Hokkaido Mathematical Journal Vol. 11 (1982) p. 301-327

Infinitesimal automorphisms of $\mathfrak{g}$-structures and certain intransitive infinite Lie algebra sheaves

By Kazuhiro KISO

(Received January 19, 1982)

Introduction

Let $\mathscr{L}$ be an intransitive infinite Lie algebra sheaf on a manifold M and $\rho: M\rightarrow N$ be the fibered manifold of invariants of $\mathscr{L}$. For $t\in N$ we denote by $\mathscr{L}(t)$ the transitive Lie algebra sheaf on the fiber $\rho^{-1}(t)$ induced by $\mathscr{L}$. In [1] we determined $\mathscr{L}$ under the condition that for certain $o\in N$ $\mathscr{A}=\mathscr{L}(o)$ is "simple".

On the other hand T. Morimoto [2] determined the intransitive formal Lie algebras over C whose transitive parts are primitive. In this paper we consider everything in the framework of the C^{∞}-category and consider such $\mathscr{L}$ that for certain $o\in N$ $\mathscr{A}=\mathscr{L}(o)$ is one of the following Lie algebra sheaves:

(1) $\mathscr{L}_{gl(\mathbf{R}^{r}),sl(\mathbf{R}^{r})^{(1)}}$; the Lie algebra sheaf of all vector fields with constant divergence.

(2) $\mathscr{L}_{csp(\mathbf{R}^{r})}$; the Lie algebra sheaf of all vector fields which preserve a symplectic form up to constant factors.

In (2) r is assumed to be even. $\mathscr{A}$ is primitive and is not simple. Notice that besides the above $\mathscr{A}$ there are four primitive Lie algebra sheaves which are not simple. (See [3]). These cases will be treated in a future paper.

Let Ω be the volume element or the symplectic form on $\mathbf{R}^{r}$. Let N be a manifold. Let X be a local vector field on $N\times\mathbf{R}^{r}$ tangent to the fibers of the fibering $N\times\mathbf{R}^{r}\rightarrow N$. Let $Q_{k_0}\subset J_{k_0}(N\times\mathbf{R})$ be a formally integrable and integrable homogeneous linear differential equation on N, where $J_{k_0}(N\times\mathbf{R})$ means the bundle of k_0-jets of cross sections of the trivial vector bundle $N\times\mathbf{R}\rightarrow N$. Let $\mathscr{A}[N;Q_{k_0}]$ denote the sheaf of germs of all vector fields X satisfying the following condition; there exists a local solution f_X of Q_{k_0} such that $L_X\Omega=f_X\Omega$, where L_X means the Lie derivative along the fibers.

Then we will prove under certain conditions the following (Theorem 2): Let $\mathscr{L}$ be an intransitive Lie algebra sheaf whose parameter space is N.

Suppose that for certain $o \in N$ $\mathscr{A} = \mathscr{L}(o)$ is one of the Lie algebra sheaves (1) and (2). Then there exists a formally integrable and integrable homogeneous linear differential equation Q_{k_0}, and $\mathscr{L}$ is locally equivalent to $\mathscr{A}[N; Q_{k_0}]$.

In § 1 we review the fundamental facts on $\mathfrak{g}$-structures and in § 2 we consider the infinitesimal automorphisms of $\mathfrak{g}$-structures. In § 3 we give the definition of continuous Lie lagebra sheaves and state the main theorem. §4∼§ 7 are devoted to the proof of the main theorem.

§ 1. $\mathfrak{g}$-structures

In the following we always assume the differentiability of class C^∞ unless otherwise stated.

In this section we will review the fundamental facts on $\mathfrak{g}$-structures and their structure functions. For the details we refer to [1].

A fibered manifold means a triple (M, N, ρ) of differentiable manifolds M, N and a differentable map $\rho: M \to N$ whose rank is equal to the dimension of N at any point. Let (M, N, ρ) be a fibered manifold and let $m = \dim M$, $n = \dim N$. Let $V = \mathbf{R}^m$ and $W = \mathbf{R}^{m-n}$ ($\subset V$). Let h be a Lie algebra and $\mathfrak{g}$ be a subbundle of the trivial vector bundle $N \times h \to N$. Set

$$\mathfrak{g}(t) = \left\{ A \in h \,\middle|\, (t, A) \in \mathfrak{g} \right\}.$$

Then $\mathfrak{g}$ is called an N-subalgebra of h if $\mathfrak{g}(t)$ is a subalgebra of h for all $t \in N$.

For a manifold M, TM (resp. T^*M) denotes the tangent bundle (resp. the cotangent bundle) of M. We denote by $F(M)$ the frame bundle of M, which is a principal bundle with structure group $GL(V)$ over M. Let $\pi: F(M) \to M$ be the natural projection. $A \in gl(V)$ defines a vertical vector field A^* on $F(M)$ induced from the right action of $GL(V)$ on $F(M)$. A local transformation ϕ of M induces local transformation $\tilde{\phi}$ of $F(M)$ defined by $\tilde{\phi}(p)(v) = \phi_* p(v)$, where $p \in F(M)$ and $v \in V$. Hence a local vector field X on M defines a local vector field on $F(M)$. We denote it by $\tilde{X}$. Let $\rho: M \to N$ be a fibered manifold. Then, V and W being as above, we set

$$F(M, N) = \left\{ p \in F(M) \,\middle|\, \rho_* p(W) = 0 \right\}.$$

Let $\mathfrak{g}$ be an N-subalgebra of $W \otimes V^*$.

Definition 1.1. *A submanifold P of $F(M, N)$ is called a $\mathfrak{g}$-structure if it satisfies the following conditions:*

(1) *$\pi: P \to M$ is a fibered manifold.*

(2) *For $p \in P$ and $A \in gl(V)$, $A^*_p \in T_p P$ if and only if $A \in \mathfrak{g}(t)$, where*

$t=\rho\circ\pi(p)$.

Let $P\subset F(M, N)$ be a $\mathfrak{g}$-structure.

DEFINITION 1. 2. *A local vector field X on M is called an infinitesimal automorphism of P if and only if $\rho_* X=0$ and $\tilde{X}$ is tangent to P.*

Let θ be the fundamental form of $F(M)$, i. e., θ is a V-valued form on $F(M)$ defined by

$$\theta_p(X)=p^{-1}\pi_* X$$

where $p\in F(M)$ *and* $X\in T_p F(M)$.

Let $P\subset F(M, N)$ be a $\mathfrak{g}$-structure. For $p\in P$ with $\rho\circ\pi(p)=t$, define $\tilde{\rho}(p): V\to T_t N$ by $\tilde{\rho}(p)v=\rho_* p(v)$ for $v\in V$. $\tilde{\rho}$ is called the structure function of the first kind of P.

Let $p\in P$. An m-dimentional subsapce $H\subset T_p P$ is called a horizontal subspace if $\theta|_H: H\to V$ is isomorphic. For a horizontal subspace H there exists a unique $v_H\in H$ for $v\in V$ such that $\theta(v_H)=v$. Then define $c_H\in V\otimes\wedge^2 V^*$ by

$$c_H(v, w)=d\theta(v_H, w_H),$$

where $v, w\in V$. The equivalence class $c(p)=[c_H]$ in $V\otimes\wedge^2 V^*/\delta(\mathfrak{g}(t)\otimes V^*)$ is independent of the choice of H, where $t=\rho\circ\pi(p)$. c is called the structure function of the second kind of P. Note that $\mathfrak{g}(t)$ acts naturally on $V\otimes\wedge^2 V^*$, i. e., for $A\in\mathfrak{g}(t)$ and $S\in V\otimes\wedge^2 V^*$ $A\cdot S$ is defined by

$$(A\cdot S)(v, w)=A\big(S(v, w)\big)-S(Av, w)-S(v, Aw).$$

This induces an action of $\mathfrak{g}(t)$ on $V\otimes\wedge^2 V^*/\delta(\mathfrak{g}(t)\otimes V^*)$.

$\tilde{\rho}$ (resp. c) is called N-constant if for $p, q\in P$ such that $\rho\circ\pi(p)=\rho\circ\pi(q)$, $\tilde{\rho}(p)=\tilde{\rho}(q)$ (resp. $c(p)=c(q)$) holds. In the following we consider only P whose structure functions are N-constant. For $t\in N$ the common value of $\tilde{\rho}$ (resp. c) on $(\rho\circ\pi)^{-1}(t)$ is denoted by $\tilde{\rho}_t$ (resp. $c(t)$).

Recall that $\{\mathfrak{g}(t)\}_{t\in N}$ is a family of Lie subalgebras of $W\otimes V^*$. We can define its infinitesimal deformation as follows: Let $A\in\mathfrak{g}(t)$ and take a cross section σ of $\mathfrak{g}$ such that $\sigma(t)=A$. Then for $v\in V$ define $\tau_v: \mathfrak{g}(t)\to W\otimes V^*/\mathfrak{g}(t)$ by

$$\tau_v(A)\equiv\tilde{\rho}_t(v)\,\sigma \qquad \mathrm{mod}\ \mathfrak{g}(t)$$

where right hand means the derivative of the $W\otimes V^*$-valued function σ by the vector $\tilde{\rho}_t(v)\in T_t N$.

Let $\underline{c}$ be the $V\otimes\wedge^2 V^*$-valued function on N such that $[\underline{c}(t)]=c(t)$. Define $\sigma(\underline{c})\in V\otimes\wedge^2 V^*$ by

$$\sigma(\underline{c})(v_1, v_2, v_3) = \sum_S \underline{c}\left(\underline{c}(v_1, v_2), v_3\right)$$

where $v_i \in V$ and $\sum_S$ means the cyclic sum. Further define $\tilde{\underline{\gamma}} \in V \otimes \Lambda^3 V^*$ by

$$\tilde{\underline{\gamma}}(v_1, v_2, v_3) = \sum_S \tilde{\rho}(v_1)\left(\underline{c}(v_2, v_3)\right).$$

Then we have the following structure equations. (See [1])

$$(1.1) \qquad A \cdot c - \delta\tau(A) = 0.$$

$$(1.2) \qquad \sigma(\underline{c}) + \tilde{\underline{\gamma}} = \delta T,$$

where T is a $\mathfrak{g} \otimes \Lambda^2 V^*$-valued function on N.

Next we will consider the second order structue. Let $P \subset F(M, N)$ be $\mathfrak{g}$-structure whose structure functions $\tilde{\rho}$ and c are N-constant. Let g be the standard vector space over R whose dimension is equal to $\dim \mathfrak{g}(t)$. Take a trivialization $\lambda: N \times g \to \mathfrak{g}$ of the vector bundle $\mathfrak{g}$. Since we consider only local properties of P, we assume the existence of such a trivialization. Let $\lambda_t: g \to \mathfrak{g}(t)$ be the restriction of λ to the fiber over $t \in N$.

