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On H-separable extensions of two sided simple rings

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§1. **Introduction.** Throughout this paper A is a ring with the identity 1, and B is a subgring of A such that $1\in B$. Each B-module (or A-module) is unitary, and each $A-A$-module M satisfies that $(am)b=a(mb)$ for a, $b\in A$ and $m\in M$. In addition we will set $C=V_A(A)$, the center of A, and $D=V_A(B)$ the centralizer of B in A

We say that A is an H-separable extension of B in the case where ${}_AA\otimes_BA_A<\oplus_A(A\oplus A\oplus\cdots\oplus A)_A$ (direct summand of a finite direct sum of copies of A). As for some characterizations and properties of H-separable extension see for example [3], [6], [9] and [10].

In this paper we will deal with H-separable extensions of two sided simple rings. In particular, in the case where B is a two sided simple ring we will show that A is right B-finitely generated projective and an H-separable extension of B, if and only if A is a two sided simple ring, $V_A(V_A(B))=B$ and $V_A(B)$ is a simple C-algebra (Theorem 1). Furthermore, under the conditions of Theorem 1 we will show that for any simple C-subalgebra T of D, $V_A(T)$ is two sided simple, $V_A(V_A(T))=T$ and A is an H-separable extension of $V_A(T)$ and right $V_A(T)$-finitely generated projective (Proposition 2). Finally, under the same conditions we will obtain a duality on two sided simple subrings, which is similar to the well known classical inner Galois theory on simple (artinian) rings (Theorem 2).

§2. We say that A is a two sided simple ring in case A has no proper two sided ideal except 0, and a right artinian two sided simple ring with 1 is called a simple ring. Whenever we call A a simple algebra over a field K, A shall be a K-algebra which is two sided simple and $[A:K]<\infty$. Hereafter we will call each two sided ideal simply an ideal.

Given a right A-module M, set $\Omega=\mathrm{Hom}(M_A, M_A)$. Then, as is well known, M is an $\Omega-A$-module, and we have an $A-A$-map

$$\tau:\ \mathrm{Hom}(M_A, A_A)\otimes_\Omega M\longrightarrow A$$

such that $\tau(f\otimes m)=f(m)$ for $f\in\mathrm{Hom}(M_A, A_A)$ and $m\in M$. $\mathrm{Im}\,\tau$ is an ideal of A, and $\mathrm{Im}\,\tau=A$ if and only if M is a right A-generator. Therefore if A is two sided simple and $\mathrm{Hom}(M_A, A_A)\neq 0$, we have $0\neq\mathrm{Im}\,\tau=A$. Thus we have

REMARK 1. Let A be a two sided simple ring and M a right A-module such that $\mathrm{Hom}(M_A, A_A)\neq 0$. Then M is a generator of the category of right A-modules.

PROPOSITION 1. *Let B be a two sided simple ring and A an H-separable extension of B such that* $\mathrm{Hom}(A_B, B_B)\neq 0$. *Then A is also a two sided simple ring.*

PROOF. By Remark 1 A is a right B-generator. Hence $B_B<\oplus A_B$ (right B-direct summand) by Lemmal [4]. Then for any ideal $\mathfrak{a}$ of A, we have $\mathfrak{a}=(\mathfrak{a}\cap B)A$ by Theorem 4.1 [10]. But B is two sided simple. Hence $\mathfrak{a}\cap B=0$. Thus $\mathfrak{a}=0$. Hence A is also two sided simple.

COROLLARY 1. (Corollary 3.1 [10]). *Let B be a two sided simple ring, and suppose that A is an H-separable extension of B. Then if A is left or right B-projective, A is also a two sided simple ring.*

REMARK 2. Theorem 4.1 [10] has already shown that, if A is an H-separable extension of a two sided simple ring B such that $B_B<\oplus A_B$ (or $B_B<\oplus_B A$), then A is also a two-sided simple ring.

Given an $A-A$-module M and a subset X of A, we set

$$M^A=\left\{m\in M\,\middle|\, am=ma \text{ for all } a\in A\}\right\}$$

$$V_A(X)=\left\{a\in A\,\middle|\, xa=ax \text{ for all } x\in X\right\}$$

respectively.

