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Invariant flat projective structures on homogeneous spaces

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Introduction

Let M be an n-dimensional ($n\geqq 2$) manifold with a projective structure. It is well known that there exists a unique projective normal Cartan connection which induces the original projective structure. E. Cartan proved this fact locally by his method of moving frames ([1], [2]), and later it was settled in the rigorous form using principal fibre bundles by Tanaka, Kobayashi and Nagano ([5], [6], [11]).

In this paper we shall study, as an application of the theory of Cartan connections, invariant flat projective structures (which we abbreviate IFPS) on a homogeneous space $M=L/K$. Our first main result is that there exists a natural one-to-one correspondence between the set of IFPS on $M=L/K$ and the set of projective equivalence classes of Lie algebra homomorphisms $f: \mathfrak{l}\rightarrow\mathfrak{sl}(n+1, \boldsymbol{R})$ ($\mathfrak{l}$ is a Lie algebra of L) satisfying certain conditions (Theorem 2. 12). This is a natural generalization of the classical theory concerning invariant affine connections on $M=L/K$ (cf. Vol. II [7]). Using this correspondence we can determine the existence or non-existence of IFPS on many real simple Lie groups and irreducible Riemannian symmetric spaces. It will be proved that, among them, $M=SO(3)$, $SL(m, \boldsymbol{R})$ ($m\geqq 2$), $SU^{*}(2m)$ ($m\geqq 2$), $SL(m, \boldsymbol{R})/SO(m)$ ($m\geqq 2$) and $SU^{*}(2m)/Sp(m)$ ($m\geqq 2$) admit an IFPS. We determine the number of IFPS on these spaces (Theorem 5. 3, 5. 11, 7. 1 and 8. 5). For example $M=SL(m, \boldsymbol{R})/SO(m)$ ($m\geqq 3$) admits two projectively flat invariant affine connections and neither of them is the canonical (Riemannian) connection.

In [5] and [6] a projective structure on M is defined by a reduction of structure group of $P^{2}(M)$, the bundle of 2-jet frames over M, to a certain subgroup of $G^{2}(n)$. But for the later convenience we take the standpoint of Tanaka [12], not using the jet theory. For this reason, in § 1 we review the theory of projective Cartan connections and fix our notations, following [12]. In § 2 we prove the first main result in this paper (Theorem 2. 12). Let $M=L/K$ be an n-dimensional homogeneous space with an invariant projective structure and let (P, ω) be the corresponding projective normal Cartan connection. Then a natural bundle map $j: L \to P$ is constructed and an $\mathfrak{sl}(n+1, \boldsymbol{R})$-valued 1-form $j^*\omega$ on L is left invariant. If the projective structure on M is flat, $j^*\omega: \mathfrak{l} \to \mathfrak{sl}(n+1, \boldsymbol{R})$ is a Lie algebra homomorphism and thus we obtain a homomorphism corresponding to the original projective structure. In § 3 we study the case where an IFPS on $M=L/K$ admits an invariant affine connection. We prove that there is a one-to-one correspondence between the set of projectively flat invariant affine connections on $M=L/K$ and the set of Lie algebra homomorphisms $f: \mathfrak{l} \to \mathfrak{sl}(n+1, \boldsymbol{R})$ satisfying certain conditions (Theorem 3. 5) and that a projectively flat invariant affine connection on M is affinely flat if and only if the corresponding homomorphism is $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued. (Note that $\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ has a graded Lie algebra structure: $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ and $\mathfrak{g}_{-1}+\mathfrak{g}_0$ is isomorphic to the affine Lie algebra $\mathfrak{a}(n, \boldsymbol{R})$. See § 1.) In § 4 we show that for each projective equivalence class of Lie algebra homomorphisms there exists a unique normalized one (which we call an (N)-homomorphism). As an application it will be proved that an IFPS on a reductive homogeneous space admits an invariant affine connection. Furthermore we give an algorism to obtain all (N)-homomorphisms for $M=L/K$ (Proposition 4. 8). Since we know all the real irreducible representations for each real simple Lie algebra, we can determine the existence or non-existence of an IFPS on many homogeneous spaces $M=L/K$ where L is real simple. In § 5, 6 and 7 we treat the case where $M=L$ is a Lie group and in § 8 the case where $M=L/K$ is an irreducible Riemannian symmetric space of the classical type.

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Preliminary remarks

Throughout this paper we always assume the differentiability of class C^∞. We denote by $\mathfrak{X}(M)$ the set of all vector fields on a manifold M and $\mathfrak{g}^c$, f^c the complexification of a real Lie algebra $\mathfrak{g}$ and a real Lie algebra homomorphism f respectively. We assume that the dimension of M is always not less than two.

§ 1. Projective Cartan connections

In this section we shall review the theory of projective Cartan connections and fix our notations and terminology, following Tanaka [12]. For the proof of some known facts, see [12].

1.1. Let M be a manifold of dimension $n(\geqq 2)$ and let ∇, ∇' be two torsionfree affine connections on M. We say that ∇ and ∇' are projectively equivalent (which we denote by $\nabla \sim \nabla'$) if there exists a 1-form ϕ on M such that

$$\nabla_X Y - \nabla'_X Y = \phi(X)\, Y + \phi(Y)\, X$$

for all $X, Y \in \mathfrak{X}(M)$. Obviously $\sim$ is an equivalence relation and an equivalence class $[\nabla]$ containing ∇ is called a projective structure on M.

Let M (resp. M') be a manifold of dimension n and $[\nabla]$ (resp. $[\nabla']$) be a projective structure on M (resp. M'). A diffeomorphism $f: M \to M'$ is said to be a projective isomorphism if $f^*\nabla' \sim \nabla$ where $f^*\nabla'$ is an affine connection on M defined by $(f^*\nabla')_X Y = f_*^{-1}(\nabla'_{f_* X} f_* Y)$ for $X, Y \in \mathfrak{X}(M)$.

Let ∇_0 be the standard affine connection on $\boldsymbol{R}^n$ defined by $\nabla_{0X_i} X_j = 0$ $(X_i = \partial/\partial x_i)$, for $i, j = 1, \cdots, n$. A projective structure $[\nabla]$ on M is said to be projectively flat if for each point p of M there exists a neighborhood U of p and a diffeomorphism f from U to an open subset of $\boldsymbol{R}^n$ such that $f^*\nabla_0 \sim \nabla$ on U.

REMARK 1.1. Let (M, g) be a pseudo-Riemannian manifold. It is well known that a projective structure on M defined by the Levi-Civita connection of (M, g) is projectively flat if and only if (M, g) is a space of constant curvature (cf. § 34 [3]).

There exists a standard projective structure on an n-dimensional real projective space $P^n(\boldsymbol{R})$ defined by a constant curvature metric.

1.2. Let G be a projective transformation group of $P^n(\boldsymbol{R})$ and G' be an isotropy subgroup at the point $o = [0, \cdots, 0, 1] \in P^n(\boldsymbol{R})$.

$$G = PGL(n, \boldsymbol{R}) = GL(n+1, \boldsymbol{R})/\text{center}\,,$$

$$G' = \left\{ a \in G \mid a \cdot o = o \right\}.$$

Since G acts transitively on $P^n(\boldsymbol{R})$, we regard $G/G' = P^n(\boldsymbol{R})$. The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{sl}(n+1, \boldsymbol{R})$ and has a graded Lie algebra structure $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ given by

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \middle|\, v \text{ is a column } n\text{-vector} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -\mathrm{Tr}\, A \end{pmatrix} \middle|\, A \in \mathfrak{gl}(n, \boldsymbol{R}) \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \middle|\, \xi \text{ is a row } n\text{-vector} \right\}.$$

Let V be the n-dimensional real vector space of column n-vectors and V^* be the dual space of V consisting of row n-vectors. Then $\mathfrak{g}$ may be identified with the Lie algebra $V+\mathfrak{gl}(V)+V^*$ under the correspondence;

$$\mathfrak{g}_{-1}\ni\begin{pmatrix}0 & v\\ 0 & 0\end{pmatrix}\longleftrightarrow v\in V,$$

$$\mathfrak{g}_0\ni\begin{pmatrix}A & 0\\ 0 & -\mathrm{Tr}\ A\end{pmatrix}\longleftrightarrow A+\mathrm{Tr}\,A\cdot I_n\in\mathfrak{gl}(V),$$

$$\mathfrak{g}_1\ni\begin{pmatrix}0 & 0\\ \xi & 0\end{pmatrix}\longleftrightarrow\xi\in V^*.$$

(For the bracket operation of $V+\mathfrak{gl}(V)+V^*$, see p. 133 [5]). In particular a subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(V)$ and the Lie algebra $\mathfrak{g}'$ of G' is $\mathfrak{g}_0+\mathfrak{g}_1$. Since $G/G'=P^n(\boldsymbol{R})$, the tangent space of $P^n(\boldsymbol{R})$ at o may be identified with the vector space $\mathfrak{g}_{-1}=V$. The linear isotropy representation $\rho: G'\to GL(V)$ of $P^n(\boldsymbol{R})=G/G'$ is given by

$$\rho\begin{pmatrix}A & 0\\ \xi & a\end{pmatrix}=\frac{1}{a}A\in GL(V)\qquad\text{for}\quad\begin{pmatrix}A & 0\\ \xi & a\end{pmatrix}\in G'$$

and is a surjective homomorphism. (We express the equivalence class of $B\in GL(n+1,\boldsymbol{R})$ by the same letter $B\in G$.) Let G'' be the kernel of ρ and $\tilde{G}$ be a general linear group of V. The group G'/G'' is naturally isomorphic to $\tilde{G}$ and the Lie algebra of G'' is $\mathfrak{g}_1$. The induced Lie algebra homomorphism $\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1\to\mathfrak{g}_0$ of $\rho: G'\to\tilde{G}$ is a natural projection and we denote it by the same letter ρ. We define an injective homomorphism $\iota:\tilde{G}\to G'$ by

$$\iota(A)=\begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}\qquad\text{for}\quad A\in\tilde{G}.$$

ι satisfies $\rho\circ\iota=id$ and the induced Lie algebra homomorphism ι: $\mathfrak{g}_0\to\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1$ is a natural inclusion. It is known that an element g of G' is uniquely expressed in the form $g=\iota(\tilde{g})\cdot\exp\xi$ where $\tilde{g}\in\tilde{G}$, $\xi\in\mathfrak{g}_1$ and exp is the exponential map of G' (p. 109 [12]). In particular $G'/\iota(\tilde{G})$ is diffeomorphic to $\boldsymbol{R}^n$. The group $\tilde{G}=GL(V)$ acts naturally on $\mathfrak{g}_{-1}=V$ (resp. $\mathfrak{g}_1=V^*$) on the left (resp. on the right).