Let $\pi_1: F(P) \to P$ be the frame bundle of P. Denote by $F(P; M, N)$ the subbundle of $F(P)$ consisting of the frames z such that the following hold:

$$(1.3) \qquad z(A) = \left[\lambda_t(A)\right]^*_{\pi_1(z)} \qquad \text{for} \quad A \in g.$$

$$(1.4) \qquad \theta\left(z(v)\right) = v \qquad \text{for} \quad v \in V.$$

The Lie algebra of the structure group of $F(P; M, N)$ is $g \otimes V^*$. Let $\mathfrak{g}_1$ be an N-subalgebra of $g \otimes V^* (\subset gl(V+g))$. Set

$$\bar{\mathfrak{g}}_1(t) = (\lambda_t \otimes id)\, \mathfrak{g}_1(t)$$

where id means the identity map of V^*. $\bar{\mathfrak{g}}_1(t)$ is a subspace of $W \otimes V^* \otimes V^*$. We denote by $\bar{\mathfrak{g}}_1$ the vector bundle over N whose fiber over $t \in N$ is $\bar{\mathfrak{g}}_1(t)$.

Suppose $\bar{\mathfrak{g}}_1(t) \subset \mathfrak{g}(t)^{(1)}$, where $\mathfrak{g}(t)^{(1)}$ means the first prolongation of $\mathfrak{g}(t)$. Let $P_1 \subset F(P; M, N)$ be a $\mathfrak{g}_1$-structure. Set $\rho_1 = \rho \circ \pi$ and let $\tilde{\rho}_1$ be the structure function of the first kind of P_1. By (1.4) we have $\rho_{1*} z(v) = \rho_* p(v)$ for $v \in V$, where $z \in P_1$ and $p = \pi_1(z)$. Note that $\tilde{\rho}(w) = 0$ for $w \in W$ and $\tilde{\rho}_1(w) = 0$ for $w \in W + g$. Since $V/W \cong (V+g)/(W+g)$, we can identify $\tilde{\rho}_1$ with $\tilde{\rho}$.

Let θ_1 be the fundamental form of $F(P)$ and c_1 be the structure function of the second kind of P_1. c_1 is a function on P_1 having its values in the space

$$(V+g) \otimes \Lambda^2 (V+g)^* / \delta\left(\mathfrak{g}_1 \otimes (V+g)^*\right).$$

In the following we assume that c_1 is N-constant. Then the structure equations (1. 1) and (1. 2) applied to P_1 are satisfied. Finally, recalling $\mathfrak{g}_1(t)\subset g\otimes V^*$, we denote by α_1 (resp. β_1) the $g\otimes\wedge^2(V+g)^*/\delta(\mathfrak{g}_1\otimes(V+g)^*)$-component (resp. $V\otimes\wedge^2(V+g)^*$-component) of c_1.

Let $z\in P_1$ and $\pi_1(z)=p$. Then for $X\in T_zP_1$, we have

$$\begin{aligned}\theta_1(X) &= z^{-1}(\pi_{1*}X)\\ &\equiv p^{-1}\pi_*\pi_{1*}X \qquad \text{mod } g \quad (\text{by } (1.\ 4))\\ &= (\pi_1^*\theta)(X)\,.\end{aligned}$$

That is, the V-component of θ_1 is $\pi_1^*\theta$. Note that $H_z=z(V)$ is a horizontal subspace at p.

PROPOSITION 1. 1. (cf. [3]) *Let* $z\in P_1$, $\pi_1(z)=p$ *and* $\rho\circ\pi(p)=t$. *then the following hold*:

(1) $\beta_1(t)(v, w)=C_{H_z}(v, w)$ *for* $v, w\in V$.

(2) $\beta_1(t)(A, v)=-\lambda_t(A)v$ *for* $A\in g$ *and* $v\in V$.

(3) $\alpha_1(t)(A, B)=-\lambda_t^{-1}([\lambda_t(A), \lambda_t(B)])$ *for* $A, B\in g$.

PROOF. Let H be a horiznotal space at z. For $v\in V$ and $A\in g$, we have $\pi_{1*}v_H=v_{H_z}$ and $\pi_{1*}A_H=\lambda_t(A)^*$. Hence we have

$$\begin{aligned}\beta_1(z)(v, w) &= d(\pi_1^*\theta)(v_H, w_H)\\ &= d\theta(v_{H_z}, w_{H_z})\\ &= c_{H_z}(v, w)\,.\end{aligned}$$

This proves (1). (2) is shown as follows. Let σ be a cross section of $\mathfrak{g}$ such that $\sigma(t)=\lambda_t(A)$. Then we have

$$\begin{aligned}\beta_1(t)(A, v) &= d(\pi_1^*\theta)(A_H, v_H)\\ &= d\theta\big(\lambda_t(A)^*, v_{H_z}\big)\\ &= (L_{\sigma^*}\theta)(v_{H_z})-d\big(\theta(\sigma^*)\big)(v_{H_z})\\ &= -\lambda_t(A)\,\theta(v_{H_z})\\ &= -\lambda_t(A)\,v\,,\end{aligned}$$

where L_{σ^*} means the Lie derivative. Similarly (3) follows from (1. 3).

q. e. d.

We can also consider higher order structures as follows. Let l be a positive integer and $\{d_k\}_{0\leqq k\leqq l}$ be a sequence of positive integers. For $0\leqq k\leqq l$ let g_k be the standard vector space over $\mathbf{R}$ of dimension d_k. For $k=-1$ set $g_{-1}=W$. Let $\mathfrak{g}_k$ be an N-subalgebra of $g_{k-1}\otimes V^*$ whose fiver dimension is d_k. Let $\lambda_k: g_k \times N \to \mathfrak{g}_k$ be a trivialization of the vector bundle $\mathfrak{g}_k$. We have an injection

$$\lambda_{k-1} \otimes id : \mathfrak{g}_k \longrightarrow \mathfrak{g}_{k-1} \otimes V^* .$$

Set

$$\bar{\mathfrak{g}}_k = (\lambda_0 \otimes id_k) \circ (\lambda_1 \otimes id_{k-1}) \circ \cdots \circ (\lambda_{k-1} \otimes id)\, \mathfrak{g}_k$$

where id_j means the identity map of $\overset{j}{\otimes} V^*$. $\bar{\mathfrak{g}}_k$ is an N-subalgebra of $W \otimes (\overset{k}{\otimes} V^*)$.

Definition 1.2. *$\mathcal{G} = \{(\mathfrak{g}_k, \lambda_k)\}_{l \geqq k \geqq 0}$ is called an l-sequence of N-subalgebras if for $k \geqq 1$ $\bar{\mathfrak{g}}_k$ is a subbundle of $(\bar{\mathfrak{g}}_{k-1})^{(1)}$.*

Let $\mathcal{G} = \{(\mathfrak{g}_k, \lambda_k)\}_{l \geqq k \geqq -1}$ be an l-sequence of N-subalgebras. Let

$$\mathcal{P} : P_l \xrightarrow{\pi_l} P_{l-1} \xrightarrow{\pi_{l-1}} \cdots \longrightarrow P_0 \xrightarrow{\pi_0} M$$

be a sequence of fibered manifolds such that each P_k is a subbundle of the frame bundle of P_{k-1} with fiber dimension d_k. Then an element p_k of P_k can be considered as a linear isomorphism from $V_{k-1} = V + g_0 + \cdots + g_{k-1}$ to $T_{p_{k-1}} P_{k-1}$, where $p_{k-1} = \pi_k(p_k)$. Let θ_k denote the fundamental form of P_k.

Definitioy 1.3. *The sequence of fibered manifold $\mathcal{P}$ is called a $\mathcal{G}$-structure if the following conditions are satisfied:*

(1) *$\pi_k : P_k \to P_{k-1}$ is a $\mathfrak{g}_k$-structure.*

(2) *Let $p_k \in P_k$ with $\pi_k(p_k) = p_{k-1}$. Then for $v \in V_{k-2}$,*

$$\theta_{k-1}\big(p_k(v)\big) = v .$$

(3) *Let $p_k \in P_k$ and $(\rho \circ \pi_0 \circ \cdots \circ \pi_k)(p_k) = t$. Then for $A \in g_{k-1}$*

$$p_k(A) = \big[\lambda_{k-1}(t)(A)\big]^*_{p_{k-1}} .$$

(Recall that $\lambda_{k-1}(t)(A)$ is an element of $g_{k-2} \otimes V^* \subset GL(V_{k-2})$).

Let X be a local vector field on M. Suppose that X is an infinitesimal automorphism of P_0. Then $\tilde{X}$ is tangent to P_0. The prolongation of $\tilde{X}$ to $F(P_0)$ is denoted by $\tilde{X}^{(1)}$. Then X is called an infinitseimal automorphism of P_1 if $\tilde{X}^{(1)}$ is tangent to P_1. Inductively, X is called an infinitesimal automorphism of P_k if X is an infinitesimal automorphism of P_{k-1} and $\tilde{X}^{(k)}$, the prolongation of $\tilde{X}^{(k-1)}$ to $F(P_{k-1})$, is tangent to P_k. An infinitesimal automorphism of P_l is called an infinitesimal automorphism of $\mathcal{P}$.

