Hokkaido Mathematical Journal Vol. 11 (1982) p. 201-204

On s-distance subsets in real hyperbolic space

By Eiichi BANNAI

(Received June 29, 1981)

Abstract

It is shown that if X is an s-distance subset in real hyperbolic space H^{d}, then

$$|X|\leq\binom{d+s}{s}+\binom{d+s-1}{s-1}.$$

Introduction

A subset X in a metric space M is called an s-distance subset in M if there are s distinct distances $\alpha_{1}, \alpha_{2}, \cdots, \alpha_{s}$, and all the α_{i} are realized. Delsarte-Goethals-Seidel [6] have shown that the cardinality $|X|$ of an s-distance subset X in the d-dimensional unit sphere $S^{d}=\{(x_{1}, x_{2}, \cdots, x_{d+1})|x_{1}^{2}+x_{2}^{2}+\cdots+x_{d+1}^{2}=1\}\subset \boldsymbol{R}^{d+1}$ is bounded from above as

$$|X|\leq\binom{d+s}{s}+\binom{d+s-1}{s-1}. \tag{1}$$

Larman-Rogers-Seideal [9] and Bannai-Bannai [1] have shown that the same upper bound (1) is obtained for the cardinality of an s-distance subset in real Euclidean space $\boldsymbol{R}^{d}$. In this paper we prove that the same bound (1) is also true for an s-distance subset in the real hyperbolic space H^{d} of (topological) dimension d. That is:

THEOREM 1. *If X is an s-distance subset in H^{d}, then*

$$|X|\leq\binom{d+s}{s}+\binom{d+s-1}{s-1}.$$

1. PROOF OF THEOREM 1

The basic idea of the proof is the same as that of Delsarte-Goethals-Seidel [6] and Koornwinder [8]. Here we need a proper realization of the hyperbolic space H^{d} in $\boldsymbol{R}^{d+1}$.

(i) It is known that the hyperbolic space H^{d}, which is also called Lobatschewsky and Bolyai space, of dimension d is realized in a Euclidean space of $\boldsymbol{R}^{d+1}$ as

$$H^d=\left\{(x_1,\cdots,x_{d+1})\in \boldsymbol{R}^{d+1}\,|\,x_1^2-x_2^2-\cdots-x_{d+1}^2=1,\ x_1>0\right\}$$

with the distance $d(x,y)$ for $x=(x_1,x_2,\cdots,x_{d+1})$ and $y=(y_1,y_2,\cdots,y_{d+1})\in H^d$ being given by

$$d(x,y)=\operatorname{arc\,cosh}(x_1y_1-x_2y_2-\cdots-x_dy_d-x_{d+1}y_{d+1})\,.$$

(See, for example, [5, page 209], [4, pages 375-6].)

(ii) Let X be an s-distance subset in H^d and let $\alpha_1,\alpha_2,\cdots,\alpha_s$ be the distances. For each $y\in X$ let us define

$$F_y(x)=\prod_{i=1}^{s}\frac{((x,y)-\cosh\alpha_i)}{(1-\cosh\alpha_i)}\,,\quad \text{for } x\in H^d,$$

where $(x,y)=x_1y_1-x_2y_2-\cdots-x_{d+1}y_{d+1}$. Since $F_y(x)=\delta_{x,y}$ for $x\in X$, the set $\{F_y(x)|y\in X\}$ is linearly independent. Also note that each $F_y(x)$ is a polynomial of degree s in $x_1,\cdots,x_{d+1}$.

(iii) In order to complete the proof of Theorem 1, we have only to show that the dimension of the space spanned by the set $\{F_y(x)|y\in X\}$ is bounded by the right hand side of (1). Now we use the following lemma:

LEMMA 2. *Let H_j be the space of homogeneous polynomials of degree j in $x_1,x_2,\cdots,x_{d+1}$, and let $\Delta^{(1,d)}$ be the differential operator defined by*

$$\Delta^{(1,d)}=\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\cdots-\frac{\partial^2}{\partial x_{d+1}^2}\,.$$

Then we have

(a) *The* map $\Delta^{(1,d)}$ *from H_j to H_{j-2} is onto, and so*

$$\text{dimension (kernel of } \Delta^{(1,d)}: H_j\to H_{j-2})=\binom{d+j}{j}-\binom{d+j-2}{j-2}.$$

(Note that $\dim H_j=\binom{d+j}{j}$.)

