RX \mid \sigma(X)=0\right\}.$$

For a module ${}_RQ$, we define an idempotent preradical t_Q and a radical k_Q for R-mod by

$$t_Q(M)=\sum\left\{\operatorname{Im}(\alpha) \mid \alpha\in\operatorname{Hom}_R(Q, M)\right\}$$

$$k_Q(M)=\cap\left\{\operatorname{Ker}(\beta) \mid \beta\in\operatorname{Hom}_R(M, Q)\right\}$$

for each $M\in R$-mod. With respect to the partial ordering in the class of all preradicals for R-mod, the identity functor 1 is the largest member and the functor 0, defined by $0(M)=0$ for all $M\in R$-mod, is the smallest one. For each preradical σ, there exists a largest idempotent preradical $\hat{\sigma}$ smaller than σ, and there exists a smallest radical $\bar{\sigma}$ larger than σ ([14, p 137]). Furthermore, there exists a left exact preradical $h(\sigma)$ defined by $h(\sigma)(M)=M\cap\sigma(E(M))$ for all $M\in R$-mod, and there exists a cotorsion radical $ch(\sigma)$ defined by $ch(\sigma)(M)=\sigma(R)M$ for all $M\in R$-mod. Recall that a preradical σ for R-mod is said to be faithful if $\sigma(P)=0$ for every projective $P\in R$-mod. It is easy to see that a preradical σ is faithful if and only if $\sigma(R)=0$. Dually, a preradical σ for R-mod is said to be *cofaithful* if $\sigma(Q)=Q$ for every injective $Q\in R$-mod ([10]). Also, a module ${}_RI$ is called *cofaithful* if I generates all injective left R-modules.

LEMMA 1.1 ([1, p 88]). *$E(R)$ is cofaithful. Hence a module ${}_RI$ is cofaithful if and only if I generates $E(R)$.*

LEMMA 1.2 ([1, p 89]). *A module ${}_RI$ is cofaithful if and only if ${}_RR$ can be embedded in some finite direct sum $\bigoplus_{i=1}^n I$.*

In case I is a left ideal of R, we have

LEMMA 1.3. *Let I be a left ideal of R and ${}_RQ$ an injective module. Then $IQ=Q$ if and only if I generates Q. Hence a left ideal I is cofaithful if and only if $IE(R)=E(R)$.*

PROOF. Suppose $IQ=Q$ and consider, for each $x\in Q$, the mapping θ_x: $I\to Q$ defined by $\theta_x(r)=rx$ for all $r\in I$. Then the induced homomorphism $\oplus\theta_x$: $\bigoplus_{x\in Q} I\to Q$ is surjective, i.e. I generates Q.

Conversely, suppose that we have a surjective homomorphism ϕ: $\bigoplus_\Lambda I\to Q$ for some index set Λ. Put α: $\bigoplus_\Lambda I\to\bigoplus_\Lambda R$ the canonical inclusion mapping, and η_i: $R\to\bigoplus_\Lambda R$ the canonical injection mapping for each $i\in\Lambda$. Since Q is injective, there exists a homomorphism ψ: $\bigoplus_\Lambda R\to Q$ such that $\psi\alpha=\phi$. Now, for each $x\in Q$, there exists some $a=\sum_{i=1}^n a_i\in\bigoplus_\Lambda I$ such that $x=\phi(a)$. Therefore $x=\psi\alpha(a)=\psi(\sum_{i=1}^n\eta_i(a_i))=\sum_{i=1}^n(\psi\eta_i)(a_i)=\sum_{i=1}^n a_i(\psi\eta_i)(1)\in IQ$, which means $Q=IQ$.

LEMMA 1.4. *The following conditions are equivalent for a preradical σ for R-mod:*

(1) *σ is cofaithful.*
(2) ([10]) $\sigma(E(R))=E(R)$.
(3) ([15]) $\sigma(E(R))\supseteq R$.
(4) $\sigma\geqq t_{E(R)}$.

PROOF. (1)$\Leftrightarrow$(2). This is done by Lemma 1.1, since $\boldsymbol{T}(\sigma)$ is closed under direct sums and factors.

(2)$\Rightarrow$(4). For each $M\in R$-mod and $f\in\mathrm{Hom}_R(E(R), M)$, we have $f(E(R))=f(\sigma(E(R)))\subseteq\sigma(M)$. Hence $t_{E(R)}(M)\subseteq\sigma(M)$, which means $t_{E(R)}\leqq\sigma$.

(4)$\Rightarrow$(3). By assumption we have $\sigma(E(R))\supseteq t_{E(R)}(E(R))=E(R)\supseteq R$.

(3)$\Rightarrow$(1). This was proved in [15, Corollary 2.2].