The principal fibre bundle $G\to G/G'=P^n(\boldsymbol{R})$ and the Maurer-Cartan form ω on G give the prototype of the (flat) projective Cartan connection described below.

Let M be an n-dimensional manifold and $\tilde{P}$ be the frame bundle of M. The structure group of $\tilde{\pi}$: $\tilde{P}\to M$ is $\tilde{G}=GL(V)$. We denote by θ

the canonical form of $\tilde{P}$. θ is a $\mathfrak{g}_{-1}$-valued 1-form on $\tilde{P}$. For later use we give an equivalent definition of projective equivalence of two torsionfree affine connections in terms of connection 1-forms on $\tilde{P}$. Let χ and χ' be two connection 1-forms on $\tilde{P}$. We say that χ and χ' are projectively equivalent if there exists a $\mathfrak{g}_1$-valued function F on $\tilde{P}$ such that $\chi'-\chi=[\theta, F]$ (cf. [8], p. 128 [12]). It is easy to see that F satisfies the equality:

$$(1.1)\qquad F(z\cdot a)=F(z)\cdot a \qquad \text{for all} \quad z\in\tilde{P},\ a\in\tilde{G}.$$

Note that $[\theta, F]$ is a $\mathfrak{g}_0$-valued 1-form on $\tilde{P}$ since $[\mathfrak{g}_{-1}, \mathfrak{g}_1]\subset\mathfrak{g}_0$. An equivalence class containing χ is denoted by $[\chi]$ and a pair $(\tilde{P}, [\chi])$ is called a projective structure on M. Let $\tilde{P}$ (resp. $\tilde{P}'$) be the frame bundle of M (resp. M') and θ (resp. θ') be the canonical form of $\tilde{P}$ (resp. $\tilde{P}'$) and let $(\tilde{P}, [\chi])$ (resp. $(\tilde{P}', [\chi'])$) be a projective structure on M (resp. M'). A bundle isomorphism $\tilde{\phi}: \tilde{P}\rightarrow\tilde{P}'$ is said to be a projective isomorphism of $(\tilde{P}, [\chi])$ onto $(\tilde{P}', [\chi'])$ if

$$\tilde{\phi}^*\theta'=\theta,$$
$$[\tilde{\phi}^*\chi']=[\chi],$$

where $\tilde{\phi}^*\chi'$ is a connection 1-form on $\tilde{P}$ induced by $\tilde{\phi}$.

1.3. Under the above notations we shall review the theory of projective Cartan connections.

Let P be a principal G' bundle over an n-dimensional manifold M. Let ω be a $\mathfrak{g}$-valued 1-form on P. Then we say that the pair (P, ω) is a projective Cartan connection if

1) $R_a^*\,\omega=\mathrm{Ad}\,a^{-1}\cdot\omega \qquad$ for $a\in G'$,
2) $\omega(A^*)=A \qquad$ for $A\in\mathfrak{g}'$,

where A^* is the fundamental vector field corresponding to A.

3) Let X be a tangent vector to P. If $\omega(X)=0$, then $X=0$.

Let (P, ω) (resp. (P', ω')) be a projective Cartan connection on M (resp. M'). A bundle isomorphism ϕ: $P\rightarrow P'$ is called an isomorphism of (P, ω) onto (P', ω') if $\phi^*\omega'=\omega$. We define a $\mathfrak{g}$-valued 2-form Ω on P by

$$(1.2)\qquad \Omega=d\omega+\frac{1}{2}[\omega, \omega],$$

and we call Ω the curvature form of (P, ω). The equation (1.2) is called the structure equation. If Ω is identically zero on P, we say that (P, ω) is a flat projective Cartan connection. We denote by ω_p (resp. Ω_p) the $\mathfrak{g}_p$-component of ω (resp. Ω).

REMARK 1.2. ω_u is a linear isomorphism of T_uP onto $\mathfrak{g}$ for all $u\in P$. Hence for any $X\in\mathfrak{g}$, we can define a vector field $\check{\omega}(X)$ by $\omega(\check{\omega}(X))=X$.

Conversely a linear map $\check{\omega}$: $\mathfrak{g}\to\mathfrak{X}(P)$ satisfying some conditions defines a projective Cartan connection ω (see p. 135 [13]). In Tanaka [12], the vector field corresponding to $\xi\in\mathfrak{g}_{-1}$ is denoted by $C(\xi)$ and a (projective) Cartan connection is called "a connection of type (L)".

1.4. Let (P,ω) be a projective Cartan connection and $\tilde{P}$ be a factor manifold P/G'' where G'' is the kernel of $\rho: G'\to\tilde{G}$. $\tilde{P}$ is a principal fibre bundle over M and the structure group of $\tilde{P}$ is $G'/G''=\tilde{G}$ (see § 1.2). We denote the projection $P\to\tilde{P}$ by the same letter ρ. $\rho: P\to\tilde{P}$ is a bundle map corresponding to the homomorphism $\rho: G'\to\tilde{G}$. The following lemma is easy to verify (cf. p. 136 [13]).

LEMMA 1.3. *Let (P,ω) be a projective Cartan connection on M. Then there exists a unique $\mathfrak{g}_{-1}$-valued 1-form θ on $\tilde{P}$ satisfying the following conditions*:

1) $R_a^*\theta=a^{-1}\cdot\theta$ *for* $a\in\tilde{G}$,

2) *Let X be a tangent vector to $\tilde{P}$. Then $\theta(X)=0$ if and only if X is a vertical vector.*

3) $\rho^*\theta=\omega_{-1}$.

By this lemma the factor manifold $\tilde{P}$ may be regarded as a frame bundle of M having the above θ as a canonical form.

Let (P,ω) be a projective Cartan connection on M satisfying $\Omega_{-1}=0$. Then (P,ω) naturally induces a projective structure on M as follows. Since $G'/\iota(\tilde{G})$ is diffeomorphic to $\boldsymbol{R}^n$, there exists a bundle map $h: \tilde{P}\to P$ corresponding to $\iota: \tilde{G}\to G'$ and satisfies $\rho\circ h=id$, where $\rho: P\to\tilde{P}=P/G''$ is a projection. Since $\rho^*\theta=\omega_{-1}$, we have $h^*\omega_{-1}=\theta$ and we can easily check that $\mathfrak{g}_0$-valued 1-form $h^*\omega_0$ on $\tilde{P}$ satisfies the usual conditions of a connection 1-form. Pulling back the $\mathfrak{g}_{-1}$-component of the structure equation (1.2) by h, we have

$$d\theta+[h^*\omega_0,\theta]=h^*\Omega_{-1}=0\,,$$

i.e., $h^*\omega_0$ is a torsionfree connection. Let h' : $\tilde{P}\to P$ be any bundle map corresponding to $\iota: \tilde{G}\to G'$ and satisfies $\rho\circ h'=id$. Then there exists a unique $\mathfrak{g}_1$-valued function F on $\tilde{P}$ which satisfies (1.1) and the equality:

$$h'(z)=h(z)\cdot\exp F(z)\qquad\text{for}\quad z\in\tilde{P}\,,\tag{1.3}$$

where exp is the exponential map of $G(\exp F(z)\in G'')$. It is easy to see that if h and h' are related by (1.3), the equality

$$h'^*\omega_0-h^*\omega_0=[\theta,F]\,,\tag{1.4}$$

holds on $\tilde{P}$ (cf. p. 126 [12]). Therefore a projective structure is induced on

M independent of the choice of h. This projective structure is called a projective structure induced by (P, ω).

1.5. Conversely every projective structure on M is induced by a unique projective Cartan connection satisfying certain curvature conditions.

Let $\{e_i^-\}$ be a base of $\mathfrak{g}_{-1}$ and let $\{e_i^+\}$ be the base of $\mathfrak{g}_1$ such that $B(e_i^-, e_j^+)=\delta_{ij}$ where B is the Killing form of $\mathfrak{g}$. Let (P, ω) be a projective Cartan connection on M. For any $X\in\mathfrak{g}$, we define a vector field $\check{\omega}(X)$ on P by $\omega(\check{\omega}(X))=X$. A $\mathfrak{g}'$-valued 1-form Ω^* on P defined by

$$\Omega_u^*(X)=\sum_i\left[e_i^+, \Omega\left(\check{\omega}(e_i^-), X\right)\right] \quad \text{for} \quad u\in P,\ X\in T_uP, \tag{1.5}$$

is called a $*$-curvature of (P, ω). Ω^* does not depend on the choice of $\{e_i^-\}$. A projective Cartan connection (P, ω) is called normal if $\Omega^*=0$. Clearly a flat projective Cartan connection is normal. If (P, ω) is normal, the $\mathfrak{g}_{-1}$-component of Ω is zero since $\Omega_0^*=0$.