§ 2. Infinitesimal automorphisms of $\mathfrak{g}$-structures

Let $\rho: M \to N$ be a fibered manifold and $\mathfrak{g}$ be an N-subalgebra of $W \otimes V^*$. Let $P \subset F(M, N)$ be a $\mathfrak{g}$-structure. Let D be a distribution on P such that for all $p \in P$ $D_p \subset T_pP$ is a horizontal subspace at p. We call such D a connection. The we have a direct sum decomposition

$$T_pP = D_p \oplus \mathrm{Ker}\,(\pi_*)_p\,.$$

For $X \in T_pP$ with $(\rho\circ\pi)(p) = t$, let X_D be the horizontal component of X and let $X - X_D = A_p^*$, where $A \in \mathfrak{g}(t)$. Define a $\mathfrak{g}$-valued 1-form ω on P by $\omega(X) = A$. Then we have

$$d\theta = c_D(\theta\wedge\theta) - \omega\wedge\theta$$

where c_D is a $V \otimes \wedge^2 V^*$-valued function on P such that $c_D(p) = c_{D_p}$ and $c_D(\theta\wedge\theta)$, $\omega\wedge\theta$ are V-valued 2-forms defined by

$$c_D(\theta\wedge\theta)(X, Y) = c_D\big(\theta(X), \theta(Y)\big)$$

and

$$(\omega\wedge\theta)(X, Y) = \omega(X)\,\theta(Y) - \omega(Y)\,\theta(X)\,.$$

Let us take a cross section $j: M \to P$ of the fibered manifold $\pi: P \to M$. Let A be a $\mathfrak{g}$-valued function on M. Then $A\cdot(j^*\theta)$ denotes the V-valued form defined by

$$A\cdot(j^*\theta)(X) = A\big((j^*\theta)(X)\big)$$

for $X \in TM$. We have

PROPOSITION 2.1. *A local vector field X on M satisfying $\rho_* X = 0$ is an infinitesimal automorphism of P if and only if there exists a $\mathfrak{g}$-valued function A on M such that the following equation holds;*

$$L_X(j^*\theta) = A\cdot(j^*\theta)\,.$$

PROOF. Let $\{v_1, \cdots, v_m\}$ be a basis of V. Let X_i be a vector field on M defined by $(X_i)_x = j(x)(v_i)$ for $x \in M$. Then X is an infinitesimal automorphism of P if and only if there exists a $\mathfrak{g}$-valued function $A = (A_{ij})$ such that

$$(2.1) \qquad L_X X_i = \sum_{j=1}^{n} A_{ij} X_j\,.$$

On the other hand we have

$$(L_X j^*\theta)(X_i) = X\big(j^*\theta(X_i)\big) - j^*\theta(L_X X_i)$$
$$= -j^*\theta(L_X X_i)\,,$$

because $j^*\theta(X_i)=v_i$. Combined with (2. 1), this proves our assertion.

q. e. d.

Let P be a $\mathfrak{g}$-structure. Notations being the same as in § 1, let $\mathfrak{g}_1$ be an N-subalgebra of $g \otimes V^*$ and P_1 be a $\mathfrak{g}_1$-structure on P. We assume that the structure functions of P and P_1 are N-constant. Take a cross section $j_1\colon P \to P_1$ of the fibered manifold $\pi_1\colon P_1 \to P$. For $p \in P$ $D_p = j_1(p)(V)$ is a horizontal space at p. Hence $D = \bigcup_{p\in P} D_p$ is a connection on P. The $V \otimes \wedge^2 V^*$-valued function c_D does not depend on the choice of j and is denoted by $c(P_1)$. Moreover

$$(D_1)_{j_1(p)} = (j_1)_*(T_p P)$$

is a horizontal space of the fibered manifold $\pi_1\colon P_1 \to P$ at $j_1(p)$. Let us denote by $c_1(j_1)$ the representative of c_1 on $j_1(P)$ determined by the horizontal spaces $\bigcup_{p\in P}(D_1)_{j_1(p)}$. $c_1(j_1)$ is a $(V+g)\otimes\wedge^2(V+g)^*$-valued function on $j_1(P)$. Let ω be the $\mathfrak{g}$-valued from on P determined by D. Let $\bar{\omega}$ be the g-valued form defined by $\bar{\omega}(X) = \lambda^{-1}(\omega(X))$ for $X \in TP$. Then the following identities are immediately shown by the definitions.

(2. 2) $\quad d\theta = c(P_1)(\theta\wedge\theta) - \omega\wedge\theta\,.$

(2. 3) $\quad d(j_1^*\theta_1) = c_1(j_1)(j_1^*\theta_1 \wedge j_1^*\theta_1)\,.$

(2. 4) $\quad j_1^*\theta_1 = \theta + \bar{\omega}\,.$

PROPOSITION 2. 2. *Let ϕ be a local transformation of P. Assume that $\phi^*\theta = \theta$ and $(\rho\circ\pi)\circ\phi = \rho\circ\pi$. Then there exists a $\mathfrak{g}^{(1)}$-valued function A on P such that the following identity holds:*

$$\phi^*\omega = A\cdot\theta + \omega\,.$$

PROOF. For $p \in P$ with $\rho\circ\pi(p) = t$ let $\phi(p) = q$, $D_p = H$ and $D_q = H'$. By (1) of Proposition 1. 1 and the fact that c_1 is N-constant, we have $c_H = c_{H'}$. On the other hand it follows from $\phi^*\theta = \theta$ that $c_H = c_{\phi_* H}$.

As before, for $v \in V$ v_H denotes the vector in H such that $\theta(v_H) = v$. Then there exists a unique $A(p) \in \mathfrak{g}(t)\otimes V^*$ such that

$$v_{\phi_* H} = v_{H'} + \big(A(p)_v\big)^*\,.$$

Hence we have $c_{\phi_* H} = c_{H'} + \delta A(p)$. Combined with the fact $c_H = c_{\phi_* H} = c_{H'}$, this proves $\delta A(p) = 0$. That is, $A(p)$ is in $\mathfrak{g}(t)^{(1)}$. We have

(2. 5) $$(\phi^*\omega)(v_H) = \omega(v_{\phi_* H}) = A(p)_v = \big(A(p)\cdot\theta\big)(v_H).$$

Note that $\phi^*\theta=\theta$ implies that there exists a local transformation ψ of M such that locally $\phi=\tilde{\psi}$. Hence we know that for $B\in\mathfrak{g}$ $\phi_* B^* = B^*$. Therefore we have

$$(\phi^*\omega)(B^*) = \omega(B^*).$$

Combined with (2. 5), this completes the proof. q. e. d.

As an infinitesimal version of this proposition, we have

PROPOSITION 2. 3. *Let X be a local vector field on P such that $(\rho\circ\pi)_* X = 0$ and $L_X\theta=0$. Then there exists a $\mathfrak{g}^{(1)}$-valued function A on P such that the following identity holds:*

$$L_X\omega = A\cdot\theta.$$

Let X be a local vector field on P such that $(\rho\circ\pi)_* X=0$. By Proposition 2. 1 and (2. 4) we know that X is an infinitesimal automorphism of P_1 if and only if there exists a $\mathfrak{g}_1$-valued function A such that

$$L_X(\theta+\bar{\omega}) = A\cdot(\theta+\bar{\omega}).$$

Since $A\cdot\bar{\omega}=0$ and $A\cdot\theta$ is a g-valued form, we have $L_X\theta=0$ and $L_X\bar{\omega}=A\cdot\theta$. This proves

THEOREM 1. *A local vector field X on P satisfying $(\rho\circ\pi)_* X=0$ is an infinitesimal automorphism of P_1 if and only if the following hold:*

(1) $L_X\theta=0$.

(2) *There exists a $\mathfrak{g}_1$-valued function A on P such that $L_X\bar{\omega}=A\cdot\theta$.*

§3. Continuous Lie algebra sheaves

Let $\mathscr{L}$ be a subsheaf of the sheaf of germs of all vector fields on M. The stalk of $\mathscr{L}$ over $x\in M$ is denoted by $\mathscr{L}_x$. For a vector field X we denote by $j_x^k(X)$ the k-jet of X at x and by $J_k(TM)$ the bundle of k-jets of all vector fields. Set

$$R_{k,x} = \left\{ j_x^k(X) \,\middle|\, X\in\mathscr{L}_x \right\} \quad \text{and} \quad R_k = \bigcup_{x\in M} R_{k,x}.$$

DEFINITION 3. 1. *$\mathscr{L}$ is called a continuous Lie algebra sheaf (CLAS) if the following conditions are satisfied:*

(1) *$\mathscr{L}_x$ is a Lie algebra with respect to the natural bracket operation for all x.*

(2) *There exists a fibered manifold ρ: $M\to N$ such that the equality $R_0=\{v\in TM \mid \rho_* v=0\}$ holds.*

(3) *R_k is a vector bundle over M for all k.*

(4) *There is an integer k_0 such that the following holds; a local vector field X is a local section of $\mathscr{L}$ if and only if $j_x^{k_0}(X)\in R_{k_0,x}$ for all x.*

It is shown in [1] that for a CLAS $\mathscr{L}$ there exists a k_0-sequence of N-subalgebras $\mathscr{G}=\{(\mathfrak{g}_k, \lambda_k)\}_{k\leq k_0}$ and $\mathscr{G}$-structure

$$\mathscr{P}: P_{k_0}\longrightarrow P_{k_0-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P\longrightarrow M$$

such that a local vector field X on M is a section of $\mathscr{L}$ if and only if X is an infinitesimal automorphism of $\mathscr{P}$.

Since the sections of $\mathscr{L}$ are tangent to the fiber $\rho^{-1}(t)$, $\mathscr{L}$ induces a transitive Lie algebra sheaf on $\rho^{-1}(t)$, which is denoted by $\mathscr{L}(t)$. Suppose that for a point $o\in N$ $\mathscr{A}=\mathscr{L}(o)$ is one of the following Lie algebra sheaves:

I. $\mathscr{L}_{gl(W),sl(W)^{(1)}}$; the sheaf of germs of all vector fields with constant divergence.

II. $\mathscr{L}_{csp(W)}$; the sheaf of germs of all vector fields which preserve a symplectic structure up to constant factors.

Let $N\times\mathbf{R}\rightarrow N$ be the trivial vector bundle over N. Let $Q_l\subset J_l(N\times\mathbf{R})$ be a linear differential equation. Q_l is called formally integrable if for any $k\geqq 1$ the k-th prolongation $Q_l^{(k)}$ of Q_l is a subbundle of $J_{l+k}(N\times\mathbf{R})$ and $Q_l^{(k)}\rightarrow Q_l^{(k-1)}$ is surjective. Moreover Q_l is called integrable if for any $r\in Q_l$ there exists a local solution of Q_l passing through r. For $k\leqq k'$ let $\varpi_k^{k'}$: $J_{k'}(N\times\mathbf{R})\rightarrow J_k(N\times\mathbf{R})$ be the natural projection. We assume the following condition:

(C. 1) For $k<l$ the image of Q_l under ϖ_k^l is a subbundle of $J_k(N\times\mathbf{R})$.

Let $\mathscr{A}$ be one of the Lie algebra sheaves I and II. Let Ω denote the standard volume on $\mathbf{R}^{m-n}$ in case I and the symplectic form in case II, where $m=\dim M$ and $n=\dim N$. For a formally integrable and integrable homogeneous linear differential equation Q_l satisfying (C. 1), let $\mathscr{A}[N; Q_l]$ be the CLAS consisting of germs of all vector fields X on $N\times\mathbf{R}^{m-n}$ satisfying the following conditions:

(1) X is tangent to the fibers of the fibered manifold $N\times\mathbf{R}^{m-n}\rightarrow N$.

(2) There exists a solution f of Q_l such that $L_X\Omega=f\Omega$.