To illustrate cofaithfulness of preradicals, we shall give two examples.

EXAMPLE 1.1. The basic one of cofaithful preradicals is an idempotent radical $t_{E(\boldsymbol{Z})}$ for $\boldsymbol{Z}$-mod, where $\boldsymbol{Z}$ is the ring of integers, whose torsion class $\boldsymbol{T}(t_{E(\boldsymbol{Z})})$ consists of the divisible modules and torsionfree class $\boldsymbol{F}(t_{E(\boldsymbol{Z})})$ consists of the reduced modules.

EXAMPLE 1.2. We shall give examples of a cofaithful cotorsion radical and anexa ct radical which is not cofaithful. Let R be the ring of 2×2 upper triangular matrices over a field K. It was shown that $E(R)=\begin{pmatrix} K & K \\ K & K \end{pmatrix}$ ([3]). Put $I=\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ and $J=\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$. Since I is an idempotent ideal of R, the radical σ for R-mod defined by $\sigma(M)=IM$ for all $M\in R$-mod is cotorsion. Because $\sigma(E(R))=\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}\begin{pmatrix} K & K \\ K & K \end{pmatrix}=\begin{pmatrix} K & K \\ K & K \end{pmatrix}=E(R)$, σ is cofaithful. Now put τ by $\tau(M)=JM$ for all $M\in R$-mod. Since J is idempotent and $(R/J)_R$ is flat, τ is an exact radical for R-mod. But τ is not cofaithful because $\tau(E(R))=\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & K \\ K & K \end{pmatrix}=\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}\neq E(R)$. Now by a routine computation, we have $t_{E(R)}(R)=I$. Since $E(R)$ is projective, $t_{E(R)}$ is a cotorsion radical. Thus we notice that $t_{E(R)}$ coincides with above σ.

In § 2, we shall study those rings R for which every nonzero preradical (or radical, cohereditary radical, etc.) σ for R-mod is cofaithful. In § 3, as a slight weaker condition, we shall consider those rings R satisfying that, for all σ as in § 2, either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$ holds. In § 4, we shall compare those rings discussed in § 2 and § 3 and give examples to distinguish them. Some of them were investigated by several authors by different approaches. So we recall the definitions of those rings. A ring R is called *left* R if $\sigma(R)=0$ for every idempotent radical $\sigma\neq 1$ for R-mod. A ring R is called *left SP* if $\sigma(R)=0$ for every left exact preradical $\sigma\neq 1$ for R-mod. A ring R is called *left CTF* if $\sigma(R)=0$ for every left exact radical $\sigma\neq 1$ for R-mod. Some equivalent properties of those rings are summarized in [11, p 71]. A ring R is called *left ChC* if $\mathrm{Hom}_R(C_1, C_2)\neq 0$ for every nonzero cyclic modules ${}_RC_1$ and ${}_RC_2$ ([8, p 96]). A ring R is called *left SSP* if every essential left ideal of R is cofaithful ([7]).

2. Several rings classified by its preradicals (I)

To begin with, we have the following proposition, in which the equivalence of (4) and (7) was proved in [4].

PROPOSITION 2.1. *The following properties are equivalent for a ring* R:

(1) *Every nonzero preradical for* R-mod *is cofaithful.*

(2) *Every nonzero idempotent preradical for* R-mod *is cofaithful.*

(3) *Every nonzero left exact preradical for* R-mod *is cofaithful.*

(4) R *is simple artinian.*

(5) *There exist only two preradicals for* R-mod.

(6) *There exist only two idempotent preradicals for* R-mod.

(7) *There exist only two left exact preradicals for* R-mod.

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). Clear.

(3)$\Rightarrow$(4). Since soc is a nonzero left exact preradical for R-mod, where $soc(M)$ denotes the socle of $M \in R$-mod, we have $soc(E(R))=E(R)$ by (3). Hence R must be completely reducible. Suppose that R is not prime. Then we have a non-trivial ring decomposition $R=S \oplus T$. Since S is an idempotent ideal such that $(R/S)_R$ is flat, the radical σ for R-mod defined by $\sigma(M)=SM$ for all $M \in R$-mod is (left) exact. Since $\sigma(S)=SS=S$, we have $\sigma \neq 0$ and so $\sigma(E(R))=E(R)$. Hence we obtain $\sigma(R)=R \cap \sigma(E(R))=R$, i.e. $\sigma=1$. But we also have $\sigma(T)=ST=0$, which is a contradiction.