THEOREM A ([12]). (1) *Let (P, ω) be a projective normal Cartan connection on M. Then (P, ω) induces a projective structure $(\tilde{P}, [\chi])$ on M. Conversely if $(\tilde{P}, [\chi])$ is a projective structure on M, there is a projective normal Cartan connection which induces the given $(\tilde{P}, [\chi])$.*

(2) *Let (P, ω) (resp. (P', ω')) be a projective normal Cartan connection on M (resp. M') and $(\tilde{P}, [\chi])$ (resp. $(\tilde{P}', [\chi'])$) be an induced projective structure. Then every isomorphism $\phi: (P, \omega)\to(P', \omega')$ induces a projective isomorphism $\tilde{\phi}: (\tilde{P}, [\chi])\to(\tilde{P}', [\chi'])$ such that $\tilde{\phi}\circ\rho=\rho'\circ\phi$ where ρ (resp. ρ') is the projection $P\to\tilde{P}$ (resp. $P'\to\tilde{P}'$). Conversely for every projective isomorphism $\tilde{\phi}: (\tilde{P}, [\chi])\to(\tilde{P}', [\chi'])$ there is a unique isomorphism $\phi: (P, \omega)\to(P', \omega')$ such that $\tilde{\phi}\circ\rho=\rho'\circ\phi$.*

(3) *A projective normal Cartan connection (P, ω) is flat if and only if the induced projective structure on M is projectively flat.*

For the proof of this theorem, see [11], [12] (cf. [6]).

REMARK 1.4. (1) In Tanaka [12], the $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$)-component of the $*$-curvature is denoted by S^* (resp. W^*).

(2) We may regard the principal G' bundle $P\to M$ as the extended bundle of $\tilde{P}\to M$ by the injective homomorphism $\iota: \tilde{G}\to G'$ (see p. 119, 129 [12]). In particular the construction of the principal bundle $P\to M$ is independent of the projective structure on M. The treatment in [5], [6] is somewhat different from ours, namely in [5], [6] a subbundle Q of $P^2(M)$ with structure group $\dot{G}'$ is called a projective structure on M, where $P^2(M)$ is the bundle of 2-jet frames over M. Using the canonical form on $P^2(M)$, a projective normal Cartan connection is constructed on Q uniquely. But once a projective structure (in our sense) on M is given, there exists a

unique bundle isomorphism $\phi: P \to Q$ which preserves the projective normal Cartan connection. For later convenience we adopt the former standpoint.

In the proof of Theorem A (Theorem 9.1 in [12]) the following fact is proved. It plays an important role in our argument.

PROPOSITION B. *Let $\tilde{P}$ be the frame bundle of M, P be a principal G' bundle on M and let $h: \tilde{P} \to P$ be a bundle map corresponding to $\iota: \tilde{G} \to G'$. Then for each torsionfree connection χ on $\tilde{P}$, there exists a unique projective normal Cartan connection (P, ω) satisfying the following conditions:*

$$h^* \omega_0 = \chi , \qquad h^* \omega_{-1} = \theta , \tag{1.6}$$

where θ is the canonical form of $\tilde{P}$.

§ 2. Invariant flat projective structures on L/K and the main theorem

In this section we shall prove the first main theorem (Theorem 2.12) of this paper.

Let L be a Lie group and K be a connected closed subgroup of L and we set $M = L/K$ ($\dim M = n$). Let $\mathfrak{l}$ (resp. $\mathfrak{k}$) be the Lie algebra of L (resp. K).

DEFINITION 2.1. A projective structure on $M = L/K$ is invariant if the left action $L_a: M \to M$ of L is a projective transformation for any $a \in L$.

Let $\tilde{P}$ be the frame bundle of $M = L/K$ and $h: \tilde{P} \to P$ be an extension of the principal bundle $\tilde{P}$ by the injective homomorphism $\iota: \tilde{G} \to G'$ and let $\rho: P \to \tilde{P}$ be the bundle map corresponding to $\rho: G' \to \tilde{G}$ such that $\rho \circ h = id$. In the standpoint of [5], [6], the map $\rho: P \to \tilde{P}$ corresponds to the restriction of the canonical projection $P^2(M) \to P^1(M) = \tilde{P}$ to the subbundle P of $P^2(M)$. We fix a linear frame $\tilde{o}$ at the origin $o \in M$ throughout. Then there is a natural bundle map $\tilde{j}: L \to \tilde{P}$ corresponding to the linear isotropy representation $\rho_0: K \to \tilde{G}$ of $M = L/K$, i.e.,

$$\tilde{j}(a) = (L_a)_*(\tilde{o}) \qquad \text{for} \quad a \in L .$$

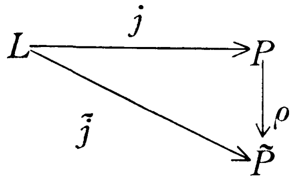
We set $o' = h(\tilde{o}) \in P$. The group L acts on $\tilde{P}$ on the left and the map $\tilde{j}$ is compatible with this left action. We denote by $\tilde{L}_a: \tilde{P} \to \tilde{P}$ the left action of $a \in L$ on $\tilde{P}$.

Let $[\chi]$ be an invariant projective structure on $M = L/K$ and χ be a (not necessary invariant) connection belonging to $[\chi]$. We first construct a linear map from $\mathfrak{l}$ to $\mathfrak{g} = \mathfrak{sl}(n+1, \boldsymbol{R})$ depending on the choice of χ. By

Proposition B, there is a unique projective normal Cartan connection ω on P such that $h^*\omega_0=\chi$ and $h^*\omega_{-1}=\theta$. Since $L_a: M\to M$ is a projective transformation for any $a\in L$, there is a unique bundle isomorphism $L'_a: P\to P$ such that $L'^*_a\omega=\omega$ and $\rho\circ L'_a=\tilde{L}_a\circ\rho$ (Theorem A). We define a map j: $L\to P$ by $j(a)=L'_a(o')$ for $a\in L$. The equality $L'_a\circ L'_b=L'_{ab}$ holds for any a, $b\in L$ and $L'_e=id$ where e is the unit element of L. Therefore $\{L'_a\}$ defines a left action of L on P and the map j: $L\to P$ is compatible with this left action. For any $a\in K$, $L_a: M\to M$ fixes the origin $o\in M$, and hence $j(a)$ and o' lie in the same fibre of P. We define a map $\rho_1: K\to G'$ by $o'\cdot\rho_1(a)=j(a)$ for $a\in K$. The following two lemmas are easy to verify and we omit the proofs.

LEMMA 2.2. *ρ_1: $K\to G'$ is a group homomorphism and the map j: $L\to P$ is a bundle map corresponding to ρ_1.*

LEMMA 2.3. *The following diagram is commutative*:



In particular $\rho\circ\rho_1=\rho_0$: $K\to\tilde{G}$, where ρ: $G'\to\tilde{G}$ is the linear isotropy representation of $P^n(\boldsymbol{R})=G/G'$.

For later use, we shall write down the homomorphism ρ_1 explicitly. Since L_a: $M\to M$ is a projective transformation for any $a\in L$, there exists a $\mathfrak{g}_1$-valued function F_a on $\tilde{P}$ satisfying (1.1) and the equality:

$$\tilde{L}^*_a\chi-\chi=[\theta, F_a]\,.$$

LEMMA 2.4. *For any a, $b\in L$, we have*

$$F_{ab}=F_b+\tilde{L}^*_b F_a\,. \tag{2.1}$$

The proof is trivial.

LEMMA 2.5. $\rho_1(a)=(\iota\circ\rho_0)(a)\cdot\exp\{-F_a(\tilde{o})\}$ *for $a\in K$.*

PROOF. In the proof of Theorem 9.2 in [12], it is shown that the bundle map L'_a: $P\to P$ satisfies the relation $h\circ\tilde{L}_a=L'_a\circ h_a$ where h_a: $\tilde{P}\to P$ is a bundle map defined by $h_a(z)=h(z)\cdot\exp F_a(z)$. Hence for $a\in K$, $o'\cdot(\iota\circ\rho_0)(a)=h(\tilde{o}\cdot\rho_0(a))=(h\circ\tilde{L}_a)(\tilde{o})=(L'_a\circ h_a)(\tilde{o})\ L'_a(o'\cdot\exp F_a(\tilde{o}))=j(a)\cdot\exp F_a(\tilde{o})=o'\cdot\rho_1(a)\cdot\exp F_a(\tilde{o})$. Therefore we have $\rho_1(a)=(\iota\circ\rho_0)(a)\cdot\exp\{-F_a(\tilde{o})\}$. q. e. d.

COROLLARY 2.6. *If χ is an invariant affine connection, we have*

$\rho_1=\iota\circ\rho_0$ and $h: \tilde{P}\to P$ is compatible with the left action of L. In particular $j=h\circ\tilde{j}: L\to P$.

We consider the $\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$-valued 1-form $j^*\omega$ on L. Since the map j is compatible with the left action of L and $L_a'^*\omega=\omega$ for any $a\in L$, $j^*\omega$ is a $\mathfrak{g}$-valued left invariant form on L and defines a linear map $j^*\omega$: $\mathfrak{l}\to\mathfrak{g}$. Note that the map $j^*\omega: \mathfrak{l}\to\mathfrak{g}$ depends on the choice of χ.

LEMMA 2.7. *The projective structure defined by χ is projectively flat if and only if $j^*\omega$ is a Lie algebra homomorphism.*

PROOF. If $[\chi]$ is projectively flat, the corresponding projective normal Cartan connection ω is flat, i.e., $d\omega+\frac{1}{2}[\omega, \omega]=0$. Pulling back by j, we have $d(j^*\omega)+\frac{1}{2}[j^*\omega, j^*\omega]=0$ on L. We put left invariant vector fields X, Y on L in this equation and obtain the equality

$$-j^*\omega[X, Y]+\left[j^*\omega(X), j^*\omega(Y)\right]=0,$$

which implies that $j^*\omega$ is a Lie algebra homomorphism. Conversely if $j^*\omega$ is a Lie algebra homomorphism, we have $j^*\left(d\omega+\frac{1}{2}[\omega, \omega]\right)=0$ on L. Since $j_*(T_aL)$ contains a complement of the vertical subspace of $T_{j(a)}P$ for any $a\in L$, the equality $d\omega+\frac{1}{2}[\omega, \omega]=0$ holds on P, i.e., the corresponding projective normal Cartan connection is flat. q.e.d.