REMARK 3. For any ring A, its subring B, $C=V_A(A)$ and $D=V_A(B)$, there exists a ring homomorphism

$$\eta:\ A\otimes_C D^0\longrightarrow \mathrm{Hom}(A_B, A_B)$$

such that $\eta(a\otimes d^0)(x)=axd$, for any a, $x\in A$, $d\in D$, where D^0 is the opposite ring of D.

K. Hirata showed that if A is an H-separable extension of B, η is an isomorphism and D is C-finitely generated projective (See Theorem 2 [2] and Proposition 3.1 [3]). Furthermore, in the case where A is right B-finitely generated projective, A is an H-separable extension of B if and only if D is C-finitely generated projective and η is an isomorphism by Corollary 3 [7].

REMARK 4. Let R be a commutative artinian ring with 1 with its Jacobson radical J. Then, $R=Re_1\oplus Re_2\oplus\cdots\oplus Re_n$, where each e_i is a primitive idempotent, and Je_i is the unique maximal R-submodule of Re_i. Furthermore each isomorphism class of simple R-module is of the form Re_i/Je_i for some i (see 54.11 and 54.13 [1]). Now denote $R_i=Re_i$ and $\mathfrak{m}_i=Je_i$. Since R is commutative, each R_i is an artinian local ring with its unique maximal ideal $\mathfrak{m}_i$. R_i contains at least one minimal ideal. But there is only one isomorphism class of simple R_i-module, namely, $R_i/\mathfrak{m}_i$. Therefore each R_i contains an ideal isomorphic to $R_i/\mathfrak{m}_i$. This means that R contains each isomorphism class of simple R-modules.

THEOREM 1. *Let B be a two sided simple ring. Then, A is an H-separable extension of B which is right B-finitely generated projective, if and only following three conditions are satisfied;*

(1) *A is a two sided simple ring*

(2) $V_A(V_A(B))=B$

(3) *$V_A(B)$ is a simple C-algebra.*

PROOF. First suppose that A is an H-separable extension of B and A is right B-finitly generated projective. Then since $\mathrm{Hom}(A_B, B_B)\neq 0$, A is a right B-generator, and consequently, $B_B<\oplus A_B$, which implies that $V_A(V_A(B))=B$ by Proposition 1.2 [6]. A is a two sided simple ring by Corollary 1. Thus (1) and (2) are satisfied. Since A is a right B-progenerator, $\mathrm{Hom}(A_B, A_B)$ is also a two sided simple ring by Morita theorem. But there is a ring isomorphism of $A\otimes_C D^0$ to $\mathrm{Hom}(A_B, A_B)$ (See Remark 3). Hence $A\otimes_C D^0$ is a two sided simple ring. Then it is clear that D has no non zero proper ideal, since C is a field. Theorem 2 [2] shows that $[D: C]<\infty$. Thus we have (3). Conversely, assume conditions (1), (2) and (3). By Remark 3 there is a ring homomorphism η of $A\otimes_C D^0$ to $\mathrm{Hom}(A_B, A_B)$. We see that $A\otimes_C D^0$ is a two sided simple ring by (1) and (3) and by Proposition 4.3 [14]. Therefore $\mathrm{Ker}\,\eta=0$. Set $\Lambda=A\otimes_C D^0$, and let Z be the center of D. Z is a finite field extension of C. Hence $Z\otimes_C Z$ is a commutative artinian ring. Therefore $Z\otimes_C Z$ contains all of the isomorphism classes of its simple modules. Hence $\mathrm{Hom}({}_{Z\otimes Z}Z, {}_{Z\otimes Z}Z\otimes_C Z)\neq 0$, which means that there exists $0\neq\Sigma x_i\otimes y_i\in Z\otimes_C Z$ such that $\Sigma xx_i\otimes y_i=\Sigma x_i\otimes y_i x$ for all $x\in Z$. On the other hand since D is a central simple Z-algebra, we have $(A\otimes_C D)^Z=D(A\otimes_C D)^D$ regarding $A\otimes_C D$ as a $D-D$-module by $y(a\otimes d)z=ya\otimes dz$ for d, y, $z\in D$ and $a\in A$. Then since C is a field, $0\neq\Sigma x_i\otimes y_i\in(Z\otimes_C Z)^Z\subseteq(A\otimes_C D)^Z$. Hence $(A\otimes_C D)^D\neq 0$. This means that $\mathrm{Hom}({}_A A_D, {}_A A\otimes_C D_D)=\mathrm{Hom}({}_\Lambda A, {}_\Lambda\Lambda)\neq 0$. Therefore, A is a left Λ-generator by Remark 1, and consequently, A is right $\mathrm{Hom}({}_\Lambda A, {}_\Lambda A)$-finitely generated projective by Morita theorem. But $\mathrm{Hom}({}_\Lambda A, {}_\Lambda A)=\mathrm{Hom}({}_A A_D, {}_A A_D)\cong V_A(D)=B$ by (2).