(4)$\Rightarrow$(5). Let $\sigma \neq 1$ be a preradical for R-mod. Since $\sigma(R)$ is a proper ideal of R, we have $\sigma(R)=0$. Thus for a minimal left ideal I of R, $\sigma(\mathrm{I}) \subseteq \sigma(\mathrm{R})=0$. Now for each $M \in R$-mod, there exists an index set Λ such that $M=\oplus_\Lambda M_\lambda$, where $M_\lambda \cong I$ for each $\lambda \in \Lambda$. Therefore we have $\sigma(M)=\oplus_\Lambda \sigma(M)$ $=0$, which means $\sigma=0$.

(5)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(1). Clear.

In the next theorem, we consider the ring R in which every nonzero radical for R-mod is cofaithful. Such a ring is in fact simple artinian which will be proved in § 4.

THEOREM 2. 2. *The following properties are equivalent for a ring* R:

(1) *Every nonzero radical for* R-mod *is cofaithful.*

(2) *If* $Q \in R$-mod *satisfies* $k_Q \neq 0$, *then* k_Q *is cofaithful.*

(3) *Every nonzero factor module of an injective left R-module is a cogenerator for* R-mod.

(4) *Every nonzero factor module of* $E(R)$ *is a cogenerator for* R-mod.

PROOF. Recall that, for a $Q \in R$-mod, $k_Q=0$ if and only if Q is a cogenerator for R-mod, and k_Q is cofaithful if and only if $\mathrm{Hom}_R(E(R), Q)=0$.

(1)$\Rightarrow$(3). Let ${}_RA$ be an injective module and B its proper submodule. For a fixed $x \in A \backslash B$, there exists $g \in \mathrm{Hom}_R(E(R), A)$ such that $g(1)=x$. Hence $\mathrm{Hom}_R(E(R), A/B) \neq 0$ and so the radical $k_{A/B}$ is not cofaithful. Thus $k_{A/B}$ must be zero.

(3)$\Rightarrow$(4). Clear.

(4)$\Rightarrow$(2). Assume that k_Q is not cofaithful. Take any $f(\neq 0)\in \mathrm{Hom}_R(E(R), Q)$. Then $\mathrm{Im}(f)$ is a cogenerator for R-mod. Hence Q is also a cogenerator for R-mod, and so $k_Q=0$.

(2)$\Rightarrow$(1). Let σ be a nonzero radical for R-mod. Then there exists a class η of modules in R-mod such that $\sigma=k_\eta=\cap_{Q\in\eta}k_Q$ [11]. Hence $k_Q\neq 0$ and so $k_Q(E(R))=E(R)$ for all $Q\in\eta$. Thus $\sigma(E(R))=E(R)$ as desired.

PROPOSITION 2.3. *The following properties are equivalent for a ring* R:

(1) *Every nonzero idempotent radical for* R-mod *is cofaithful* (*in this case we call* R left IRC).

(2) *Every radical* σ *for* R-mod *with* $\boldsymbol{T}(\sigma)\neq\{0\}$ *is cofaithful.*

(3) *If* ${}_RM$ *satisfies* $\boldsymbol{T}(k_M)\neq\{0\}$, *then* $E(R)\in\boldsymbol{T}(k_M)$.

(4) *If* ${}_RA$ *and* ${}_RB$ *satisfy* $\mathrm{Hom}_R(A, B)=0$, *then either* $A=0$ *or* $\mathrm{Hom}_R(E(R), B)=0$.

(5) *Every nonzero module* ${}_RA$ *satisfies* $\boldsymbol{F}(t_A)\subseteq\boldsymbol{F}(t_{E(R)})$.

(6) *Every nonzero idempotent preradical* σ *for* R-mod *satisfies* $\boldsymbol{F}(\sigma)\subseteq\boldsymbol{F}(t_{E(R)})$.

(7) *Every nonzero idempotent radical* σ *for* R-mod *satisfies* $\boldsymbol{F}(\sigma)\subseteq\boldsymbol{F}(t_{E(R)})$.

PROOF. (1)$\Rightarrow$(2). Let σ be a radical with $\boldsymbol{T}(\sigma)\neq\{0\}$. Then we notice that $\hat{\sigma}$ is an idempotent radical for R-mod with $\boldsymbol{T}(\hat{\sigma})=\boldsymbol{T}(\sigma)$.

(2)$\Rightarrow$(3). This is clear by Lemma 1.4.

(3)$\Rightarrow$(4). Let ${}_RA\,(\neq 0)$ and ${}_RB$ satisfy $\mathrm{Hom}_R(A, B)=0$. Then $0\neq A\in\boldsymbol{T}(k_B)$ and so $E(R)\in\boldsymbol{T}(k_B)$ by (3).