Let $\mathscr{L}$ be a CLAS and $\mathscr{G}=\{(\mathfrak{g}_k, \lambda_k)\}_{k\leq k_0}$ be the k_0-sequence of N-subalgebras defined by $\mathscr{L}$. Recall that $\mathfrak{g}(t)=\mathfrak{g}_0(t)$ is a subalgebra of $W\otimes V^*$. Let $\mathfrak{h}(t)$ (resp. $\mathfrak{a}(t)$) be the image (resp. kernel) of $\mathfrak{g}(t)$ under the natural projection $W\otimes V^*\rightarrow W\otimes W^*$. We assume the following conditions:

(C. 2) $\dim\mathfrak{h}(t)$ is constant.

(C. 3) $\left\{v\in V \middle| A(v)=0,\ A\in\mathfrak{a}(t)\right\}=W$.

After these preparations we can state the main theorem.

THEOREM 2. *Let* $\mathcal{L}$ *be a* CLAS. *Assume the conditions* (C. 2), (C. 3) *and moreover assume that* $\mathcal{A}=\mathcal{L}(o)$ *is one of the Lie algebra sheaves* I *and* II. *Then there exists a formally integrable and integrable homogeneous linear differential equation* Q_{k_0} *satisfying* (C. 1) *and* $\mathcal{L}$ *is locally equivalent to* $\mathcal{A}[N; Q_{k_0}]$.

The proof will be given in the following sections. In the rest of this section we will note some facts.

Let $\mathcal{G}=\{(\mathfrak{g}_k, \lambda_k)\}_{k\leqq k_0}$ be the k_0-sequence of N-subalgebras and

$$\mathcal{P}: P_{k_0}\xrightarrow{\pi_{k_0}}P_{k_0-1}\longrightarrow\cdots\longrightarrow P_1\xrightarrow{\pi_1}P_0\xrightarrow{\pi_0}M$$

be a $\mathcal{G}$-structure determined by $\mathcal{L}$. Let $h=gl(W)$ or $csp(W)$. It follows from (C. 2) that $\{\mathfrak{h}(t)\}_{t\in N}$ is a deformation of h. Hence we can assume $\mathfrak{h}(t)=h$ by a suitable choice of P_0. This is trivial in case I. In case II this follows from $H^1(csp(W), gl(W)/csp(W))=0$. Set $h'=sl(W)$ when $h=gl(W)$ and $h'=sp(W)$ when $h=csp(W)$. Take a complement U of W in V. Let I denote the identity matrix in $gl(W)$. Let $\bar{\mathfrak{g}}_k$ be the N-subalgebra of $W\otimes(\overset{k+1}{\otimes} V^*)$ induced by $\mathfrak{g}_k$. Then by results of [2] and [3] we know

$$\begin{aligned}\bar{\mathfrak{g}}_k(t) = h'^{(k)}+h'^{(k-1)}\odot U^*+\cdots+h'\odot S^k U^*\\ +\{I\}\odot\mathfrak{b}_k(t)+W\otimes S^{k+1}U^*,\end{aligned}$$

where $\mathfrak{b}_0=\mathbf{R}$ and $\mathfrak{b}_k(t)$ is a subspace of S^kU^* such that the following holds: Let $u\in U$ and $b\in\mathfrak{b}_k$. Then

$$u \,\lrcorner\, b\in\mathfrak{b}_{k-1}.$$

Set $U_1^*(t)=\mathfrak{b}_1(t)$. Then $\mathfrak{b}_k(t)$ is a subspace of $S^kU_1^*(t)$. Set

$$U_2(t)=\left\{u\in U \middle| \alpha(u)=0,\ \alpha\in U_1^*(t)\right\}.$$

First note that, by taking a suitable $\mathcal{P}$, we can assume that $U_1^*(t)$ and hence $U_2(t)$ are independent of t. In fact this is shown as follows. Let σ be a $GL(V)$-valued function on N such that $\sigma(w)=w$ for $w\in W$. Then $P_0'=R_\sigma P_0$ is a $\mathfrak{g}_0$-structure, where $\mathfrak{g}_0=h+W\otimes U^*$. R_σ induces a bundle map $\tilde{R}_\sigma$ from the frame bundle $F(P_0)$ of P_0 to the frame bundle $F(P_0')$ of P_0'. Then $\tilde{R}_\sigma(P_1)$ is obviously a $\mathfrak{g}_1$-structure on P_0', but it is not contained in $F(P_0'; M, N)$. Hence, denoting by μ the $GL(V+\mathfrak{g}_0)$-valued function on N such that $\mu=\sigma$ on V and $\mu=id$ on $\mathfrak{g}_0$, let $P_1'=R_\mu\tilde{R}_\sigma(P_1)$. Then P_1' is an $Ad(\mu)\,\mathfrak{g}_1$-structure contained in $F(P_0'; M, N)$. P_1 is clearly a structure determined by the CLAS which is the prolongation of $\mathcal{L}$ to P_0'. Let U_2 be a subspace of U such that $\dim U_2 = \dim U_2(t)$. If we take σ such that $\sigma_t(U_2(t)) = U_2$, then we have

$$Ad(\mu)\,\mathfrak{g}_1 = h'^{(1)} + h' \odot U^* + \{I\} \odot U_1^* + W \otimes S^2 U^*,$$

where $U_1^* = \{\alpha \in U^* \mid \alpha(u) = 0,\ u \in U_2\}$. This proves our assertion. In the following we fix a complement U_1 of U_2 in U.

Let g_k be the standard vector space over R whose dimension is $\dim \bar{\mathfrak{g}}_k(t)$. Set

$$g'_k = h'^{(k)} + \cdots + h' \odot S^k U^* + W \otimes S^{k+1} U^*$$

and

$$\bar{\mathfrak{g}}''_k(t) = \{I\} \odot \mathfrak{b}_k(t).$$

Recall the bundle isomorphism

$$g_k \times N \xrightarrow{\lambda_k} \mathfrak{g}_k \xrightarrow{(\lambda_0 \otimes id_k) \cdots (\lambda_{k-1} \otimes id)} \bar{\mathfrak{g}}_k. \tag{3.1}$$

Since g'_k is independent of t, we can consider that g'_k is a subspace of g_k. Moreover we can assume that there exists a subspace g''_k of g_k which correspond to $\bar{\mathfrak{g}}''_k(t)$ in (3.1). In the following sections we often identify g''_k with $\bar{\mathfrak{g}}''_k(t)$. For example, in a equation similar to (2) or (3) of Proposition 1.1 we often omit the trivialization λ.

We put as follows:

$$W_k = W + g_0 + \cdots + g_k.$$
$$W'_k = W + g'_0 + \cdots + g'_k.$$
$$V_k = W_k + U.$$
$$V'_k = W'_k + U.$$

Let c_k be the structure function of P_k. c_k is a function on N having its value in the space

$$V_{k-1} \otimes \wedge^2 V_{k-1}^* / \delta(\mathfrak{g}_k \otimes V_{k-1}^*).$$

Note that by Proposition 2.2 of [1] we have

$$\underline{c}_k(v, w) \in W_{k-1}$$

for $v \in V_{k-1}$ and $w \in W_{k-1}$, where $\underline{c}_k$ is a representative of c_k.

Finally let α_k denote the

$$g_{k-1} \otimes \wedge^2 V_{k-1}^* / \delta(\mathfrak{g}_k \otimes V_{k-1}^*)$$

component of c_k. Similarly we denote by α'_k (resp. α''_k) the

$$g'_{k-1}\otimes\wedge^2 V^*_{k-1}/\delta(\mathfrak{g}_k\otimes V^*_{k-1}) \text{ (resp. } g''_{k-1}\otimes\wedge^2 V^*_{k-1}/\delta(\mathfrak{g}_k\otimes V^*_{k-1})$$

component of α_k.

§ 4. The infinitesimal automorphisms of P_1

Let the notations be the same in § 3. We assume that $\mathscr{P}$ has been taken as in § 3. First we have, for $u, v\in U_2$,

$$(4.1) \qquad \underset{\sim}{c}_0(u, v)\in W+U_2,$$

where $\underset{\sim}{c}_0$ is a representative of the structure function c_0 of P_0. In fact, let $A\in\{I\}\odot U_1^*$. Then by (1.1) applied to P_1 we have $A\underset{\sim}{c}_1(u, v)=(\delta T)(u, v)$, where $T\in\mathfrak{g}_1\otimes V_0^*$. This implies that the U_1-component of $\underset{\sim}{c}_1(u, v)$ is 0. Hence by (1) of Proposition 1.1 we have (4.1). (4.1) means that the distribution E on N defined by

$$E_t=\left\{\tilde{\rho}_t(u)\middle|u\in U_2\right\}$$

for $t\in N$ is completely integrable. (Recall $[\tilde{\rho}(u), \tilde{\rho}(v)]=-\tilde{\rho}(\underset{\sim}{c}_0(u, v))$ for $u, v\in U$.)

The proof of the following proposition will be given in § 5.

PROPOSITION 4.1. *We can choose the bundle P_1 so that the following hold*:

(1) $\underset{\sim}{c}_1(v, w)\in W_0'$ *for* $v, w\in W_0$.
(2) $\underset{\sim}{c}_1(w, u)\in W_0'$ *for* $w\in W_0'$ *and* $u\in U_2$.
(3) $\underset{\sim}{c}_1(u, v)\in W_0'+U_2$ *for* $u, v\in U_2$.

Now assuming the above proposition, we will consider the infinitesimal automorphisms of P_1. As in § 2, take a cross section $j_1: P_0\to P_1$. Let D be the connection determined by j_1 and ω be the g_0-valued form associated with D. (Note that $\mathfrak{g}_0(t)$ is independent of t, and hence we can consider that $\mathfrak{g}_0(t)=g_0$.) We denote by ω'' the g_0''-component of ω. Let θ (resp. θ_1) denote the fundamental form of P_0 (resp. P_1). Then (2.2)~(2.4) hold. We denote by θ_{U_1} (resp. θ_{U_2}) the U_1-component (resp. U_2-component) of θ. Then it is easy to see that the system of Pfaffian equations $\theta_{U_1}=0$ is completely integrable.