Now we define a linear map $c: \mathfrak{l}\to\mathfrak{g}_{-1}=\boldsymbol{R}^n$ as follows. We consider the frame $\tilde{o}\in\tilde{P}$ as a linear isomorphism $\tilde{o}: \mathfrak{g}_{-1}\to T_oM$. Then c is defined by

$$c(X)=\tilde{o}^{-1}(\pi_* X) \qquad \text{for} \quad X\in\mathfrak{l}=T_eL,$$

where $\pi: L\to M=L/K$ is the projection.

DEFINITION 2.8. A Lie algebra homomorphism $f: \mathfrak{l}\to\mathfrak{g}$ satisfies condition (P) (which we say that f is a (P)-homomorphism) if

1) $f_{-1}=c: \mathfrak{l}\to\mathfrak{g}_{-1}$,
2) $f(Y)\in\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1$ for all $Y\in\mathfrak{k}$,
3) $f_0(Y)=\rho_0(Y)$ for $Y\in\mathfrak{k}$,

where f_p is the $\mathfrak{g}_p$-component of f and $\rho_0: \mathfrak{k}\to\mathfrak{g}_0$ is the homomorphism induced by the linear isotropy representation $\rho_0: K\to\tilde{G}$ of $M=L/K$. Note that $\rho_0(Y)$ is expressed in the matrix form since we have fixed the frame $\tilde{o}\in\tilde{P}$ at $o\in M$.

PROPOSITION 2.9. *Let χ be a connection 1-form on $\tilde{P}$ such that the projective structure $[\chi]$ defined by χ is invariant and flat. Then the cor-*

responding Lie algebra homomorphism $j^*\omega: \mathfrak{l}\to\mathfrak{g}$ *satisfies condition* (*P*).

PROOF. We identify $\mathfrak{l}$ with the tangent space of L at e. Then for $X\in\mathfrak{l}$, we have $f_{-1}(X)=j^*\omega_{-1}(X)=(j^*\rho^*\theta)(X)=(\tilde{j}^*\theta)(X)=\theta_{\tilde{o}}(\tilde{j}_*X)=\tilde{o}^{-1}(\tilde{\pi}_*\tilde{j}_*X)=\tilde{o}^{-1}(\pi_*X)=c(X)$, where $\tilde{\pi}: \tilde{P}\to M$ is the projection. For $Y\in\mathfrak{k}(\subset T_eL)$ we have $j(\exp tY)=o'\cdot\rho_1(\exp tY)=o'\cdot\exp t\rho_1(Y)$ where $\rho_1: \mathfrak{k}\to\mathfrak{g}'$ is the induced Lie algebra homomorphism of $\rho_1: K\to G'$. Differentiating the above equality at $t=0$ we have $j_*(Y)=\rho_1(Y)^*_{o'}$ and hence $f(Y)=(j^*\omega)(Y)=\rho_1(Y)\in\mathfrak{g}'$. 3) is clear from the identity $\rho\circ\rho_1=\rho_0: \mathfrak{k}\to\mathfrak{g}_0$ and the fact that the homomorphism $\rho: \mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1\to\mathfrak{g}_0$ induced by $\rho: G'\to\tilde{G}$ is a natural projection. q. e. d.

Let χ' be a connection 1-form on $\tilde{P}$ which is projectively equivalent to χ. Then the Lie algebra homomorphism satisfying condition (*P*) is constructed in the same manner. We shall study the difference between these two homomorphisms.

DEFINITION 2.10. Let f and f' be (*P*)-homomorphisms. We say that f is projectively equivalent to f' (which we denote by $f\sim f'$) if there exists $\xi\in\mathfrak{g}_1$ such that

$$f'_0-f_0=[\xi,f_{-1}]: \mathfrak{l}\to\mathfrak{g}_0, \tag{2.2}$$

where f_p (resp. f'_p) is the $\mathfrak{g}_p$-component of f (resp. f').

Clearly $\sim$ is an equivalence relation.

PROPOSITION 2.11. *Let χ and χ' be projectively equivalent connection 1-forms on $\tilde{P}$ such that the projective structure defined by χ is invariant and flat. Let f (resp. f') be the (P)-homomorphism corresponding to χ (resp. χ'). Then f is projectively equivalent to f'.*

PROOF. Let ω (resp. ω') be the flat Cartan connection corresponding to χ (resp. χ'). For any $Z\in T_{o'}P$, $(\rho^*h^*\omega)(Z)=\omega(h_*\rho_*Z)=\omega(Z+Y^*)=\omega(Z)+Y$, for some $Y\in\mathfrak{g}_1$. Hence $\rho^*\chi=\rho^*h^*\omega_0=\omega_0$ at $o'\in P$ where ω_0 is the $\mathfrak{g}_0$-component of ω. Pulling back this equality by j, we have $j^*\omega_0=j^*\rho^*\chi=\tilde{j}^*\chi$ at $e\in L$, i. e., $f_0(X)=\chi(\tilde{j}_*X)$ for $X\in\mathfrak{l}=T_eL$. In the same way, we have $f'_0(X)=\chi'(\tilde{j}_*X)$. Since χ and χ' are projectively equivalent, there exists a function $F: \tilde{P}\to\mathfrak{g}_1$ satisfying (1.1) and $\chi'-\chi=[\theta, F]$. Then $f'_0(X)-f_0(X)=(\chi'-\chi)(\tilde{j}_*X)=[\theta(\tilde{j}_*X), F(\tilde{o})]=[c(X), F(\tilde{o})]$ for $X\in\mathfrak{l}$. Setting $\xi=-F(\tilde{o})\in\mathfrak{g}_1$. the equality (2.2) holds on $\mathfrak{l}$. q. e. d.

By this proposition we obtain the following map:

Φ: {an invariant flat projective structure on $M=L/K$}$\to${$f: \mathfrak{l}\to\mathfrak{g}|f$ is a (*P*)-homomorphism}$/\sim$.

The first main result in this paper is the following theorem.

THEOREM 2.12. *Φ is a bijective map.*

REMARK 2.13. It is well known that there is a one-to-one correspondence between the set of invariant (affinely) flat affine connections on $\tilde{P}$ and the set of Lie algebra homomorphisms $f\colon \mathfrak{l}\to\mathfrak{g}_{-1}+\mathfrak{g}_0$ satisfying certain conditions (see Vol. II [7]). Theorem 2.12 is a natural generalization of this correspondence to the projective geometry (cf. Theorem 3.7).

Proof of Theorem 2.12. We first prove that Φ is injective. For this purpose we prepare several lemmas. Let χ and χ' be connection 1-forms on $\tilde{P}$ such that $[\chi]$ and $[\chi']$ are IFPS (an abbreviation of invariant flat projective structure). Let f (resp. f') be the (P)-homomorphism corresponding to χ (resp. χ'). We assume that f is projectively equivalent to f', i.e., $f'_0-f_0=[\xi, f_{-1}]$ for some $\xi\in\mathfrak{g}_1$. We shall prove that χ' is projectively equivalent to χ. Since $[\chi]$ and $[\chi']$ are invariant, there exist $\mathfrak{g}_1$-valued functions F_a and F'_a depending on $a\in L$ which satisfy (1.1) and

$$\text{(2.3)}\qquad \begin{aligned}\tilde{L}_a^*\chi-\chi&=[\theta, F_a],\\ \tilde{L}_a^*\chi'-\chi'&=[\theta, F'_a]\end{aligned}\qquad \text{for}\quad a\in L.$$

Note that F_a and F'_a satisfy (2.1).

LEMMA 2.14. *If* $\eta\in\mathfrak{g}_1$ *and if* $[\eta, v]=0$ *for all* $v\in\mathfrak{g}_{-1}$, *then* $\eta=0$.
The proof is easy.

LEMMA 2.15. $\xi\cdot\rho_0(a)=\xi+F_a(\tilde{o})-F'_a(\tilde{o})$ *for* $a\in K$.

PROOF. In the proof of Proposition 2.11, we have already shown the equations $f_0(X)=\chi(\tilde{j}_*X)$ and $f'_0(X)=\chi'(\tilde{j}_*X)$ for $X\in\mathfrak{l}=T_eL$. Hence we have $\chi(\tilde{j}_*X)-\chi'(\tilde{j}_*X)=f_0(X)-f'_0(X)=[f_{-1}(X), \xi]=[\theta(\tilde{j}_*X), \xi]$ at $e\in L$ and therefore $\chi-\chi'=[\theta, \xi]$ at $\tilde{o}\in\tilde{P}$. Applying (2.3), for $a\in L$, we have

$$\tilde{L}_a^*\chi-\tilde{L}_a^*\chi'=\left[\theta, \xi+F_a(\tilde{o})-F'_a(\tilde{o})\right]\qquad \text{at } \tilde{o}\in\tilde{P}.$$

For $X\in T_{\tilde{o}}\tilde{P}$ and for $a\in K$, we have $\tilde{L}_a^*\chi(X)=\mathrm{Ad}\rho_0(a)^{-1}\chi(R_{\rho_0(a)^{-1}*}\tilde{L}_{a*}X)$. Hence

$$\begin{aligned}\tilde{L}_a^*\chi(X)-\tilde{L}_a^*\chi'(X)&=\mathrm{Ad}\rho_0(a)^{-1}(\chi-\chi')(R_{\rho_0(a)^{-1}*}\tilde{L}_{a*}X)\\ &=\mathrm{Ad}\rho_0(a)^{-1}\left[\theta(R_{\rho_0(a)^{-1}*}\tilde{L}_{a*}X), \xi\right]=\mathrm{Ad}\rho_0(a)^{-1}\left[\rho_0(a)\cdot\theta(X), \xi\right]\\ &=\left[\theta(X), \xi\cdot\rho_0(a)\right]\end{aligned}$$

for $a\in K$. Therefore $[\theta, \xi\cdot\rho_0(a)]=[\theta, \xi+F_a(\tilde{o})-F'_a(\tilde{o})]$ at $\tilde{o}\in\tilde{P}$. The lemma follows from Lemma 2.14, since $\theta\colon T_{\tilde{o}}\tilde{P}\to\mathfrak{g}_{-1}$ is a surjective map.

q.e.d.