(4)$\Rightarrow$(5). Let ${}_RA\neq 0$ and ${}_RB\in\boldsymbol{F}(t_A)$. Then $\mathrm{Hom}_R(A, B)=0$ which implies $\mathrm{Hom}_R(E(R), B)=0$ by (4). Thus we have ${}_RB\in\boldsymbol{F}(t_{E(R)})$.

(5)$\Rightarrow$(6). Let σ be a nonzero idempotent preradical for R-mod. Then there exists a class ζ of modules in R-mod such that $\sigma=\sum_{Q\in\zeta}t_Q$ [11]. Since $\sigma\neq 0$, there exists a ${}_RQ\in\zeta$ satisfying $t_Q\neq 0$. Note that $Q\neq 0$. Now for each ${}_RX\in\boldsymbol{F}(\sigma)$, $\sigma(X)=0$ implies $t_Q(X)=0$. Hence $X\in\boldsymbol{F}(t_Q)\subseteq\boldsymbol{F}(t_{E(R)})$ by (5).

(6)$\Rightarrow$(7). Clear.

(7)$\Rightarrow$(1). Assume that σ is a nonzero idempotent radical for R-mod. Then we have $\boldsymbol{F}(\sigma)\subseteq\boldsymbol{F}(t_{E(R)})=\boldsymbol{F}(\bar{t}_{E(R)})$. Thus $\boldsymbol{T}(\sigma)\supseteq\boldsymbol{T}(\bar{t}_{E(R)})$ and so $\sigma\geqq\bar{t}_{E(R)}\geqq t_{E(R)}$. Hence σ is cofaithful by Lemma 1.4.

We called that a module ${}_RQ$ is QF-3′ if k_Q is left exact ([12]). Every cogenerator for R-mod is QF-$3'$. The next theorem deals with a ring satisfying the converse statement.

THEOREM 2.4. *The following properties are equivalent for a ring R:*

(1) *Every nonzero left exact radical for* R-mod *is cofaithful.*

(2) *Every nonzero QF-$3'$ left R-module is a cogenerator for* R-mod.

(3) *Every nonzero injective left R-module is a cogenerator for* R-mod.

(4) *There exist only two left exact radicals for* R-mod.

(5) ([6]) *R is left semiartinian and all simple left R-modules are isomorphic.*

(6) ([2]) *R is left CTF and* $\mathrm{soc}({}_RR)\neq 0$.

(7) ([5]) *R is isomorphic to the ring of all $n\times n$ matrices over a local right perfect ring for some n.*

(8) ([8]) *R is left ChC.*

PROOF. (1)$\Rightarrow$(2). Let ${}_RQ$ be a nonzero QF-$3'$ module. Then k_Q is left exact. If $k_Q\neq 0$, by (1) $Q=k_Q(Q)$, a contradiction. Hence $k_Q=0$.

(2)$\Rightarrow$(3). Clear.

(3)$\Rightarrow$(4). Let σ be a left exact radical for R-mod. It is well known that there exists an injective module ${}_RQ$ such that $\sigma=k_Q$. If $Q=0$, then $\sigma=k_0=1$. If $Q\neq 0$, then (3) implies $\sigma=k_0=0$.

(4)$\Rightarrow$(1). Clear.

In the references [6], [2], [5] and [8] it was proved that (4) is equivalent to each of (5), (6), (7) and (8).

PROPOSITION 2.5. *The following properties are equivalent for a ring R:*

(1) *Every nonzero cohereditary radical for* R-mod *is cofaithful.*

(2) *If I is a nonzero ideal of R, then $IE(R)=E(R)$.*

(3) *If L is a nonzero left ideal of R, then $LE(R)=E(R)$.*

(4) *Every nonzero left ideal of R is cofaithful.*

(5) ([16]) *Every nonzero left ideal of R generates $E(R)$.*

(6) *R is left SP.*

Note that the word "left ideal" in conditions (3), (4) *and* (5) *can be replaced by "cyclic left ideal".*

PROOF. (1)$\Leftrightarrow$(2). A preradical σ for R-mod is cohereditary if and only if $\sigma(M)=\sigma(R)M$ for all $M\in R$-mod. In this case $\sigma\neq 0$ if and only if $\sigma(R)\neq 0$. Hence we have the equivalence (1)$\Leftrightarrow$(2).

(2)$\Leftrightarrow$(3). Clear.

(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5). This is clear by Lemmas 1.1 and 1.3.

(5)$\Leftrightarrow$(6). This was proved in [16].

Now we have an analogous characterization for left *SSP* rings.

PROPOSITION 2.6. *The following properties are equivalent for a ring* R:

(1) *R is left SSP.*

(2) ([13]) *If L is an essential left ideal of R, then $LE(R)=E(R)$.*

(3) *Every essential left ideal of R generates $E(R)$.*

PROOF. This is clear by Lemmas 1.1 and 1.3.