By (2.3) and (2.4) we have

$$\begin{aligned} d\theta+d\omega &= c_1(j_1)\left((\theta+\omega)\wedge(\theta+\omega)\right) \\ &= c_1(j_1)(\theta\wedge\theta)+c_1(j_1)(\theta\wedge\omega)+c_1(j_1)(\omega\wedge\omega)\,. \end{aligned}$$

It follows from Proposition 4.1 that

(4. 2) $$d\omega'' = \alpha_1''(j_1)(\theta\wedge\theta_{U_1}) + \alpha_1''(j_1)(\theta_{U_1}\wedge\omega) + \alpha_1''(j_1)(\theta_{U_2}\wedge\omega''),$$

where $\alpha_1''(j_1)$ is the $g_0''\otimes\wedge^2 V_0^*$-component of $c_1(j_1)$. Hence we know that the system of Pfaffian equations $\omega''=\theta_{U_1}=0$ is completely integrable. This is interpreted as follows. Define a distribution F on P_0 by

$$F_p = \{p_1(v) + A^* \mid v\in W+U_2,\ A\in g_0'\}$$

where $p_1\in P_1$ with $\pi_1(p_1)=p$. Then the above means that F is completely integrable.

Let X be a local vector field on P_0 satisfying $L_X\theta=0$ and $(\rho\circ\pi)_*X=0$. Then there exists a local vector field Y on M such that $X=\tilde{Y}$. Recall that X is an infinitesimal automorphism of P_1 if and only if there exists a $\mathfrak{g}_1$-valued function A on P_0 such that

$$L_X\omega = A\cdot\theta .$$

Since $g_1'=g_0'^{(1)}$, this holds if and only if there exists a $\{I\}\otimes U_1^*$-valued function B such that

(4. 3) $$L_X\omega'' = B\cdot\theta .$$

On the other hand, by Proposition 2. 3, (4. 3) holds provided that B is a $\{I\}\otimes U^*$-valued function. It follows that $L_X\omega''$ is a linear combination of θ_{U_1} and θ_{U_2}. We have

(4. 4) $$L_X\omega'' = di(X)\omega'' + i(X)\,d\omega'' .$$

By (4. 2) $i(X)\,d\omega''$ is a linear combination of θ_{U_1} and θ_{U_2}. Therefore we know that $\omega''(X)$ is a function on N. Set $\omega''(X)=\phi I$, where ϕ is a function on N. Define a U_2^*-valued function f on N by

$$\alpha_1''(j_1)(I, u) = f(u)\, I$$

where $u\in U_2$. Then, since (4. 3) holds if and only if the θ_{U_2}-component of $L_X\omega''$ is 0, it follows from (4. 4) that X is an infinitesimal automorphism of P_1 if and only if

(4. 5) $$\tilde{\rho}(u)\,\phi + f(u)\,\phi = 0$$

where $u\in U_2$. Define $\tilde{f}\in\Gamma(E^*)$ by $\tilde{f}(\tilde{\rho}(u))=f(u)$. Recalling that the distribution E on N is completely integrable, we denote by d' the exterior differentiation with respect to E. Then (4. 5) means

(4. 6) $$d'\phi + \phi\tilde{f} = 0 .$$

Moreover, it is not difficult to see that $dd\omega''=0$ implies

(4. 7) $$d'\tilde{f} = 0 .$$

(cf. § 5) (4. 7) is the integrability condition of (4. 6). We note that by the theory of partial defferential equations of the first order, (4. 6) satisfying (4. 7) is the general form of the formally integrable and integral homogeneous linear differential equation of the first order.

Recall that the distribution F on P_0 is completely integral. We will denote by $\mathscr{I}_p$ the integral manifold of F passing through $p\in P$. Let N' be a submanifold of N transversal to the distribution E. Let $\dot{p}: N'\to P_0$ be a map satisfying $(\rho\circ\pi)\,\dot{p}(t)=t$. Set

$$\mathscr{I}=\bigcup_{t\in N}\mathscr{I}_{\dot{p}(t)}\,.$$

Then, by the definition of F, $\mathscr{I}$ is a g_0'-structure on M. Therefore $\mathscr{I}$ determines a volume element or a symplectic form $\Omega(t)$ on each fiber $\rho^{-1}(t)$ corresponding to $h'=sl(W)$ or $sp(W)$. First we will consider the case when $h'=sl(W)$. As before, let $X=\tilde{Y}$ be an infinitesimal automorphism of P_1. We can prove

(4. 8) $$L_Y\Omega=r\phi\Omega\,,$$

where $r=m-n$ is the fiber dimension of the fibered manifold $\rho: M\to N$ and $\omega''(X)=\phi I$. In fact this is shown as follows. Let ψ_s be the 1-parameter transformation generated by Y, where s moves in a neighborhood of 0 in R. Let Ω_0 denote the standard volume element in W. For $x\in M$ let $\{Z_1, \cdots, Z_r\}$ be a basis of the subspace of T_xM consisting of the vectors tangent to the fiber. Then we have

(4. 9) $$\begin{aligned}(\psi_s^*\Omega)_x(Z_1,\cdots,Z_r)&=\Omega_{\psi_s(x)}(\psi_{s*}Z_1,\cdots,\psi_{s*}Z_r)\\&=\Omega_0(p_s^{-1}\psi_{s*}Z_1,\cdots,p_s^{-1}\psi_{s*}Z_r)\\&=\Omega_0\Big(\big\{\tilde{\psi}_{-s}(p_s)\big\}^{-1}Z_1,\cdots,\big\{\tilde{\psi}_{-s}(p_s)\big\}^{-1}Z_r\Big)\end{aligned}$$

where p_s is a curve in I such that $\pi(p_s)=\psi_s(x)$. There exists a curve a_s in $GL(W)$ satisfying

(4. 10) $$\tilde{\psi}_{-s}(p_s)=p_0a_s^{-1}\,.$$

By (4. 9) we have

(4. 11) $$(\psi_s^*\Omega)_x=(\det a_s)\,\Omega_x\,.$$

Let $A=\frac{d}{ds}a_s|_{s=0}$. Then differentiating (4. 11), we get

(4. 12) $$(L_Y\Omega)_x=(\mathrm{Tr}\,A)\,\Omega_x\,.$$

On the other hand (4. 10) implies $p_s=\tilde{\psi}_s(p_0)\,a_s^{-1}$. Let $X'=\frac{d}{ds}p_s|_{s=0}\in T_{p_0}I$. We have

$$X' = X_{p_0} - A^*_{p_0}\,,$$

and

$$A = \omega(A^*_{p_0}) = \omega(X_{p_0}) - \omega(X')\,.$$

Since $\mathrm{Tr}\,(\omega(X'))=0$ and $\mathrm{Tr}\,(\omega(X_{p_0}))=\mathrm{Tr}\,(\omega''(X_{p_0}))=r\phi$, we have $\mathrm{Tr}\,A=r\phi$. Combined with (4. 12), this proves (4. 8).

Similarly, in case $h'=sp(W)$, we can prove

(4. 13) $$L_Y\Omega = 2\phi\Omega\,.$$

Therefore we know that a local vector field Y on M is an infinitesimal automorphism of P_1 if and only if there exists a solution ϕ of (4. 5) such that $L_Y\Omega=\phi\Omega$. Hence Dalboux's theorem implies that Theorem 2 holds when $k_0=1$.

§ 5. Proof of Proposition 4. 1.

First we prepare two lemmas on $sp(W)$ and $sl(W)$. It is known that $sp(W)^{(k)}\cong S^{k+2}W^*$. This implies that every basis of W is a regular basis for $sp(W)^{(k)}$. For $w\in W$ we denote by $i(w)$ the contraction $\overset{k+1}{\otimes} W^*\to\overset{k}{\otimes} W^*$. Define an action of $GL(W)$ on $W\otimes S^k W^*$ by

$$(gT)\,(w_1, \cdots, w_k) = gT(g^{-1}w_1, \cdots, g^{-1}w_k)$$

where $g\in GL(W)$, $T\in W\otimes S^k W^*$ and $w_i\in W$. Then, since $sl(W)$ is an ideal of $gl(W)$, we know that $sl(W)^{(k)}$ is invariant under this action of $GL(W)$. Moreover it is easy to see that the following diagram is commutative:

(5. 1) $$\begin{array}{ccc} sl(W)^{(k+1)} & \xrightarrow{\;g\;} & sl(W)^{(k-1)} \\ {\scriptstyle i(w)}\downarrow & & \downarrow{\scriptstyle i(gw)} \\ sl(W)^{(k)} & \xrightarrow[\;g\;]{} & sl(W)^{(k)} \end{array}$$

On the other hand it is known that $sl(W)$ is involutive and hence generic basis of W are regular for $sl(W)^{(k)}$. Combined with (5. 1) this implies that every basis is regular for $sl(W)^{(k)}$. In particular we have

LEMMA 5. 1. *Let $h'=sl(W)$ or $sp(W)$. Let v, w be linearly independent vectors in W. Then the following two maps are surjective:*

$$i(v):\ h'^{(k+1)}\longrightarrow h'^{(k)}\,.$$

$$i(w):\ \left\{A\in h'^{(k+1)}\,\middle|\, i(v)\,A=0\right\}\longrightarrow\left\{A\in h'^{(k)}\,\middle|\, i(v)\,A=0\right\}.$$

LEMMA 5. 2. *Let v, w be linearly independent vectors in W. Let A, B*

$\in h'^{(k)}+h'^{(k-1)}\odot U^*+\cdots+h'\odot S^k U^*$. *Assume* $A_v=B_w$. *Then there exists* $X\in g'_{k+1}$ *such that* $X_w=A$ *and* $X_v=B$.

PROOF. First assume $A, B\in h'^{(k)}$. By conditions there exists $Y\in h'^{(k+1)}$ such that $Y_v=B$. Since

$$i(v)(A-Y_w)=A_v-B_w=0,$$

there exists $Z\in h'^{(k+1)}$ such that $Z_v=0$ and $Z_w=A-Y_w$. Then $X=Y+Z$ has the prescribed property. The other cases reduce to the above case.

q. e. d.