Now we construct a $\mathfrak{g}_1$-valued function F on $\tilde{P}$ as follows. Any point $z\in\tilde{P}$ is expressed in the form $a\cdot\tilde{o}\cdot g$ where $a\in L$ and $g\in\tilde{G}$. We set $F(z)=$

$\xi\cdot g+F_a(\tilde{o})\cdot g-F'_a(\tilde{o})\cdot g\in\mathfrak{g}_1$. Then by Lemma 2.4 and Lemma 2.15, F is well defined on $\tilde{P}$ and satisfies (1.1). It is easy to see that the equality $\chi-\chi'=[\theta, F]$ holds on $\tilde{P}$ and thus χ and χ' are projectively equivalent, i.e., Φ is injective.

Next we shall prove that Φ is surjective. Let f: $\mathfrak{l}\to\mathfrak{g}$ be a (P)-homomorphism. We shall construct a torsionfree affine connection χ on $\tilde{P}$ such that $[\chi]$ is an IFPS and the corresponding (P)-homomorphism is f. For this purpose we shall prove two lemmas.

LEMMA 2.16. *The notation being as above, there exists a unique Lie group homomorphism ρ_1: $K\to G'$ such that $\rho\circ\rho_1=\rho_0$: $K\to\tilde{G}$ and the differential of ρ_1 is the same as the restriction of f to $\mathfrak{k}$.*

PROOF. The uniqueness is evident since K is connected. We shall prove the existence. First we show that for any $a\in K$, there exists a unique $\xi(a)\in\mathfrak{g}_1$ satisfying the equality:

$$\text{(2.4)}\qquad \mathrm{Ad}\ \rho_0(a)^{-1}\circ f_0\circ\mathrm{Ad}\ a-f_0=\left[f_{-1}, \xi(a)\right]: \mathfrak{l}\to\mathfrak{g}_0\,.$$

Uniqueness of $\xi(a)$ follows from Lemma 2.14. For $a\in K$, which is expressed in the form $a=\exp Y$ $(Y\in\mathfrak{k})$, we set

$$\xi(a)=-\sum_{k=1}^{\infty}\frac{1}{(k+1)\,!}\,\mathrm{ad}\left(-f_0(Y)\right)^k\cdot f_1(Y)\,.$$

Then $\xi(a)$ satisfies the equality (2.4) and it is well defined on $\exp\mathfrak{k}\subset K$ by the uniqueness of $\xi(a)$. If $\xi(a)$ and $\xi(b)$ $(a, b\in K)$ satisfy (2.4), we set

$$\text{(2.5)}\qquad \xi(ab)=\xi(b)+\xi(a)\cdot\rho_0(b)\,.$$

Then $\xi(ab)$ also satisfies (2.4) and thus we obtain the map ξ: $K\to\mathfrak{g}_1$ since K is connected. Using ξ we define a map ρ_1: $K\to G'$ by

$$\rho_1(a)=(\iota\circ\rho_0)(a)\cdot\exp\left\{-\xi(a)\right\}\qquad\text{for}\quad a\in K.$$

It is easily checked that ρ_1 is a group homomorphism and satisfies the desired conditions. q.e.d.

LEMMA 2.17. *The notations being as above, there exists a map j: $L\to P$ satisfying the following conditions:*

1) $j(z\cdot a)=j(z)\cdot\rho_1(a)$ *for* $z\in L$, $a\in K$,
2) $\rho\circ j=\tilde{j}$: $L\to\tilde{P}$,
3) $j(e)=o'$.

PROOF. Let $M=\bigcup_{\alpha\in A}U_\alpha$ be a locally finite open covering of M such that the bundle π: $L\to M$ is trivial on U_α and let $\{f_\alpha\}$ be a partition of unity subordinating to $\{U_\alpha\}$. For each $\alpha\in A$, we fix a section $\sigma_\alpha: U_\alpha\to L$ satisfying $\sigma_\alpha(o)=e\in L$ if $o\in U_\alpha$. We define a map $j_\alpha: \pi^{-1}(U_\alpha)\to P$ by $j_\alpha(\sigma_\alpha(x)\cdot a)=h\circ\bar{j}\circ\sigma_\alpha(x)\cdot\rho_1(a)$ for $x\in U_\alpha$ and $a\in K$. Obviously j_α satisfies the conditions 1), 2) and 3) locally. If $U_\alpha\cap U_\beta\neq\phi$, there exists a uniqu function $\tau_{\alpha\beta}: \pi^{-1}(U_\alpha\cap U_\beta)\to\mathfrak{g}_1$ such that $j_\alpha(z)=j_\beta(z)\cdot\exp\tau_{\alpha\beta}(z)$ for $z\in\pi^{-1}(U_\alpha\cap U_\beta)$. $\tau_{\alpha\beta}$ satisfies the equality $\tau_{\alpha\beta}(z\cdot a)=\operatorname{Ad}\rho_1(a)^{-1}\cdot\tau_{\alpha\beta}(z)$ for $z\in\pi^{-1}(U_\alpha\cap U_\beta)$ and $a\in K$, and if $U_\alpha\cap U_\beta\cap U_\gamma\neq\phi$, then $\tau_{\alpha\gamma}=\tau_{\alpha\beta}+\tau_{\beta\gamma}$. We define a map $j: L\to P$ by

$$j(z)=j_\alpha(z)\cdot\exp\left(-\sum_\gamma f_\gamma\big(\pi(z)\big)\cdot\tau_{\alpha\gamma}(z)\right)\qquad\text{for}\quad z\in\pi^{-1}(U_\alpha)\,.$$

It is easy to see that j is well defined on L and satisfies the conditions 1), 2) and 3). q. e. d.

Now we show that Φ is surjective. Let $j: L\to P$ be the bundle map constructed in Lemma 2. 17. First we define a flat projective Cartan connection ω on P such that $j^*\omega$ is the given (P)-homomorphism f. For any $a\in L$, a tangent vector at $j(a)$ can be expressed in the form j_*X+A^* where $X\in\mathfrak{l}=T_aL$ and $A\in\mathfrak{g}'$. We set $\omega_{j(a)}(j_*X+A^*)=f(X)+A\in\mathfrak{g}$ and extend it to any point of P by $\omega_{j(a)\cdot g}=\operatorname{Ad}g^{-1}\cdot R_{g^{-1}}^{\ *}\omega_{j(a)}$ for $a\in L$ and $g\in G'$. ω is well defined on P and satisfies the conditions of a flat Cartan connection. It is easy to verify that $h^*\omega_{-1}=\theta$ and $h^*\omega_0$ is a torsionfree connection on $\tilde{P}$. We show that the projective structure defined by $\chi=h^*\omega_0$ is invariant. Any point of P is expressed in the form $j(z)\cdot g$ $(z\in L, g\in G')$. For each $a\in L$ we define a map $L'_a: P\to P$ by $L'_a(j(z)\cdot g)=j(a\cdot z)\cdot g$. Then L'_a is well defined and is a bundle isomorphism. $\{L'_a\}$ defines a left action of L on P and j is compatible with this action. $L'_a: P\to P$ satisfies the equality $\rho\circ L'_a=\tilde{L}_a\circ\rho: P\to\tilde{P}$ and preserves ω, i. e., $L'^*_a\omega=\omega$. Therefore, by Theorem A, $\tilde{L}_a: (\tilde{P}, [\chi])\to(\tilde{P}, [\chi])$ is a projective isomorphism for all $a\in L$ and hence $[\chi]$ is an IFPS. It is easy to check that the homomorphism corresponding to χ is $j^*\omega=f$ and therefore Φ is surjective. q. e. d.

§ 3. The case where an IFPS admits invariant affine connections

Let $[\chi]$ be an IFPS on a homogeneous space $M=L/K$. In general an IFPS $[\chi]$ does not admit an invariant affine connection. For example the model space $P^n(\boldsymbol{R})=G/G'$ does not admit an invariant affine connection since the dimension of $G(=n^2+2n)$ exceeds the largest dimension n^2+n of the affine transformation group. In this section we shall study the case where an IFPS admits invariant affine connections. First we answer the question: when does an IFPS on $M=L/K$ admit an invariant affine connection (Proposition 3. 1) and next we show that there is a one-to-one correspondence between the set of projectively flat invariant affine connections on $\tilde{P}$ and the set of homomorphisms $f: \mathfrak{l} \to \mathfrak{g}$ which satisfy certain conditions (Theorem 3. 5).

Let χ be an affine connection on $M = L/K$ such that $[\chi]$ is an IFPS and let $\rho_1: K \to G'$ be the corresponding homomorphism constructed in § 2. We have already shown in Corollary 2. 6 that if χ is an invariant affine connection, then $\rho_1 = \iota \circ \rho_0$. Conversely we have

PROPOSITION 3. 1. *If $\rho_1 = \iota \circ \rho_0$, then there is a unique invariant connection χ' such that χ' is projectively equivalent to χ and the (P)-homomorphism $f': \mathfrak{l} \to \mathfrak{g}$ corresponding to χ' is the same as that of χ.*

COROLLARY 3. 2. *If M is a Lie group, i. e., $K = \{e\}$, every IFPS on M admits an invariant affine connection.*

First we prove the following lemma.