PROPOSITION 2.7. *The following properties are equivalent for a ring* R:

(1) *Every nonzero cotorsion radical for R*-mod *is cofaithful* (*in this case we call R* left CC).

(2) *If I is a nonzero idempotent ideal of R, then $IE(R)=E(R)$.*

(3) *Every nonzero idempotent ideal of R generates $E(R)$.*

(4) *Every nonzero idempotent ideal of R is cofaithful.*

PROOF. Recall that a preradical σ for R-mod is cotorsion if and only if $\sigma(R)$ is an idempotent ideal and $\sigma(M)=\sigma(R)M$ for all $M\in R$-mod. Hence we have the equivalences by Lemmas 1.1 and 1.3.

PROPOSITION 2.8. *The following properties are equivalent for a ring* R:

(1) *Every nonzero exact radical for R*-mod *is cofaithful.*

(2) *There exist only two exact radicals for R*-mod (*i.e., R is left* E2 [11, Theorem 5.4]).

PROOF. (2)$\Rightarrow$(1). Clear.

(1)$\Rightarrow$(2). Let σ be a nonzero exact radical for R-mod. Since σ is left exact and cofaithful, $\sigma(R)=R\cap\sigma(E(R))=R\cap E(R)=R$, proving $\sigma=1$.

3. Several rings classified by its preradicals (II)

Let ${}_RM$ be a module. A submodule ${}_RN$ of M is called a *torsion submodule* of M if $N=\sigma(M)$ for some idempotent radical σ for R-mod. It is known that N is a torsion submodule of M if and only if $\mathrm{Hom}_R(N, M/N)=0$ ([2, Proposition 2.6] or [9, Lemma 1]). Obviously if R is left *IRC*, then every injective left R-module has no non-trivial torsion submodules. It was proved that a ring R is left R if and only if ${}_RR$ has no non-trivial torsion submodules ([2, Proposition 1.10]). Now the next proposition is clear.

PROPOSITION 3.1. *The following properties are equivalent for a ring R:*

(1) *$E(R)$ has no non-trivial torsion submodules, i.e., if σ is an idempotent radical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$ (in this case we call R* left DR).

(2) *If K is a non-trivial submodule of $E(R)$, then* $\mathrm{Hom}_R(K, E(R)/K) \neq 0$.

THEOREM 3.2. *The following properties are equivalent for a ring R:*

(1) *If σ is a preradical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(2) *If σ is an idempotent preradical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(3) *If σ is a left exact preradical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$, i.e., σ is either faithful or cofaithful.*

(4) *If σ is a radical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(5) *If σ is a cohereditary radical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(6) *R is left SP.*

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). Clear.

(3)$\Rightarrow$(4). Let σ be a radical for R-mod. Then a left exact preradical $h(\sigma)$ satisfies either $h(\sigma)(E(R))=0$ or $h(\sigma)(E(R))=E(R)$ by (3). Hence we obtain (4) by noticing that $h(\sigma)(E(R))=\sigma(E(R))$.

(4)$\Rightarrow$(5). Clear.

(5)$\Rightarrow$(6). Let I be a nonzero ideal of R. Consider a cohereditary radical σ for R-mod defined by $\sigma(M)=IM$ for each $M\in R$-mod. Since $\sigma\neq 0$, we have $IE(R)=\sigma(E(R))=E(R)$ by (5). Hence R is left SP by Proposition 2.5.

(6)$\Rightarrow$(1). Let σ be a preradical for R-mod. Since $h(\sigma)$ is left exact, we have either $h(\sigma)(R)=0$ or $h(\sigma)=1$. In the former case we have $\sigma(E(R))=0$. In the latter case we have $\sigma(E(R))=h(\sigma)(E(R))=E(R)$.

PROPOSITION 3.3. *The following properties are equivalent for a ring R:*

(1) *If σ is a left exact radical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(2) *Every nonzero injective left R-module cogenerates $E(R)$.*

(3) *Every nonzero injective left R-module cogenerates all projective left R-modules.*

(4) *R is left CTF.*

PROOF. (1)$\Rightarrow$(2). Let ${}_RQ$ be a nonzero injective module. Then we obtain the left exact radical $k_Q \neq 1$ for R-mod. By assumption $k_Q(E(R))=0$, which means that Q cogenerates $E(R)$.

(2)$\Rightarrow$(3). This is clear by noticing that, if a module cogenerates ${}_RR$, then it also cogenerates all (free and so) projective left R-modules.