After these preparations we will prove Proposition 4. 1. First we have

$$c_1(A, B)=-[A, B]\in g'_0 \tag{5. 2}$$

for $A, B\in g_0$. Let $v, w\in W$ and $X\in g'_1$. By (1. 1) applied to P_1 we have

$$Xc_1(w, v)-c_1(X_w, v)-c_1(w, X_v)=(\delta T)(w, v)$$

where $T\in\mathfrak{g}_1\otimes\wedge^2 V_0^*$. Recall that $\mathfrak{g}_1(t)\subset g_0\otimes V^*\subset gl(V+g_0)$ and $X_w\in g'_0$ for $X\in\mathfrak{g}_1$, $w\in W$. Hence we have

$$\alpha''_1(X_w, v)=\alpha''_1(X_v, w) \tag{5. 3}$$

where α''_1 is the $g''_0\otimes\wedge^2 V_0^*$-component of c_1. Define $\sigma\in GL(V_0)$ by

$$\sigma=id \qquad \text{on} \quad U+g_0$$

and

$$\sigma(w)=w+\alpha''_1(v, A) \qquad \text{for} \quad w\in W$$

where $v(\in W)$ and $A(\in g'_0)$ satisfy $A_v=w$. Suppose $B_u=w$ for $B\in g'_0$ and $u\in W$. Then, if v and u are linearly independent, there exists $X\in g'_1$ such that $X_u=A$ and $X_v=B$. Therefore (5. 3) implies

$$\alpha''_1(A, v)=\alpha''_1(B, u).$$

In case v and w are not linearly independent, we can similarly prove that $\alpha''_1(v, A)$ does not depend on the choice of v and A. R_σ denoting the right action of $GL(V_0)$ on $F(P_0)$, let $P'_1=R_\sigma P_0$ and c'_1 be the structure function of P'_1. Let $w\in W$ and $A\in g'_0$. Then, by (2. 8) of [1], we have

$$\begin{aligned} c'_1(w, A) &= \sigma^{-1}c_1(\sigma w, \sigma A) \\ &= \sigma^{-1}c_1\Big(w+\alpha''_1(v, B), A\Big) \\ &= \sigma^{-1}\Big\{A_w+\alpha_1(w, A)\Big\} \\ &= A_w-\alpha''_1(w, A)+\alpha_1(w, A) \\ &\equiv 0 \qquad \text{mod } W'_0 \end{aligned}$$

where $v \in W$ and $B \in g'_0$ satisfy $B_v = w$. If we denote by P_1 the conjugate bundle P'_1, we have

(5. 4) $$c_1 : W \otimes g'_0 \longrightarrow W'_0 .$$

Next let $w \in W$, $A \in g_0$ and $B \in g'_0$. By (1. 2) we have

$$c_1\Big(c_1(w, A), B\Big) + c_1\Big(c_1(A, B), w\Big) + c_1\Big(c_1(B, w), A\Big) = (\delta T)(w, A, B)$$

where $T \in \mathfrak{g}_1 \otimes \Lambda^2 V_0^*$. By (5. 4) the first term and the second term belong to W'_0. On the other hand the right hand is equal to $T(A, B) w$ and in W'_0. Hence we have

$$c_1(B_w, A) \equiv 0 \qquad \text{mod } W .$$

This implies

(5. 5) $$c_1 : W \otimes g_0 \longrightarrow W'_0 .$$

To prove

(5. 6) $$c_1 : W \otimes W \longrightarrow W'_0$$

let $v, w \in W$ and $A \in g_0$. Substituting v, w and A into (1. 2), we have

$$c_1(A_w, v) - c_1(A_v, w) \equiv 0 \qquad \text{mod } W'_0 .$$

If we put $A = I$ in this equation, we have (5. 6). This completes the proof of (1).

Let $u \in U_2$ and A, $B \in g'_0$. By (1. 2) we have

$$c_1\Big([A, B], u\Big) \equiv 0 \qquad \text{mod } W'_0 .$$

Combined with the fact $[g'_0, g'_0] = g'_0$, this implies

(5. 7) $$c_1 : g'_0 \otimes U_2 \longrightarrow W'_0 .$$

Next, substituting $u \in U_2$, $w \in W$ and $A \in g_0$ again into (1. 2), we have

$$c_1(A_w, u) - c_1(A_u, w) \equiv 0 \qquad \text{mod } W'_0 .$$

Setting $A = I$ in this equation, we have

(5. 8) $$c_1 : W \otimes U_2 \longrightarrow W'_0 .$$

(5. 7) and (5. 8) prove (2).

Finally we will prove (3). For $u \in U_2$ set $\alpha''_1(I, u) = f(u) I$. f is a U_2^*-valued function on N. Let $c_0(P_1)$ be the representative of c_0 uniquely determined by P_1 (cf. Proposition 1. 1), and denote by $\eta(u, v)$ the U_2-component of $c_0(P_1)(u, v)$ for $u, v \in U_2$. Then, substituting $u, v \in U_2$ and I into (1. 2), we have

(5. 9) $$f\left(\eta(u, v)\right)+\tilde{\rho}(u)f(v)-\tilde{\rho}(v)f(u)=0\,.$$

Define $\tilde{f}\in\Gamma(E^{*})$ by $\tilde{f}(\tilde{\rho}(u))=f(u)$. As in § 4, we will denote by d' the exterior differentiation with repsect to E. Then (5. 9) means

(5. 10) $$d'\tilde{f}=0\,.$$

For $u, v\in U_2$ set

$$\underline{\alpha}_1''(u, v)=h(u, v)\,I\,.$$

h is a $\wedge^2 U_2^*$-valued function on N. Let $u_i\in U_2$. ($i=1, 2, 3$.) By (1. 2) we have

(5. 11) $$\sum_S\left\{h\left(\eta(u_1, u_2), u_3\right)+h(u_1, u_2)f(u_3)+\tilde{\rho}(u_1)\,h(u_2, u_3)\right\}=0\,.$$

Define $\tilde{h}\in\Gamma(\wedge^2 E^{*})$ by $\tilde{h}(\tilde{\rho}(u), \tilde{\rho}(v))=h(u, v)$. Then (5. 11) means

(5. 12) $$d'\tilde{h}+\tilde{h}\wedge\tilde{f}=0\,.$$

We assert that under the conditions (5. 10) and (5. 12), there exists a local solution $\tilde{\sigma}\in\Gamma(E^{*})$ of the following differential equation:

(5. 13) $$d'\tilde{\sigma}-\tilde{\sigma}\wedge\tilde{f}-\tilde{h}=0\,.$$

In fact this is shown as follows. First there exists a function a such that $\tilde{f}=d'a$. Since

$$d'(e^{a}\tilde{h})=e^{a}(d'a\wedge\tilde{h}+d'\tilde{h})=0\,,$$

there exists $\xi\in\Gamma(E^{*})$ such that $e^{a}\tilde{h}=d'\xi$. Then for any function b, $\tilde{\sigma}=e^{-a}(\xi+d'b)$ satisfies (5. 13).

Let $\tilde{\sigma}$ be a solution of (5. 13) and σ be the U_2^*-valued function determined by $\tilde{\sigma}$. Define $\mu\in GL(V_0)$ by

$$\mu=id \qquad \text{on} \quad W+U_1+g_0$$

and

$$\mu(u)=u+\sigma(u)\,I \qquad \text{for} \quad u\in U_2\,.$$

Set $P_1'=R_\mu P_1$. For $u, v\in U_2$ we have

$$\begin{aligned}\underline{c}_1'(u, v)&=\mu^{-1}\underline{c}_1\left(\mu(u), \mu(v)\right)-\mu^{-1}\left\{\tilde{\rho}(u)\,\mu(v)\right\}+\mu^{-1}\left\{\tilde{\rho}(v)\,\mu(u)\right\}\\&\equiv\mu^{-1}\left\{\eta(u, v)+h(u, v)\,I+\sigma(u)f(v)\,I-\sigma(v)f(u)\,I\right\}\\&\qquad\qquad -\left(\tilde{\rho}(u)\,\sigma(v)\right)I+\left(\tilde{\rho}(v)\,\sigma(u)\right)I \qquad \bmod W_0'\\&=\eta(u, v)+\left\{-\sigma\left(\eta(u, v)\right)+h(u, v)+\sigma(u)f(v)-\sigma(v)f(u)\right.\\&\qquad\qquad \left.-\tilde{\rho}(u)\,\sigma(v)+\tilde{\rho}(v)\,\sigma(u)\right\}I\\&=\eta(u, v)\,.\end{aligned}$$

The last equality follows from (5. 13). Then, denoting by P_1 the conjugate bundle P_1', this prove (3).

§ 6. The higher order cases.

We will consider the higher order cases. Let the notation be the same as in the previous sections. The proof of the following proposition will be given in § 7.

PROPOSITION 6. 1. *We can choose the $\mathcal{G}$-structure $\mathcal{P}$ so that the following hold for* $1 \leqq k \leqq k_0$:

(1) *Let* $v, w \in W_{k-1}$. *Then* $\underline{c}_k(v, w) \in W'_{k-1}$.

(2) *Let* $u \in U_1$ *and* $v \in W'_{k-1}$. *Then the* g''_{k-2}*-component of* $\underline{c}_k(u, v)$ *is* 0.

Assuming the above proposition, we will prove Theorem 2. In the first place let $j_{k+1}: P_k \to P_{k+1}$ be a cross section. Let D_k be the distribution on P_k determined by j_{k+1}, i. e., for $p_k \in P_k$

$$(D_k)_{p_k} = j_{k+1}(p_k)(V_{k-1}) .$$

D_k defines a $\mathfrak{g}_k$-valued form ω_k on P_k. Set $\bar{\omega}_k = \lambda_k^{-1} \cdot \omega_k$, which is a g_k-valued form. Let θ_k be the fundamental form of P_k. Let $c_k(P_{k+1})$ be the representative of c_k uniquely determined by P_{k+1}, and let $c_{k+1}(j_{k+1})$ be the representative of c_{k+1} on $j_{k+1}(P_k)$ determined by the horizontal space $(j_{k+1})_* P_k$. Then by (2. 2)~(2. 4) we have

(6. 1) $$d\theta_k = c_k(P_{k+1})(\theta_k \wedge \theta_k) - \omega_k \wedge \theta_k .$$

(6. 2) $$d(j^*_{k+1}\theta_{k+1}) = c_{k+1}(j_{k+1})(j^*_{k+1}\theta_{k+1} \wedge j^*_{k+1}\theta_{k+1}) .$$

(6. 3) $$j^*_{k+1}\theta_{k+1} = \theta_k + \bar{\omega}_k .$$

Let X be a local vector field on M and suppose that X is an infinitesimal automorphism of P_k. Set $Y = \tilde{X}^{(k)}$. By Proposition 2. 3 we have

$$L_Y \omega_k = T \cdot \theta_k ,$$

where T is a $\mathfrak{g}_k^{(1)}$-valued function on P_k. (Note that $\mathfrak{g}_k^{(1)} = (\mathfrak{g}_k \otimes V^* \cap g_{k-1} \otimes S^2 V^*) \subset g_{k-1} \otimes V^* \otimes V^*$.) Let ω''_k be the $\mathfrak{g}''_k$-component of ω_k and S be the $\mathfrak{g}''^{(1)}_k$-component of T. Then we have