LEMMA 3. 3. *Let f and f' be (P)-homomorphisms. If $f_0 = f'_0$, then $f = f'$.*

PROOF. For $X_1, X_2 \in \mathfrak{l}$, we have

$$\begin{aligned}\left[f_{-1}(X_1), f_1(X_2)\right] - \left[f_{-1}(X_2), f_1(X_1)\right] &= f_0[X_1, X_2] - \left[f_0(X_1), f_0(X_2)\right] \\ &= f'_0[X_1, X_2] - \left[f'_0(X_1), f'_0(X_2)\right] \\ &= \left[f_{-1}(X_1), f'_1(X_2)\right] - \left[f_{-1}(X_2), f'_1(X_1)\right],\end{aligned}$$

i. e.,

$$\left[f_{-1}(X_1), (f_1 - f'_1)(X_2)\right] = \left[f_{-1}(X_2), (f_1 - f'_1)(X_1)\right].$$

Since $f_{-1}: \mathfrak{l} \to \mathfrak{g}_{-1}$ is surjective, the above equality implies that $f_1 = f'_1$ on $\mathfrak{l}$ and hence $f = f'$. q. e. d.

Proof of Proposition 3. 1. We first show the uniqueness. Let ω (resp. ω') be the flat Cartan connection corresponding to χ (resp. χ'). We have already proved that $j^* \omega_0 = \tilde{j}^* \chi$ and $j'^* \omega'_0 = \tilde{j}^* \chi'$ at $e \in L$. Therefore if $f = f'$, we have $\chi = \chi'$ at $\tilde{o} \in \tilde{P}$. Since χ' is invariant, χ' is uniquely determined on $\tilde{P}$. Now we show the existence. Let F_a $(a \in L)$ be a $\mathfrak{g}_1$-valued function on $\tilde{P}$ defined by $\tilde{L}_a^* \chi - \chi = [\theta, F_a]$. Since $\rho_1 = \iota \circ \rho_0$, we have $F_a(\tilde{o} \cdot g) = 0$ for $a \in K$ and $g \in \tilde{G}$ (Lemma 2. 6). We define a $\mathfrak{g}_0$-valued 1-form χ' on $\tilde{P}$ by $\chi'_{a \cdot \tilde{o} \cdot g} = \tilde{L}_{a^{-1}}^*(\chi_{\tilde{o} \cdot g})$ for $a \in L$ and $g \in \tilde{G}$. Then χ' is a well defined connection 1-form on $\tilde{P}$ and is invariant by the left action of L. We define a $\mathfrak{g}_1$-valued function F on $\tilde{P}$ by $F(a \cdot \tilde{o} \cdot g) = F_{a^{-1}}(a \cdot \tilde{o} \cdot g)$ for $a \in L$ and $g \in \tilde{G}$. F is well defined and satisfies (1. 1) and the equality $\chi' - \chi = [\theta, F]$. Hence χ' is projectively equivalent to χ. Since $\chi = \chi'$ at $\tilde{o} \in \tilde{P}$ and $j^* \omega_0 = \tilde{j}^* \chi$, $j'^* \omega'_0 = \tilde{j}^* \chi'$ at $e \in L$, we have $f_0 = f'_0$. Therefore by Lemma 3. 3, we have $f = f'$. q. e. d.

In § 4, we shall show that every IFPS on a reductive homogeneous space admits an invariant affine connection (Proposition 4. 5).

DEFINITION 3. 4. Let f: $\mathfrak{l}\rightarrow\mathfrak{g}$ be a Lie algebra homomorphism. We say that f satisfies condition (A) (or f is an (A)-homomorphism) if

1) $f_{-1}=c: \mathfrak{l}\rightarrow\mathfrak{g}_{-1}$,

2) $f(Y)=\rho_0(Y)\in\mathfrak{g}_0$ for all $Y\in\mathfrak{k}$,

where ρ_0 is the linear isotropy representation of $M=L/K$.

It is obvious that an (A)-homomorphism is a (P)-homomorphism.

If $\rho_1=\iota\circ\rho_0$: $K\rightarrow G'$, then the induced Lie algebra homomorphism ρ_1: $\mathfrak{k}\rightarrow\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1$ coincides with the linear isotropy representation ρ_0 since ι: $\mathfrak{g}_0\rightarrow\mathfrak{g}'$ is a natural inclusion. If χ is a projectively flat invariant affine connection, then the corresponding (P)-homomorphism f: $\mathfrak{l}\rightarrow\mathfrak{g}$ satisfies condition (A) (Corollary 2. 6). Thus we obtain the following map:

Ψ: {a projectively flat invariant affine connection χ on $M\}\rightarrow\{f$: $\mathfrak{l}\rightarrow\mathfrak{g}|f$ is an (A)-homomorphism}.

THEOREM 3. 5. *Ψ is a bijective map.*

PROOF. We first prove that Ψ is injective. Let χ and χ' be projectively flat invariant affine connections on M and let f and f' be the corresponding (A)-homomorphisms. Since χ and χ' are invariant, we have $\tilde{L}_a^*\chi=\chi$, $\tilde{L}_a^*\chi'=\chi'$ for all $a\in L$ and hence $\tilde{j}^*\chi$ and $\tilde{j}^*\chi'$ are $\mathfrak{g}_0$-valued left invariant forms on L. If $f=f'$, then we have $\tilde{j}^*\chi=j^*\omega_0=f_0=f'_0=j'^*\omega'_0=\tilde{j}^*\chi'$ at $e\in L$ where ω and ω' be the corresponding flat Cartan connections on P. Since $\tilde{j}^*\chi$ and $\tilde{j}^*\chi'$ are left invariant, we have $\tilde{j}^*\chi=\tilde{j}^*\chi'$ on L and hence $\chi=\chi'$ on $\tilde{P}$.

Next we prove that Ψ is surjective. Let f: $\mathfrak{l}\rightarrow\mathfrak{g}$ be an (A)-homomorphism. Then in the proof of Theorem 2. 12, we showed that there is an affine connection χ such that $[\chi]$ is invariant and flat and the corresponding homomorphism is f. Let ρ_1: $K\rightarrow G'$ be the Lie group homomorphism determined by χ. Then the differential of ρ_1 is the restriction of f to $\mathfrak{k}$ and hence we have $\rho_1=\iota\circ\rho_0$ by the uniqueness in Lemma 2. 16. Thus by Proposition 3. 1, there exists an invariant affine connection χ' such that $[\chi']=[\chi]$ and the homomorphism corresponding to χ' is f, i. e., $\Psi(\chi')=f$.

q. e. d.

COROLLARY 3. 6. *Let $[\chi]$ be an IFPS on M. Then $[\chi]$ admits an invariant affine connection if and only if the projective equivalence class $\Phi([\chi])$ contains an (A)-homomorphism.*

The following theorem implies that the correspondence Ψ is a natural generalization of the correspondence in the affine geometry explained in Remark 2. 13 to the projective geometry.

THEOREM 3. 7. *Let χ be a projectively flat invariant affine connection. Then χ is affinely flat* $\left(i.e., d\chi+\frac{1}{2}[\chi, \chi]=0 \text{ and } d\theta+[\chi, \theta]=0 \text{ on } \tilde{P}\right)$ *if and only if the (A)-homomorphism $\Psi(\chi): \mathfrak{l}\to\mathfrak{g}$ is $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued.*

PROOF. Let ω be the corresponding flat Cartan connection on P. Then by Corollary 2. 6, we have $j^*\omega_0=\tilde{j}^*h^*\omega_0=\tilde{j}^*\chi$. If $\Psi(\chi)$ is $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued, $j^*\omega_1=0$ on L and hence $0=j^*\left(d\omega_0+\frac{1}{2}[\omega_0, \omega_0]+[\omega_1, \omega_{-1}]\right)=j^*\left(d\omega_0+\frac{1}{2}[\omega_0, \omega_0]\right)$ $=\tilde{j}^*\left(d\chi+\frac{1}{2}[\chi, \chi]\right)$. Therefore we have $d\chi+\frac{1}{2}[\chi, \chi]=0$ on $\tilde{P}$. Conversely if χ is affinely flat, the linear map $f': \mathfrak{l}\to\mathfrak{g}_{-1}+\mathfrak{g}_0$ defined by $f'(X)=\tilde{j}^*(\theta+\chi)(X)$ (note that $\tilde{j}^*\theta$ and $\tilde{j}^*\chi$ are left invariant 1-forms on L) is a Lie algebra homomorphism satisfying condition (*A*). Since $\tilde{j}^*\chi=j^*\omega_0$, the $\mathfrak{g}_{-1}+\mathfrak{g}_0$-component of $\Psi(\chi)$ coincides with f'. Thus by Lemma 3. 3, we have $f'=f$. In particular f is $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued. q. e. d.

§ 4. A normalization of Lie algebra homomorphisms

In § 2, we have proved that there is a natural one-to-one correspondence between the set of IFPS on $M=L/K$ and the set of projective equivalence classes of (*P*)-homomorphisms $f: \mathfrak{l}\to\mathfrak{g}$. In this section we first show that for each projective equivalence class of (*P*)-homomorphisms, there exists a unique normalized homomorphism ((*N*)-homomorphism) which satisfies an additional condition (Proposition 4. 2) and give the procedure to obtain all (*N*)-homomorphisms for many classes of homogeneous spaces. As an application of Proposition 4. 2, we show that any IFPS on a reductive homogeneous space admits an invariant affine connection (Proposition 4. 5).

We fix, once for all, a complementary subspace $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{l}$:

$$\mathfrak{l}=\mathfrak{k}+\mathfrak{m} \quad (\text{direct sum}).$$

Since we have fixed the frame $\tilde{o}: \boldsymbol{R}^n\to T_oM$ at the origin $o\in M$, there is a base $\{X_1, \cdots, X_n\}$ of $\mathfrak{m}$ such that $c(X_k)=e_k$ $(k=1, \cdots, n)$, where $\{e_1, \cdots, e_n\}$ is the standard base of $\mathfrak{g}_{-1}=\boldsymbol{R}^n$. Note that if f is a (*P*)-homomorphism, $f_{-1}(X_k)=e_k$ for $k=1, \cdots, n$.

DEFINITION 4. 1. Let f be a (*P*)-homomorphism. We say that f satisfies condition (*N*) (or f is an (*N*)-homomorphism) if the $(n+1, n+1)$-component of $f(X)$ is zero foı all $X\in\mathfrak{m}$, namely, $f(X_k)a_{n+1}=a_k$ for $k=1, \cdots, n$, where $\{a_1, \cdots, a_{n+1}\}$ is the standard base of $\boldsymbol{R}^{n+1}$.