(3)$\Rightarrow$(4). For each left exact radical $\sigma \neq 1$ for R-mod, there exists an injective module ${}_RQ$ such that $\sigma = k_Q$. Since ${}_RQ \neq 0$, R is cogenerated by Q. Using $Q \in \boldsymbol{F}(\sigma)$, we have $R \in \boldsymbol{F}(\sigma)$.

(4)$\Rightarrow$(1). Let σ be a left exact radical for R-mod. By assumption either $\sigma=1$ or $\sigma(R)=0$. Thus we obtain (1), since $\boldsymbol{F}(\sigma)$ is closed under injective hulls.

Recall that, if τ is a left exact preradical for R-mod, then $\tau(M)$ is essential in $\bar{\tau}(M)$ for all $M \in R$-mod.

COROLLARY 1. *Let R be a left CTF ring. If σ is a preradical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))$ is essential in $E(R)$.*

PROOF. For a preradical σ for R-mod, consider the left exact radical $\overline{h(\sigma)}$ for R-mod. Then either $\overline{h(\sigma)}(E(R))=0$ or $\overline{h(\sigma)}(E(R))=E(R)$. In the former case we have $\sigma(E(R))=h(\sigma)(E(R))=0$. In the latter case, since $h(\sigma)$ is left exact, $\sigma(E(R))=h(\sigma)(E(R))$ is essential in $E(R)$.

Recall also that a ring R is *left Kasch* if and only if $E(R)$ is a cogenerator for R-mod.

COROLLARY 2. (1) *A ring R is left ChC if and only if R is left CTF and left Kasch.*

(2) *Let R be a left CTF ring. Then R is left Kasch if and only if* $\mathrm{soc}({}_RR) \neq 0$.

PROOF. (1) The "only if" part is clear by Theorem 2.4. To prove the "if" part, let ${}_RQ \neq 0$ be an injective module. Since Q cogenerates $E(R)$ by Proposition 3.3 and $E(R)$ is a cogenerator for R-mod, Q is a cogenerator for R-mod. Hence R is left *ChC* by Theorem 2.4.

(2) This is clear by Theorem 2.4 combined with above (1).

PROPOSITION 3.4. *The following properties are equivalent for a ring R:*

(1) *If σ is a cotorsion radical for* R-mod, *then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(2) *R is left CC.*

PROOF. This is clear by Proposition 2. 7.

PROPOSITION 3. 5. *The following properties are equivalent for a ring R:*

(1) *If σ is an exact radical for R-mod, then either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$.*

(2) *R is left E2.*

PROOF. The same proof of Proposition 2. 8 is valid.

4. Some relations and examples

In this section, we shall compare those rings discussed in the previous sections. First of all we shall prove the following

PROPOSITION 4. 1. (1) *A left IRC ring is left DR.*

(2) *A left SP ring is left DR.*

(2) *A left DR ring is left CTF.*

(4) *A left DR ring is left CC.*

PROOF. (1). Clear.

(2). Suppose that R is a left SP ring and σ is an idempotent radical such that $\sigma(E(R))\neq 0$. Since $h(\sigma)$ is left exact and $h(\sigma)(R)=R\cap\sigma(E(R))\neq 0$, $h(\sigma)$ must be 1 by assumption. Hence we have $E(R)=h(\sigma)(E(R))=\sigma(E(R))$, which means that R is left DR.

(3). Let σ be a left exact radical for R-mod. Since σ is an idempotent radical, we have either $\sigma(E(R))=0$ or $\sigma(E(R))=E(R)$. In the former case we have $\sigma(R)=0$. In the latter case, since σ is left exact, we have $\sigma(R)=R\cap\sigma(E(R))=R$, i. e. $\sigma=1$.

(4). Let R be a left DR ring. Assume that I is a nonzero idempotent ideal of R. Consider the cotorsion radical σ for R-mod defined by $\sigma(M)=IM$ for all $M\in R$-mod. Since σ is an idempotent radical and $\sigma(E(R))=IE(R)\supseteq IR=I\neq 0$, we have $IE(R)=\sigma(E(R))=E(R)$ by assumption.

In the next proposition, we give another characterizations of left IRC rings.

PROPOSITION 4. 2. *The following properties are equivalent for a ring R:*

(1) *R is left IRC.*

(2) *R is left DR and left ChC.*

(3) *R is left DR and* $\mathrm{soc}({}_RR)\neq 0$.

(4) *R is left DR and left Kasch.*

PROOF. (1)$\Rightarrow$(2). Clear.

(2)$\Rightarrow$(3). Each left *ChC* ring R satisfies $soc({}_RR)\neq 0$ by Theorem 2.4.

(3)$\Leftrightarrow$(4). Since "left *DR*" implies "left *CFT*" by Proposition 4.1, this is proved by applying Corollary 2 of Proposition 3.3.