(6. 4) $$L_Y \omega''_k = S \cdot \theta_k = S \cdot \theta_{U_1} ,$$

where we identified the U_1-component of θ_k with θ_{U_1}. Let $F_{k+1}: \mathfrak{g}''_k \otimes U_1^* \to g''_k \otimes U_1^* / \mathfrak{g}''_{k+1}$ be the composition of the maps $\lambda_k^{-1} \otimes id: \mathfrak{g}''_k \otimes U_1^* \to g''_k \otimes U_1^*$ and $g''_k \otimes U_1^* \to g''_k \otimes U_1^* / \mathfrak{g}''_{k+1}$. Since $g'_{k+1} = g'^{(1)}_k$, we know that Y is an infinitesimal automorphism of P_{k+1} if and only if

$$F_{k+1}(S)=0\,.$$

We will prove that $\bar{\omega}''_k(Y)$ is a function on N. This is shown as follows. First note that by (6.4) $L_Y\bar{\omega}''_k$ is a linear combination of θ_{U_1}. Let $\alpha_{k+1}(j_{k+1})$ be the $g_k\otimes\wedge^2 V^*_k$-component of $c_{k+1}(j_{k+1})$ and $\alpha''_{k+1}(j_{k+1})$ be the $g''_k\otimes\wedge^2 V^*_k$-component of $\alpha_{k+1}(j_{k+1})$. Then we have

$$L_Y\bar{\omega}''_k=di(Y)\,\bar{\omega}''_k+i(Y)\,d\bar{\omega}''_k \tag{6.5}$$

and

$$d\bar{\omega}''_k=\alpha''_{k+1}(j_{k+1})\Big((\theta_k+\bar{\omega}_k)\wedge(\theta_k+\bar{\omega}_k)\Big) \tag{6.6}$$

by (6.2). Since $(\theta_k+\bar{\omega}_k)(Y)\in W_k$, Proposition 6.1 implies that $i(Y)\,d\bar{\omega}''_k$ is a linear combination of θ_{U_1} and θ_{U_2}. Therefore we know that $d\bar{\omega}''_k(Y)$ is a linear combination of θ_{U_1} and θ_{U_2}, and hence $\bar{\omega}(Y)$ is a function on N.

As in § 4, let $\bar{\omega}''_0(\tilde{X}^{(0)})=\phi I$, where ϕ is a function on N. We prove

LEMMA 6.2. *There exists a linear differential operator Ψ_k of order k on N such that $\bar{\omega}''_k(Y)=\Psi_k(\phi)$.*

PROOF. Suppose that for $l<k$ there exists a linear differential operator of order l such that $\bar{\omega}''_l(\tilde{X}^{(l)})=\Psi_l(\phi)$. Let $p, q\in P_k$ with $\pi_{k-1}(p)=\pi_{k-1}(q)=p'$ and $(\rho\circ\pi_0\circ\cdots\circ\pi_{k-2})(p')=t$. Let $G_k\subset GL(V_{k-1})$ be the connected Lie group whose Lie algebra is $\mathfrak{g}_k(t)$ $(\subset g_{k-1}\otimes V^*\subset GL(V_{k-1}))$. Then there exists $a\in G_k$ satisfying $q=pa$. We have

$$(\theta_k)_q(Y_q)=q^{-1}(\tilde{X}^{(k-1)}_{p'})=a^{-1}\Big\{(\theta_k)_p(Y_p)\Big\}\,. \tag{6.7}$$

On the other hand, since the V-component of $\theta_k(Y)$ is in W, we know that

$$a^{-1}\Big\{(\theta_k)_p(Y_p)\Big\}\equiv(\theta_k)_p(Y_p)\qquad \mathrm{mod}\ g'_{k-1}\,.$$

Combined with (6.7), this means that the g''_{k-1}-component of $\theta_k(Y)$ is constant on the fibers of the fibered mainfold $\pi_{k-1}: P_k\to P_{k-1}$. Let us denote by ζ''_k the g''_{k-1}-component of θ_k. Then we have

$$\zeta''_k(Y)=\zeta''_k(j_{k*}\tilde{X}^{(k-1)})=\bar{\omega}''_{k-1}(\tilde{X}^{(k-1)})=\Psi_{k-1}(\phi)\,.$$

By similar considerations we can prove that for $l<k$
the g''_l-component of $\theta_k(Y)=\bar{\omega}''_l(\tilde{X}^{(l)})=\Psi_l(\phi)$.

Let θ'_k be the V'_{k-1}-component of θ_k and $\theta''_k=\theta_k-\theta'_k$ be the $(g''_0+\cdots+g''_{k-1})$ -component of θ_k. Then it follows that

$$\theta''_k(Y)=\sum_{l=0}^{k-1}\Psi_l(\phi)\,.$$

For $u\in U_1$, let u_{D_k} be the cross section of D_k satisfying $\theta_k(u_{D_k})=u$.

Since $L_Y\theta_k=0$, we have

$$d\theta_k(Y, u_{D_k}) = \big(i(Y)\, d\theta_k\big)(u_{D_k})$$
$$= -d\big(\theta_k(Y)\big)(u_{D_k}).$$

In particular, we have

$$d\zeta''_k(Y, u_{D_k}) = -\tilde{\rho}(u)\,\Psi_{k-1}(\phi).$$

Let us substitute Y, u_{D_k} into (6. 1) and consider its g''_{k-1}-component. By Proposition 1. 1 and Proposition 6. 1 in which k is replaced by $k+1$, we know

$$\alpha''_k(P_{k+1})\big(\theta'_k(Y), u\big) = 0.$$

Hence it follows from above arguments that

$$(6.8)\qquad -\tilde{\rho}(u)\,\Psi_{k-1}(\phi) = \alpha''_k(P_{k+1})\big(\theta''_k(Y), u\big) - \omega''_k(Y)\,u$$
$$= \sum_{l=0}^{k-1} \alpha''_k(P_{k+1})\big(\Psi_l(\phi), u\big) - \omega''_k(Y)\,u.$$

Let $\{u_i\}$ be a basis of U_1 and $\{u_i^*\}$ be its dual basis of U_1^*. Define $A_i \in g''_{k-1}$ by

$$(6.9)\qquad A_i = \tilde{\rho}(u_i)\,\Psi_{k-1}(\phi) + \sum_{l=0}^{k-1} \alpha''_k(P_{k+1})\big(\Psi_l(\phi), u_i\big).$$

Then we have

$$(6.10)\qquad \omega''_k(Y) = \sum_i A_i \otimes u_i^*.$$

Recall that $\bar{\omega}''_k(Y)$ is a function on N. It follows from (6. 8) that $\alpha''_k(P_{k+1})$ is a function on N. Therefore the right hand of (6. 9) defines a k-th order linear differential operator acting on ϕ. Then (6. 10) and the definition of $\bar{\omega}''_k$ imply our assertion. q. e. d.

After these preparations we will prove Theorem 2. First note that Proposition 6. 1 in which k is replaced by $k+2$ implies that the $g''_k \otimes W'^*_k \otimes U_1^*/\delta(\mathfrak{g}''_{k+1} \otimes W'^*_k)$-component of c_{k+1} is 0. Hence we have

$$\alpha''_{k+1}(j_{k+1})\big(\theta'_k(Y) + \bar{\omega}'_k(Y), u\big) = T_{\theta'_k(Y)+\bar{\omega}'_k(Y)}\,u$$

where $T \in \mathfrak{g}''_{k+1} \otimes W'^*_k$. By (6. 6) we have

$$d\bar{\omega}''_k(Y, u_{D_k}) = T_{\theta'_k(Y)+\bar{\omega}'_k(Y)}\,u + \alpha''_{k+1}(j_{k+1})\big(\theta''_k(Y) + \bar{\omega}''_k(Y), u\big)$$
$$= T_{\theta'_k(Y)+\bar{\omega}'_k(Y)}\,u + \sum_{l=0}^{k} \alpha''_{k+1}(j_{k+1})\big(\Psi_l(\phi), u\big)$$

Define $S'_{k+1}(\phi)_i \in g''_k$ by

(6. 11) $$S'_{k+1}(\phi)_i = \tilde{\rho}(u_i)\,\Psi_k(\phi) + \sum_{l=0}^{k} \alpha''_{k+1}(j_{k+1})\left(\Psi_l(\phi), u_i\right)$$

and set

$$S'_{k+1}(\phi) = \sum_i \lambda_k\left(\left(S'_{k+1}(\phi)_i\right) \otimes u_i^*\right).$$

Then by (6. 5) and the above arguments we know that

$$S \equiv S'_{k+1}(\phi) \qquad \mathrm{mod}\ \mathfrak{g}''_{k+1},$$

where S is the $\mathfrak{g}''^{(1)}_k$-valued function satisfying (6. 4). This proves that Y is an infinitesimal automorphism of P_{k+1} if and only if

(6. 12) $$F_{k+1}(S'_{k+1}(\phi)) = 0 .$$

By (6. 11) this is a $(k+1)$-th order differential equation with respect to ϕ.

From the above arguments it follows that a local vector field X on M is an infinitesimal automorphism of P_{k+1} if and only of

(6. 13) $$F_1\left(S'_1(\phi)\right) = F_2\left(S'_2(\phi)\right) = \cdots = F_{k+1}\left(S'_{k+1}(\phi)\right) = 0 .$$

Let $Q_{k+1} \subset J_{k+1}(N \times \mathbf{R})$ be the differential equation defined by (6. 13) and q_{k+1} be the symbol of Q_{k+1}. It is not difficult to see that q_{k+1} can be identified with $\mathfrak{h}_{k+1}$. For $p_k \in P_k$, let

$$\mathscr{L}_{p_k} = \left\{X \in \mathscr{L} \,\middle|\, (\tilde{X}^{(k)})_{p_k} = 0\right\}.$$

Let $\bar{\omega}''_0(\tilde{X}) = \phi I$. Then we have a surjective map

$$\mathscr{L}_{p_k} \ni X \longrightarrow j_t^{k+1}(\phi) \in (q_{k+1})_t$$

where $t = (\rho \circ \pi_0 \circ \cdots \circ \pi_k)(p_k)$. These show that for any k, Q_k is a subbundle of $J_k(N \times \mathbf{R})$ and moreover Q_k is integrable. This completely proves the theorem.

§ 7. Proof of Proposition 6. 1.

We will prove Proposition 6. 1 by induction. By Proposition 4. 1, (1) of Proposition 6. 1 holds for $k=1$. (2) is trivial. Assume that we can choose the sequence of bundles

$$P_{k-1} \longrightarrow P_{k-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M$$

so that Proposition 6. 1 holds. For $-1 \leqq i \leqq k-1$ consider the following statement:

$(7.1)_i$ (1) Let $A \in g_i$ and $w \in W_{k-1}$. Then

$$\underline{c}_k(A, w) \in W'_{k-1} .$$

(2) Let $A\in g'_i$ and $u\in U_1$. Then the g_2-component of $\underline{c}_k(u, A)$ is 0.