Then we have

PROPOSITION 4. 2. *Let f be a (P)-homomorphism. Then there exists* *a unique* (N)*-homomorphism* f' *such that* f' *is projectively equivalent to* f.

Combining with Theorem 2.12, we have

COROLLARY 4.3. *There is a one-to-one correspondence between the set of IFPS on* $M=L/K$ *and the set of* (N)*-homomorphisms.*

We first prove the following lemma.

LEMMA 4.4. *Let* f *be a* (P)*-homomorphism and let* ξ *be an element of* $\mathfrak{g}_1$. *We define linear maps* f'_{-1} *and* f'_0 *by*

$$f'_{-1}=f_{-1}: \mathfrak{l}\longrightarrow\mathfrak{g}_{-1}$$

and

$$f'_0=f_0+[\xi,f_{-1}]: \mathfrak{l}\longrightarrow\mathfrak{g}_0 .$$

Then there exists a unique (P)*-homomorphism* f' *such that the* $\mathfrak{g}_{-1}$ *(resp.* $\mathfrak{g}_0$*)-component of* f' *is* f'_{-1} *(resp.* f'_0*). In particular there is a one-to-one correspondence between the elements of* $\mathfrak{g}_1$ *and the set of* (P)*-homomorphisms* f' *that are projectively equivalent to* f.

PROOF. The uniqueness is clear from Lemma 3.3. We prove the existence. We define a linear map $f'_1: \mathfrak{l}\rightarrow\mathfrak{g}_1$ by

$$f'_1(X)=f_1(X)+\left[\xi,f_0(X)\right]+\frac{1}{2}\left[\xi,\left[\xi,f_{-1}(X)\right]\right]$$

for $X\in\mathfrak{l}$, and set $f'=f'_{-1}+f'_0+f'_1: \mathfrak{l}\rightarrow\mathfrak{g}$. Then direct calculations show that f' is a Lie algebra homomorphism and satisfies the desired conditions. By Lemma 2.14, it follows that the correspondence $\xi\mapsto f'$ is a bijective map from $\mathfrak{g}_1$ to the set of (P)-homomorphisms that are projectively equivalent to f.

q. e. d.

Proof of Proposition 4.2. Let f be a (P)-homomorphism. We denote by ξ^k $(k=1,\cdots,n)$ the $(n+1,n+1)$-component of $-f(X_k)$ and we set $\xi=(\xi^1,\cdots,\xi^n)\in\mathfrak{g}_1$. Using f and ξ, we construct a (P)-homomorphism f' as in Lemma 4.4. Then it is easy to check that the $(n+1,n+1)$-component of $f'(X_k)$ is zero for $k=1,\cdots,n$, i.e., f' is an (N)-homomorphism. The uniqueness is evident. q. e. d.

As applications of Proposition 4.2, we prove the following two propositions.

PROPOSITION 4.5. *Every IFPS on a reductive homogeneous space admits an invariant affine connection.*

PROOF. Since $M=L/K$ is reductive, we can choose the complementary subspace $\mathfrak{m}$ satisfying $[\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}$. Let f be the (N)-homomorphism corresponding to the given projective structure. For $X\in\mathfrak{m}$ and $Y\in\mathfrak{k}$, the matrix $f(X)$ and $f(Y)$ are expressed in the form:

$$f(X)=\begin{pmatrix} f_0(X) & f_{-1}(X) \\ f_1(X) & 0 \end{pmatrix}\begin{matrix} \}\, n \\ \}\, 1 \end{matrix}$$

and

$$f(Y)=\begin{pmatrix} h(Y) & 0 \\ g(Y) & -\mathrm{Tr}\ h(Y) \end{pmatrix}.$$

By a direct calculation, the $(n+1, n+1)$-component of $[f(Y), f(X)]$ is $g(Y)\cdot f_{-1}(X)$ and the $(n+1, n+1)$-component of $f[Y, X]$ is zero since $[Y, X]\in\mathfrak{m}$. Therefore $g(Y)\cdot f_{-1}(X)=0$ for all $X\in\mathfrak{m}$ and $Y\in\mathfrak{k}$. Since $f_{-1}(X)$ takes any value of $\mathfrak{g}_{-1}$, we have $g(Y)=0$ for all $Y\in\mathfrak{k}$, i.e., f satisfies condition (A). Thus by Corollary 3.6, $M=L/K$ admits an invariant affine connection belonging to the given projective structure. q.e.d.

PROPOSITION 4.6. *Let $M=L/K$ be a reductive homogeneous space and let $\mathfrak{m}$ be a complementary subspace of $\mathfrak{k}$ in $\mathfrak{l}$ such that $[\mathfrak{m}, \mathfrak{k}]\subset\mathfrak{m}$. Suppose that M satisfies the following conditions*:

1) $\mathrm{Tr}\rho_0(Y)=0$ *for all* $Y\in\mathfrak{k}$,
2) $\mathfrak{m}=\{\Sigma\rho_0(Y)\cdot X \mid Y\in\mathfrak{k}, X\in\mathfrak{m}\}$.

Then any (A)-homomorphism f is an (N)-homomorphism. In particular every IFPS on M admits a unique invariant affine connection.

PROOF. The latter part of this proposition follows from the uniqueness of an (N)-homomorphism in Proposition 4.2. Let f be an (A)-homomorphism. Then for $X\in\mathfrak{m}$ and $Y\in\mathfrak{k}$, the matrix $f(X)$ and $f(Y)$ are expressed in the form:

$$f(X)=\begin{pmatrix} g(X) & f_{-1}(X) \\ f_1(X) & -\mathrm{Tr}\ g(X) \end{pmatrix}\begin{matrix} \}\, n \\ \}\, 1 \end{matrix},$$

$$f(Y)=\begin{pmatrix} \rho_0(Y) & 0 \\ 0 & 0 \end{pmatrix}.$$

The $(n+1, n+1)$-component of $[f(Y), f(X)]$ is zero and hence the $(n+1, n+1)$-component of $f[Y, X]$ is zero. By condition 2), the elements of the form $[Y, X]$ $(Y\in\mathfrak{k}, X\in\mathfrak{m})$ span $\mathfrak{m}$. Therefore the $(n+1, n+1)$-component of $f(X)$ is zero for $X\in\mathfrak{m}$, i.e., f is an (N)-homomorphism. q.e.d.

Note that irreducible Riemannian symmetric spaces satisfy the coditions in Proposition 4.6. As an another application of Proposition 4.2, we prove the following theorem.

THEOREM 4.7. *Let $M=L/K$ be a homogeneous space such that* Tr

$\rho_0(Y)=0$ *for all* $Y\in\mathfrak{k}$, *and admits a projectively flat invariant affine connection. Let* f *be the corresponding* (A)*-homomorphism. If* f *satisfies condition* (N), *the homogeneous space* $M\times \boldsymbol{R}^1=L\times\boldsymbol{R}^1/K\times\{e\}$ (*or* $M\times S^1=L\times S^1/K\times\{e\}$) *admits an invariant* (*affinely*) *flat affine connection.*

PROOF. Let $\mathfrak{l}\oplus\boldsymbol{R}$ be the direct sum of two Lie algebras $\mathfrak{l}$ and $\boldsymbol{R}$ and let $\{Z\}$ be a base of $\boldsymbol{R}$. We have only to construct a $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued (A)-homomorphism $\tilde{f}$: $\mathfrak{l}\oplus\boldsymbol{R}\to\mathfrak{g}=\mathfrak{sl}(n+2,\boldsymbol{R})$ (cf. Theorem 3. 7). Since f is an (N)-homomorphism and $\mathrm{Tr}\rho_0(Y)=0$ for $Y\in\mathfrak{k}$, the matrix $f(X)$ $(X\in\mathfrak{l})$ is expressed in the form:

$$f(X)=\begin{pmatrix} f_0(X) & f_{-1}(X) \\ f_1(X) & 0 \end{pmatrix}\begin{matrix}\}\,n\\ \}\,1\end{matrix} \qquad \text{for} \quad X\in\mathfrak{l}.$$

We define a linear map $\tilde{f}$: $\mathfrak{l}\oplus\boldsymbol{R}\to\mathfrak{sl}(n+2,\boldsymbol{R})$ by

$$\tilde{f}(X)=\begin{pmatrix} f_0(X) & f_{-1}(X) & f_{-1}(X) \\ f_1(X) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{matrix}\}\,n\\ \}\,1\\ \}\,1\end{matrix} \qquad \text{for} \quad X\in\mathfrak{l},$$

and

$$\tilde{f}(Z)=\begin{pmatrix} \frac{1}{n+2}I_n & 0 & 0 \\ 0 & \frac{1}{n+2} & 1 \\ 0 & 0 & -\frac{n+1}{n+2} \end{pmatrix}.$$

Then it is easily checked that $\tilde{f}$ is a Lie algebra homomorphism satisfying condition (A). Obviously $\tilde{f}$ is $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued. q. e. d.

Note that if M is a Lie group (i. e., $K=\{e\}$) or satisfies the conditions in Proposition 4. 6, then the conditions in Theorem 4. 7 are fulfilled.

In the rest of this section we assume that a homogeneous space $M=L/K$ satisfies one of the following conditions:

A) $K=\{e\}$, i. e., M is a Lie group.

B) $M=L/K$ is reductive and $\mathrm{Tr}\rho_0(Y)=0$ for all $Y\in\mathfrak{k}$.

(In this case we assume that a complementary space $\mathfrak{m}$ satisfies $[\mathfrak{m},\mathfrak{k}]\subset\mathfrak{m}$.)