(4)$\Rightarrow$(1). Let σ be an idempotent radical for R-mod with $\sigma(E(R))=0$. We have to show that $\sigma=0$. Since $E(R)\in \boldsymbol{F}(\sigma)$ and $E(R)$ is a cogenerator for R-mod, every left R-module is σ-torsionfree.

THEOREM 4.3. *The following properties are equivalent for a ring* R:

(1) *Every nonzero radical for* R-mod *is cofaithful* (*see* Theorem 2.2).

(2) *R is left SP and left ChC.*

(3) *R is left SP and left Kasch.*

(4) *R is left SP and* $\mathrm{soc}({}_RR)\neq 0$.

(5) *R is simple artinian.*

(6) *There exist only two radicals for* R-mod.

(7) *R is left ChC and left weakly regular.*

PROOF. (1)$\Rightarrow$(2). This is clear by Proposition 2.5 and Theorem 2.4.

(2)$\Rightarrow$(3). Every left *ChC* ring is left Kasch by Theorem 2.4.

(3)$\Rightarrow$(4). Since "left *SP*" implies "left *CTF*" by definitions, this is proved by applying Corollary 2 of Proposition 3.3.

(4)$\Rightarrow$(5). By Proposition 2.5, $soc({}_RR)$ generates $E(R)$. Hence $E(R)$ is completely reducible, and so is R. But since R is prime, we see that R is simple artinian.

(5)$\Rightarrow$(6). This is clear by Proposition 2.1.

(5)$\Rightarrow$(7) and (6)$\Rightarrow$(1). Clear.

(7)$\Rightarrow$(2). Since "left *ChC*" implies "left $E2$", R is a simple ring by [11, Corollary 5.7]. Thus R is left *SP*.

PROPOSITION 4.4. *The following properties are equivalent for a ring* R:

(1) *R is left SP.*

(2) *R is left SSP and left CTF.*

PROOF. (1)$\Rightarrow$(2). Clear.

(2)$\Rightarrow$(1). Let $\sigma\neq 1$ be a left exact preradical for R-mod. Since R is left *SSP*, we obtain $\bar{\sigma}\neq 1$ by [7, Theorem 1]. Now since R is left *CTF*, we have $\bar{\sigma}(R)=0$ and so $\sigma(R)=0$. Hence R is left SP.

Finally, we shall give some examples to distinguish those rings to be refered.

EXAMPLE 4.1. We have an example of left IRC ring which is not left SP. Put $R=\left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \middle| a, b \in K\right\}$, where K is a field. Then R is a commutative artinian ring with a unique non-trivial ideal $J=\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$. Moreover R is self-injective and satisfies $\mathrm{Hom}_R(J, R/J)\neq 0$. Thus R is (left) DR, and so is IRC by Proposition 4.2. But since R is not semiprime, R is not SP.

EXAMPLE 4.2. Consider the ring $\boldsymbol{Z}$ of integers. Clearly $\boldsymbol{Z}$ is SP but not ChC.

EXAMPLE 4.3. There exists a left DR ring which is not left SSP. To prove this, let $R=\boldsymbol{Z}/(p^n)$, where p is a prime and n is an integer greater than 1. Since R is left R ([11, Example 3.9]) and $E(R)=R$, we see that R is (left) DR. But R is not SSP, because $IE(R)\neq E(R)$ for some (and every) proper essential ideal I of R.

EXAMPLE 4.4. There exists a left SSP ring which is not left CC. Let $R=\boldsymbol{Z}\times\boldsymbol{Z}$. Since $\boldsymbol{Z}$ is (left) SP, R is left SSP ([7, Theorem 1]). By a routine verification, we have $E(R)=\boldsymbol{Q}\times\boldsymbol{Q}$, where $\boldsymbol{Q}$ is the additive group of rational numbers. Now $I=(\boldsymbol{Z}, 0)$ is a nonzero idempotent ideal and does not generate $E(R)$, and so R is not CC.

EXAMPLE 4.5. We shall give an example of left CC ring which is not left CTF. Consider $R=\boldsymbol{Z}\times\boldsymbol{Q}/\boldsymbol{Z}$. Define the addition on R by component wise and the multiplication on R by