(b) *Each $f\in H_j$ is uniquely expressed as*

$$f=f_j+(x_1^2-x_2^2-\cdots-x_{d+1}^2)f_{j-2}+(x_1^2-x_2^2-\cdots-x_{d+1}^2)^2f_{j-4}$$
$$+\cdots+(x_1^2-x_2^2-\cdots-x_{d+1}^2)^{[\frac{j}{2}]}f_{j-2[\frac{j}{2}]}\,,$$

where $\quad f_{j-2i}\in(\text{kernel of } \Delta^{(1,d)}: H_{j-2i}\to H_{j-2(i+1)})\,.$

(c) *The dimension of the space of polynomial functions on H^d of degree $\leq s$ in $x_1,x_2,\cdots,x_{d+1}$ is bounded from above by*

$$\sum_{j=0}^{s}\binom{d+j}{j}-\binom{d+j-2}{j-2}=\binom{d+s}{s}+\binom{d+s-1}{s-1}.$$

PROOF OF LEMMA 2 Proof is almost identical with the proof of the expansion of a polynomial using harmonic polynomials (cf [7, Vol. 2, page 237]), that is with respect to the Laplacian

$$\Delta=\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\cdots+\frac{\partial^2}{\partial x_{d+1}{}^2}\,.$$

To prove (a) we have only to show that

$$\Delta^{(1,d)}\left\{(x_1^2-x_2^2-\cdots-x_{d+1}^2)f\right\}\neq 0$$

for any non-zero polynomial f. This is straightforwardly proved as $\Delta\{(x_1^2+x_2^2+\cdots+x_{d+1}^2)f\}\neq 0$ is proved for any non-zero polynomial f. The rest of the statements in Lemma 2 are easy consequences of this.

Now Lemma 2 (c) completes the proof of Theorem 1.

REMARKS

(i) It would be interesting to know how much the common bound (1) can be improved for each S^d, $\boldsymbol{R}^d$, H^d.

(a) For spherical case Bannai-Damerell [2] proved that the equality does not hold if $s\geq 3$ and $d\geq 2$. For $s=2$ it is still an open problem when the equality is attained. (Such examples exist for $d=1$, 5 and 21, cf. [6, 11].)

(b) For Euclidean case Bannai-Bannai [1] proved that the equality never holds. Recently Blokhuis [3] has shown that the bound is improved. His argument easily reduces the bound (1) by $d+1$ for any $s\geq 2$. (Further improvement for larger s will be discussed later.)

(c) Problem: How much the bound (1) can be improved for hyperbolic case? (At the time of this writing I do not know whether the bound (1) is attained in the hyperbolic case.)

(ii) Neumaier [10] tries to get similar type of results by introducing a notion of "dimension d for a set X". However it seems that his notion of "dimension d" is not directly related to the topological dimension d of the space used here, and that his dimension d is generally larger than the topological dimension d (for the case H^d). Problem: Is it possible to find some meaningful relations between these two dimensions (for the case H^d)?

References

[1] BANNAI, E. and BANNAI, E.: "An upper bound for the cardinality of an s-distance subset in real Euclidean space," Combinatorica (Hungary), (1981), 1, 99–102.

[2] BANNAI, E. and DAMERELL, R. M.: "Tight spherical designs", I. J. of Math. Soc. Japan, 31 (1979), pp. 199–207.

[3] BLOKHUIS, A.: "A new upper bound for the cardinality of 2-distance sets in Euclidean space", Eindhoven University of Technology, Memorandum 1981–4, (February 1981).

[4] BUSEMANN, H.: *The Geometry of Geodesic*, Academic Press, New York, 1955.

[5] COXETER, H. S. M.: *Non-Euclidean Geometry*, The University of Toronto Press, 1942.

[6] DELSARTE, P., GOETHALS, J. M. and SEIDEL, J. J.: "Spherical codes and designs", Geometriae Dedicata 6 (1977), pp. 363–388.

[7] ERDÉRYI, A., MAGNUS, OBERHETTINGER, F. and TRICOMI, F. G.: *Higher Transcendental Functions* (Bateman manuscript project) McGraw-Hill, 1953.

[8] KOORNWINDER, T.: "A note on the absolute bound for systems of lines", Indag. Math. 38 (1976), pp. 152–153.

[9] LARMAN, D. G., ROGERS, C. A. and SEIDEL, J. J.: "On two-distance sets in Euclidean space", Bull. London Math. Soc. 9 (1977), pp. 261–267.

[10] NEUMAIER, A.: "Distance matrices, demension, and conference graphs", Indag. Math. (to appear).

[11] SEIDEL, J. J.: "Graphs and two-distance sets", in Springer Lecture Notes 884 (1981), 90–98.

Department of Mathematics
The Ohio State University