Thus we see that A is right B-finitely generated projective. Finally, since A is a left Λ-generator and $B\cong\mathrm{Hom}({}_{\Lambda}A, {}_{\Lambda}A)$, we have an isomorphism $\Lambda\cong\mathrm{Hom}(A_B, A_B)$ by Morita theorem. This isomorphism is exactly equal to η. Then, since η is an isomorphism and A is right B-finitely generated projective, A is an H-separable extension of B by Corollary 3 [7].

Theorem 1 includes Theorem (1. 5) [8] and Theorem 2. 1 [10], which have intimate relations with the "Fundamental theorem on simple rings".

COROLLARY 2 (Theorem (1. 5) [8], Theorem 2. 1 [10]). *Let B be a simple (artinian) ring. Then A is an H-separable extension of B, if and only if following three conditions are satisfied;*

(1) *A is a simple ring*

(2) $V_A(V_A(B))=B$

(3) *$V_A(B)$ is a simple C-algebra.*

PROOF. Since B is a simple ring, B is left (as well as right) B-injective. Therefore, we have ${}_BB<\oplus{}_BA$ (and $B_B<\oplus A_B$). Therefore, if A is an H-separable extension of B, A is right (as well as left) B-finitely generated by Theorem 4. 1 [10]. Hence A is artinian, and A is right B-finitely generated projecive. Thus we have (1), (2) and (3). The converse is also clear.

REMARK 5. Theorem 2 [12] shows that, under the same conditions as Theorem 1, all ring automorphisms of A which fixes all elements of B are inner automorphisms. This fact has been well known under the conditions of Corollary 2.

REMARK 6. Theorem 1 shows that, in the case where A is an H-separable extension of a two sided simple ring B, A is right B-finitely generated projective if and only if A is left B-finitely generated projective.

§ **3.** In this section we will deal with simple C-subalgebras of D under the conditions of Theorem 1.

PROPOSITION 2. *Let B be a two sided simple ring and A an H-separable extension of B, and suppose that A is right B-finitely generated projective. Then for any simple C-subalgebra T of D, we have*

(1) *$V_A(T)$ is a two sided simple ring*

(2) $V_A(V_A(T))=T$

(3) *A is an H-separable extension of $V_A(T)$ and right $V_A(T)$-finitely generated projective.*

PROOF. Since T is simple, D is right (as well as left) T-finitely generated projective. Therefore, $A\otimes_C D^0$ is left $A\otimes_C T^0$-finitely generated projective.

But A is left $A\otimes_C D^0$-finitely generated projective, because A is a right B-generator and $A\otimes_C D^0\cong\mathrm{Hom}(A_B, A_B)$. Then, A is left $A\otimes_C T^0$-finitely generated projective. Set $\Gamma=A\otimes_C T^0$ and $S=V_A(T)$. Γ is a two sided simple ring, since A and T are so and $C=$ the center of A. Hence A is a left Γ-generator by Remark 1. Then by Morita theorem A is a right $\mathrm{Hom}({}_\Gamma A, {}_\Gamma A)$-progenerator and $\mathrm{Hom}({}_\Gamma A, {}_\Gamma A)$ is also a two sided simple ring. But $\mathrm{Hom}({}_\Gamma A, {}_\Gamma A)\cong V_A(T)=S$. Thus we have shown that A is right S-finitely generated projective and that S is a two sided simple ring. Furthermore, since A is a left Γ-generator and $S\cong\mathrm{Hom}({}_\Gamma A, {}_\Gamma A)$, we have an isomorphism $\Gamma\cong\mathrm{Hom}(A_S, A_S)$. This isomorphism is given by $\eta'(a\otimes t^0)(x)=axt$, for a, $x\in A$ and $t\in T$. Set $T'=V_A(V_A(T))$, and consider the following maps