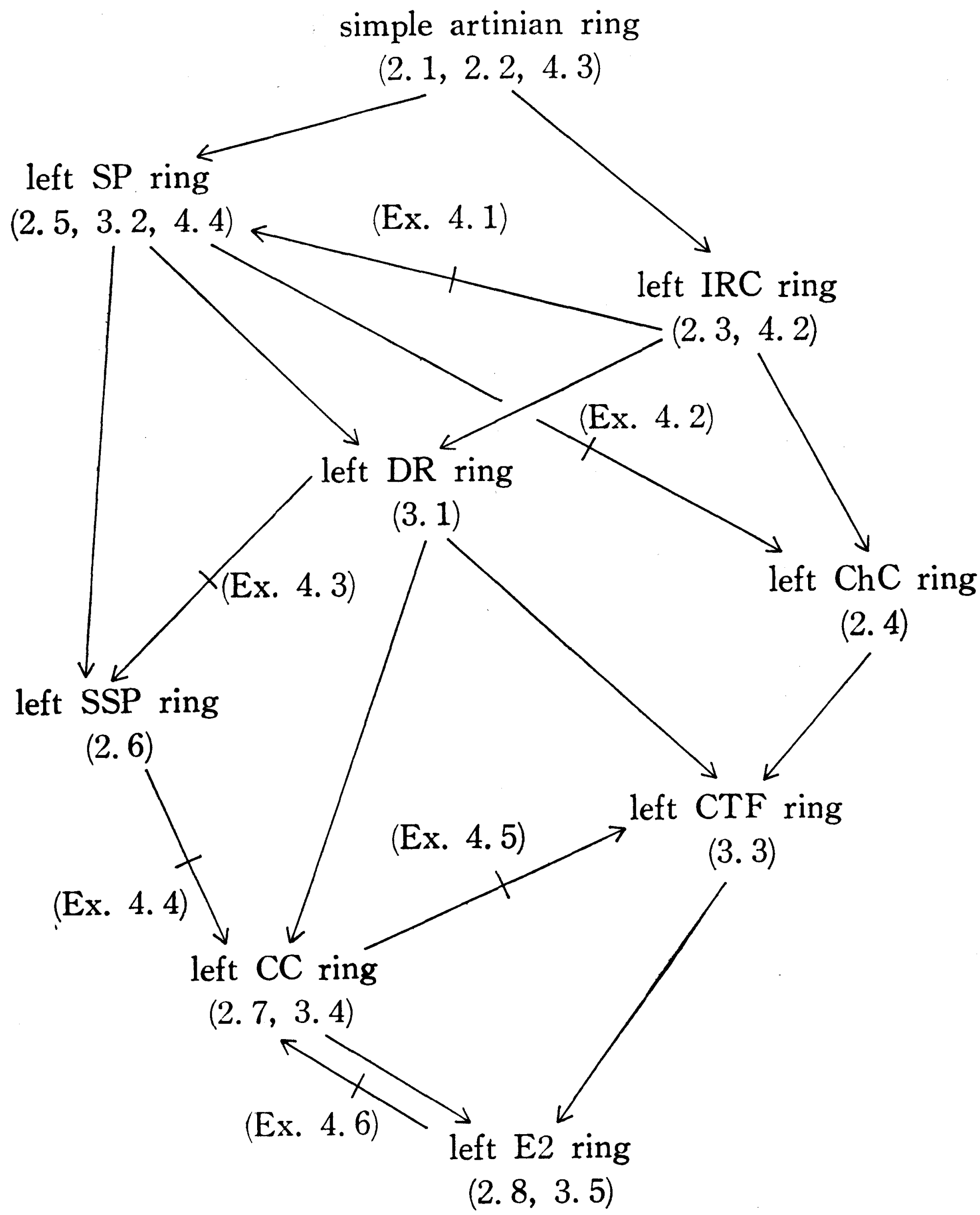
$$(a, x)\cdot(b, y)=(ab, ay+bx).$$

Then R is a commutative ring ([14, p 45]) without non-trivial idempotent ideals, and so R is CC. Since for $0\neq w=(0, 1/2+\boldsymbol{Z})\in R$, $\mathrm{Ann}_R(w)=(2Z, \boldsymbol{Q}/\boldsymbol{Z})$ is not T-nilpotent, R is not CTF by [2, Corollary 1.4].

EXAMPLE 4.6. Every left CC ring is left $E2$. But the converse is not true. For a counter example, let $R=\left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a \end{pmatrix} \middle| a, b, c, d \text{ and } e\in K\right\}$, where K is a field. It was proved that R is a left $E2$ ring [11, Example 5.9]. By a routine computation, we have $E(R)=\left\{\begin{pmatrix} a & f & 0 \\ b & c & 0 \\ d & e & a \end{pmatrix} \middle| \right.$ a, b, c, d, e and

$f \in K\Big\}$. Since $I=\begin{pmatrix} 0 & 0 & 0 \\ K & K & 0 \\ K & K & 0 \end{pmatrix}$ is an idempotent ideal of R and $IE(R)=I \neq E(R)$, we see that R is not left CC.

A table of rings



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