We will prove $(7.1)_i$ by induction on i. The proof will be devided into several steps.

7.1. Let $X\in \mathfrak{g}'_k$, $w\in W$ and $A\in g_0+g_1+\cdots+g_{k-1}$. By (1.1) applied to P_k, we have

$$X\underline{c}_k(w, A)-\underline{c}_k(X_w, A)=T_A w,$$

where $T\in \mathfrak{g}_k\otimes V^*_{k-1}$. Since $X\underline{c}_k(w, A)$ and $T_A w$ are in g'_{k-1}, this implies

$$\underline{c}_k(X_w, A)\in g'_{k-1}.$$

Hence we know that

$$(7.2)\qquad \underline{c}_k:\ g'_{k-1}\otimes(g_0+\cdots+g_{k-1})\longrightarrow W'_{k-1}.$$

Similarly substituting v, $w\in W$ and $X\in g'_k$ into (1.1), we have

$$(7.3)\qquad \underline{\alpha}''_k(X_w, v)=\underline{\alpha}''_k(X_v, w).$$

Define $\sigma\in GL(V_{k-1})$ by

$$\sigma=id \qquad \text{on} \quad V+g_0+\cdots+g_{k-3}+g''_{k-2}+g_{k-1}$$

and

$$\sigma(A)=A+\underline{\alpha}''_k(w, B) \qquad \text{for} \quad A\in g'_{k-2},$$

where $w\in W$ and $B\in g'_{k-1}$ satisfy $B_w=A$. Suppose that $v\in W$ and $C\in g'_{k-1}$ also satisfy $C_v=A$. Then by Lemma 5.2 there exists $X\in g'_k$ such that $X_w=C$ and $X_v=B$. It follows from (7.3) that

$$\underline{\alpha}''_k(v, C)=\underline{\alpha}''_k(w, B).$$

Therefore σ is well-defined. Let $P'_k=R_\sigma P_k$ and c'_k be the structure function of P'_k. For $w\in W$ and $A\in g'_{k-1}$ we have

$$\begin{aligned}\underline{c}'_k(w, A)&=\sigma^{-1}\underline{c}_k(\sigma w, \sigma A)\\&=\sigma^{-1}\underline{c}_k(w, A)\\&=\sigma^{-1}\{A_w+\alpha_k(w, A)\}\\&=A_w-\alpha''_k(w, A)+\alpha_k(w, A)\\&\equiv 0 \qquad \text{mod}\ W'_{k-1}.\end{aligned}$$

Denoting by P_k the conjugate bundle P'_k, we have

$$(7.4)\qquad \underline{c}_k:\ g'_{k-1}\otimes W\longrightarrow W'_{k-1}.$$

Secondly let $u\in U_1$, $A\in g''_0+\cdots+g''_{k-1}$ and $B\in g''_{k-1}$. *By* (1.2) we have

$$(7.5)\qquad \underline{c}_k\big(\underline{c}_k(u, A), B\big)+\underline{c}_k\big(\underline{c}_k(A, B), u\big)+\underline{c}_k\big(\underline{c}_k(B, u), A\big) + \tilde{\rho}(u)\, \underline{c}_k(A, B) = T(A, B)\, u\,,$$

where $T\in\mathfrak{g}_k\otimes\Lambda^2 V^*_{k-1}$. Let us consider the g''_{k-2}-component of (7. 5). Recalling $g''_0=\{I\}$, we have

$$\underline{c}_k(u, A)\in g_0+\cdots+g_{k-1}\,.$$

Hence the first term contains no element in g''_{k-2}. (Recall (2) of Proposition 1. 1.) The g''_{k-2}-component of the second term is $\underline{\alpha}''_k(A, B)\, u$. Using the induction assumption, we know that the g''_{k-2}-component of the third term is 0. Similarly the fourth term and the right hand contain no element in g''_{k-2}. Hence we have $\underline{\alpha}''_k(A, B)\, u=0$. This implies

$$(7.6)\qquad \underline{c}_k:\ g''_{k-1}\otimes(g''_0+\cdots+g''_{k-1})\longrightarrow W'_{k-1}\,.$$

Therefore we have proved (1) of $(7.1)_{k-1}$. (2) of $(7.1)_{k-1}$ is trivial.

7. 2. Let $l\geqq 0$. Suppose that for $i\geqq l+1$ $(7.1)_i$ holds. We will prove $(7.1)_l$.

First note that for $A\in g_i$ and $B\in g_j$ $(i, j\neq -1)$ we have

$$\underline{c}_k(A, B)\in g_{\max(i,j)}+\cdots+g_{k-1}\,.$$

Then, substituting $w\in W$, $A\in g_0+\cdots+g_{k-1}$ and $B\in g'_{l+1}$ into (1. 2), we can prove

$$\underline{\alpha}''_k(B_w, A)=0\,.$$

This implies

$$(7.7)\qquad \underline{c}_k:\ g'_l\otimes(g+\cdots+g_{k-1})\longrightarrow W'_{k-1}\,.$$

Secondly, substituting v, $w\in W$ and $A\in g'_{l+1}$ into again (1. 2), we have

$$(7.8)\qquad \underline{\alpha}''_k(A_w, v)=\underline{\alpha}''_k(A_v, w)\,.$$

Define $\sigma\in GL(V_{k-1})$ by

$$\sigma=id \qquad \text{on}\quad V+\cdots+g_{l-2}+g''_{l-1}+g_l+\cdots+g_{k-1}$$

and for $A\in g'_{l-1}$

$$\sigma(A)=A+\underline{\alpha}''_k(w, B)\,,$$

where $w\in W$ and $B\in g'_l$ satisfy $B_w=A$. Then by (7. 8) and Lemma 5. 2 we can prove that σ is well-defined. Let $P'_k=R_\sigma P_k$. For $w\in W$ and $A\in g'_l$, let S be the $(g_1+\cdots+g_{k-2})$-component of $\underline{c}_k(w, A)$. By induction assumption S belongs to $g'_l+\cdots+g'_{k-2}$ and we have

$$\underline{c}_k(w, A)=A_w+S+\underline{\alpha}_k(w, A)\,.$$

Then, denoting by c'_k the structure function of P'_k, we have

$$\begin{aligned} \underline{c}'_k(w, A) &= \sigma^{-1}\underline{c}_k(\sigma w, \sigma A) \\ &= \sigma^{-1}\{A_w + S + \underline{\alpha}_k(w, A)\} \\ &= A_w - \underline{\alpha}''_k(w, A) + S + \underline{\alpha}_k(w, A) \\ &\equiv 0 \qquad \mathrm{mod}\ W'_{k-1}. \end{aligned}$$

Therefore, denoting by P_k the conjugate bundle P'_k, we have

(7. 9) $\qquad \underline{c}_k:\ g'_l \otimes W \longrightarrow W'_{k-1}.$

It follows from (7. 7) and (7. 9) that

(7. 10) $\qquad \underline{c}_k:\ g'_l \otimes W_{k-1} \longrightarrow W'_{k-1}.$

Next we will prove (2) of $(7.1)_l$. Let $w \in W$, $u \in U_1$ and $A \in g'_{l+1}$. By (1. 2) we have

$$\underline{c}_k\big(\underline{c}_k(w, u), A\big) + \underline{c}_k\big(\underline{c}_k(u, A), w\big) + \underline{c}_k\big(\underline{c}_k(A, w), u\big) + \tilde{\rho}(u)\, \underline{c}_k(A, w) = (\delta T)\,(w, u, A)$$

where $T \in \mathfrak{g}_k \otimes \wedge^2 V^*_{k-1}$. The first term is in W'_{k-1}. Since

$$\underline{c}_k(u, A) \in g'_l + g_{l+1} + \cdots + g_{k-1},$$

the second term is in W'_{k-1} by (7. 10) and the induction assumption. Similarly the fourth term is in W'_{k-1}. On the other hand the g''_{k-2}-component of the third term is the same as the g''_{k-2}-component of $-\underline{c}_k(A_w, u)$ and the right hand contains no element in g''_{k-2}. This proves (2) of $(7.1)_l$.

Finally we will prove

(7. 11) $\qquad \underline{c}_k:\ g''_l \otimes W_{k-1} \longrightarrow W'_{k-1}.$

Substituting $u \in U_1$, $A \in g''_l$ and $B \in W_{k-1}$ into (1. 2), we have

$$\underline{c}_k\big(\underline{c}_k(u, A), B\big) + \underline{c}_k\big(\underline{c}_k(A, B), u\big) + \underline{c}_k\big(\underline{c}_k(B, u), A\big) + \tilde{\rho}(u)\, \underline{c}_k(A, B) = (\delta T)\,(u, A, B)$$

where $T \in \mathfrak{g}_k \otimes \wedge^2 V^*_{k-1}$. Using the induction assumption, we know that the g''_{k-2}-component of the first term and the third term are 0. Since

$$\underline{c}_k(A, B) \in g'_l + \cdots + g'_{k-2} + g_{k-1}$$

by induction assumption, the g''_{k-2}-component of the second term is $\underline{\alpha}''_k(A, B)\,u$. The fourth term and the right hand have no element in g''_{k-2}. Hence we have $\underline{\alpha}''_k(A, B)\,u = 0$, and so $\underline{\alpha}''_k(A, B) = 0$. This proves (7. 11).

7.3. Finally we will show $(7.1)_{-1}$. It is sufficient to prove (2) of $(7.1)_{-1}$ and

$$(7.12) \qquad \underline{c}_k : W \otimes W \longrightarrow W'_{k-1}.$$

Let v, $w \in W$ and $A \in g_0$. Then by (1.1) we can prove

$$\underline{\alpha}''_k(A_w, v) = \underline{\alpha}''_k(A_v, w).$$

In particular, if we put $A=I$, we have $\underline{\alpha}''_k(w, v)=0$. This prove (7.12). (2) of $(7.1)_{-1}$ can be proved by the similar method as the proof of (2) of $(7.1)_l$. This completes the proof of Proposition 6.1.

References

[1] K. KISO: Local properties of intransitive infinite Lie algebra sheaves, Japan. J. Math. Vol. 5 (1979), 101–155.

[2] T. MORIMOTO: On the intransitive Lie algebras whose transitive parts are infinite and primitive, J. Math. Soc. Japan 29 (1977), 35–65.

[3] I. M. SINGER and S. STERNBERG: On the infinite group of Lie and Cartan, J. Analyse Math. 15 (1965), 1–114.

Department of Mathematics
Faculty of General Education
Ehime University