$$\begin{array}{ccccc} A\otimes_C T^0 & \subseteq & A\otimes_C T'^0 & \subseteq & A\otimes_C D^0 \\ \downarrow \eta' & & \downarrow \eta'' & & \downarrow \eta \\ \mathrm{Hom}(A_S, A_S) & = & \mathrm{Hom}(A_S, A_S) & \subseteq & \mathrm{Hom}(A_B, A_B) \end{array}$$

where η, η' and η'' are all defined as in Remark 3, and η and η' are isomorphisms. Then it is obvious that $A\otimes_C T'=A\otimes_C T$, and consequently, $T=T'$. Thus we have that $T=V_A(S)$ and $A\otimes_C T^0\cong\mathrm{Hom}(A_S, A_S)$ with A right S-finitely generated projective. Hence A is an H-separable extension of $V_A(T)$ by Corollary 3 [7].

Given any subring S of A, we say that S is a left relatively separable extension of B in A, if $B\subset S\subset A$ and the map π of $S\otimes_B A$ to A such that $\pi(s\otimes a)=sa$, for $s\in S$ and $a\in A$, splits as $S-A$-map. Both left and right relatively separable extensions are called simply relatively separable extensions. Now summarizing Theorem 1 and Proposition 2, we have

THEOREM 2. *Let B be a two sided simple ring and A an H-separable extension of B such that A is right, and consequently left, B-finitely generated projective. Denote by $\mathfrak{T}$ the class of all simple C-subalgebras of D, and by $\mathfrak{S}_r$ the class of all two sided simple subrings of A which are right relatively separable extensions of B in A. Then, the maps Ψ of $\mathfrak{S}_r$ to $\mathfrak{T}$ and Φ of $\mathfrak{T}$ to $\mathfrak{S}_r$ defined by $\Psi(S)=V_A(S)$, $\Phi(T)=V_A(T)$ for $S\in\mathfrak{S}_r$ and $T\in\mathfrak{T}$, are mutually inverse one to one correspondences.*

PROOF. Let $T\in\mathfrak{T}$. Then we see ${}_T T<\oplus{}_T D$ and $T_T<\oplus D_T$. Hence $V_A(T)$ is a left and right relatively separable extension of B in A by Proposition 2.1 (2) [10]. On the other hand let $S\in\mathfrak{S}_r$. Then since ${}_A A_S<\oplus {}_A A\otimes_B S_S$, A is right S-finitely generated projective, and furthermore $A\otimes_S A <\oplus(A\otimes_B S)\otimes_S A=A\otimes_B A<\oplus(A\oplus A\oplus\cdots\oplus A)$ as $A-A$-modules. Thus A is an H-separable extension of S. Therefore, we can apply Theorem 1 and

Proposition 2.

Finally, we will give some examples of ring extensions which satisfy the conditions of Theorem 1. For any two sided simple ring B with its center Z, the $n\times n$-full matrix ring $(B)_n$ over B is a trivial example. Because, $(B)_n\cong B\otimes_Z(Z)_n$, and $(Z)_n$ is a central separable Z-algebra (See Proposition 1.7 [6]). The other example is

EXAMPLE 1. Let B be a two sided simple ring such that the characteristic of its center is not 2, and set $A=B\oplus Bi\oplus Bj\oplus Bk$, where i, j and k commute with all elements of B and satisfy $i^2=j^2=k^2=-1$ and $ij=k=-ji$. Denote the center of B by Z, and set $D=Z\oplus Zi\oplus Zj\oplus Zk$. Then since char $Z\neq 2$, D is a central simple Z-algebra. In fact, by $1/4\ (1\otimes 1-i\otimes i-j\otimes j-k\otimes k)\in(D\otimes_Z D)^D$ and $1/4\ (1-i^2-j^2-k^2)=1$, we see that D is a separable Z-algebra, while we have $Z=V_D(D)$ by direct computations. Then, since $A=B\otimes_Z D$ with D central separable over Z, A is an H-separable extension of B which is left (and right) B-finitely generated projective (See Proposition 1.7 [6]). A is not artinian if B is not so.

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