Then by Corollary 3. 2 and Proposition 4. 5, every IFPS on M admits an invariant affine connection and in the proof of Proposition 4. 5, we showed that every (N)-homomorphism is necessarily an (A)-homomorphism. In particular the restriction of an (N)-homomorphism f to $\mathfrak{k}$ is a direct sum of the linear isotropy representation ρ_0 of $M=L/K$ and the 1-dimensional trivial representation, i. e.,

$$(4.1)\qquad f(Y)=\rho_0(Y)=\begin{pmatrix}\rho_0(Y) & 0\\ 0 & 0\end{pmatrix}\begin{matrix}\}\,n\\ \}\,1\end{matrix}\qquad \text{for}\quad Y\in\mathfrak{k}.$$

Let f be a representation of $\mathfrak{l}$ with degree $n+1$. We assume that f is $\mathfrak{sl}(n+1, \boldsymbol{R})$-valued. The following proposition enables us to decide whether the equivalence class of the Lie algebra representation f contains an (N)-homomorphism.

PROPOSITION 4.8. *Let f: $\mathfrak{l}\rightarrow\mathfrak{sl}(n+1, \boldsymbol{R})$ be a Lie algebra homomorphism such that the restriction of f to $\mathfrak{k}$ is a direct sum of ρ_0 and the 1-dimensional trivial representation (i.e., $f(Y)$ is expressed in the form (4.1) for $Y\in\mathfrak{k}$).*

(1) *Let P be an element of $GL(n+1, \boldsymbol{R})$. If $P^{-1}fP$: $\mathfrak{l}\rightarrow\mathfrak{g}$ satisfies condition (N), there exists an element v of $\boldsymbol{R}^{n+1}$ such that $P=(f(X_1)v, \cdots, f(X_n)v, v)$ and $Pf(Y)=f(Y)P$ for all $Y\in\mathfrak{k}$ (Note that $f(X_k)v$ is a column n-vector for $k=1, \cdots, n$).*

(2) *For $v\in\boldsymbol{R}^{n+1}$, we define an $(n+1, n+1)$-matrix P by $P=(f(X_1)v, \cdots, f(X_n)v, v)$. If $\det P\neq 0$ and $Pf(Y)=f(Y)P$ for all $Y\in\mathfrak{k}$, then $P^{-1}fP$: $\mathfrak{l}\rightarrow\mathfrak{g}$ satisfies condition (N) (and hence satisfies condition (A)).*

PROOF. (1) If $P^{-1}fP$: $\mathfrak{l}\rightarrow\mathfrak{g}$ is an (N)-homomorphism, we have $P^{-1}f(X_k)Pa_{n+1}=a_k$ for $k=1, \cdots, n$. We set $v=Pa_{n+1}\in\boldsymbol{R}^{n+1}$. Then we have $f(X_k)v=Pa_k$. Therefore $P=(Pa_1, \cdots, Pa_n, Pa_{n+1})=(f(X_1)v, \cdots, f(X_n)v, v)$. Since $P^{-1}fP$ is an (A)-homomorphism, we have $P^{-1}f(Y)P=\rho_0(Y)=f(Y)$ for $Y\in\mathfrak{k}$ and hence $f(Y)P=Pf(Y)$.

(2) Since $Pf(Y)=f(Y)P$ for all $Y\in\mathfrak{k}$, we have $P^{-1}f(Y)P=f(Y)=\rho_0(Y)$. For $k=1, \cdots, n$, we have $f(X_k)Pa_{n+1}=f(X_k)v=Pa_k$ and hence $P^{-1}fP$ is an (N)-homomorphism. q. e. d.

Using this proposition we can determine the number of (N)-homomorphisms for many classes of homogeneous spaces. We carry out this procedure in § 5, 6, 7 and 8.

PROPOSITION 4.9. *Let f: $\mathfrak{l}\rightarrow\mathfrak{g}$ be an (N)-homomorphism. If the linear isotropy representation ρ_0: $\mathfrak{k}\rightarrow\mathfrak{g}_0$ of $M=L/K$ does not contain a 1-dimensional trivial representation (i.e., if $\rho_0(Y)w=0$ for all $Y\in\mathfrak{k}$, then $w=0\in\boldsymbol{R}^n$), an (N)-homomorphism which is equivalent to f (as a representation) is necessarily identical to f.*

PROOF. Let P be an element of $GL(n+1, \boldsymbol{R})$. If $P^{-1}fP$: $\mathfrak{l}\rightarrow\mathfrak{g}$ satisfies condition (N), we have, by Proposition 4.8, $P=(f(X_1)v, \cdots, f(X_n)v, v)$ for some $v\in\boldsymbol{R}^{n+1}$. We write ${}^tv=({}^tx, y)$ where $x\in\boldsymbol{R}^n$ and $y\in\boldsymbol{R}^1$. Since $f(Y)$ is expressed in the form (4.1) for $Y\in\mathfrak{k}$, the $n+1$-th column vector of

$f(Y)P$ is ${}^t({}^t(\rho_0(Y)x), 0)$ and the $n+1$-th column vector of $Pf(Y)$ is zero. Thus $\rho_0(Y)x=0$ for all $Y\in\mathfrak{k}$. Since ρ_0 does not contain a 1-dimensional trivial representation, we have $x=0$ and hence $v={}^t(0, y)$. Therefore $P=(f(X_1)v, \cdots, f(X_n)v, v)=y(f(X_1)a_{n+1}, \cdots, f(X_n)a_{n+1}, a_{n+1})=y\cdot I_{n+1}$ and we have $P^{-1}fP=f$ if $\det P\neq 0$. q. e. d.

§ 5. The case $M=SO(3)$ and $SL(2, \boldsymbol{R})$

In the rest of this paper we shall determine the number of IFPS on many classical simple Lie groups and on the classical irreducible Riemannian symmetric spaces. In § 5, 6 and 7 we shall treat the case where M is a simple Lie group (i. e., $K=\{e\}$ and $\mathfrak{l}$ is real simple).

The following proposition is already known.

PROPOSITION C (Matsushima-Okamoto [9]). *Let L be a real semi-simple Lie group. Then L does not admit a left invariant torsionfree (affinely) flat affine connection.*

For the proof of this proposition, see [9].

Let $\mathfrak{l}$ be an n-dimensional real semi-simple Lie algebra. We fix a base $\{X_1, \cdots, X_n\}$ of $\mathfrak{l}$ once for all. The following proposition and its corollary play an important role in our argument.

PROPOSITION 5.1. *Let $\mathfrak{l}$ be an n-dimensional real semi-simple Lie algebra and let f: $\mathfrak{l}\to\mathfrak{g}$ be an (N)-homomorphism (i. e., $f(X_k)a_{n+1}=a_k$ for $k=1, \cdots, n$). We decompose the representation f to real irreducible components $f=f_1\oplus\cdots\oplus f_k$. Then none of f_i is a 1-dimensinal (trivial) representation.*

PROOF. We assume that f contains a 1-dimensional trivial representation. We define invariant subspaces W_1 and W_2 by

$$W_1=\left\{v\in\boldsymbol{R}^{n+1}|f(X)v=0 \text{ for all } X\in\mathfrak{l}\right\},$$

$$W_2=\left\{\sum_k f(Z_k)v_k|Z_k\in\mathfrak{l},\ v_k\in\boldsymbol{R}^{n+1}\right\}.$$

Since $\mathfrak{l}$ is semi-simple, we have $\boldsymbol{R}^{n+1}=W_1\oplus W_2$ (direct sum) and by the assumption we have $W_1\neq 0$. Since f is an (N)-homomorphism, W_2 contains the space $\{a_1, \cdots, a_n\}$. Therefore we have $W_2=\{a_1, \cdots, a_n\}$ and $\dim W_1=1$. Since W_2 is an invariant subspace of f, $f(X_k)a_m\in W_2$ for k, $m=1, \cdots, n$, i. e., $f(X_k)$ is expressed in the form

$$f(X_k)=\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\begin{matrix} \}\, n \\ \}\, 1 \end{matrix} \quad \text{for} \quad k=1, \cdots, n,$$

and hence f is a $\mathfrak{g}_{-1}+\mathfrak{g}_0$-valued (A)-homomorphism. Then by Theorem 3.7, a Lie group L with Lie algebra $\mathfrak{l}$ admits an left invariant flat affine connection, which contradicts to Proposition C. q. e. d.

Since a 1-dimensional representation of $\mathfrak{l}$ is a trivial representation, we have

COROLLARY 5.2. *In the same situation as Proposition* 5.1, *the complexification* f^c: $\mathfrak{l}^c \to \mathfrak{g}^c = \mathfrak{sl}(n+1, \boldsymbol{C})$ *of* f *does not contain a* 1*-dimensional representation.*

Using the above proposition we shall study the case $L=SO(3)$. We fix the base $\{X_1, X_2, X_3\}$ of $\mathfrak{o}(3)$ by

$$X_1=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } X_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We define a Riemannian metric g on $SO(3)$ by $g(X_i, X_j)=\delta_{ij}$. Then the Riemannian connection determined by g is a bi-invariant connection $\nabla_X Y = \frac{1}{2}[X, Y]$ for X, $Y \in \mathfrak{o}(3)$ and $(SO(3), g)$ is a space of (positive) constant curvature. Hence the left invariant projective structure defined by ∇ is projectively flat. The Lie algebra homomorphism corresponding to this bi-invariant connection is given by

$$f(X_1)=\begin{pmatrix} & & & 1 \\ & & \frac{1}{2} & \\ & -\frac{1}{2} & & \\ -\frac{1}{4} & & & \end{pmatrix},$$

$$(5.1) \qquad f(X_2)=\begin{pmatrix} & & -\frac{1}{2} & \\ & & & 1 \\ \frac{1}{2} & & & \\ & -\frac{1}{4} & & \end{pmatrix},$$

$$f(X_3)=\begin{pmatrix} & \frac{1}{2} & & \\ -\frac{1}{2} & & & \\ & & & 1 \\ & & -\frac{1}{4} & \end{pmatrix}.$$

Obviously f satisfies condition (N) and it is easily checked that f is a real irreducible representation. We shall prove the following theorem.

THEOREM 5.3. *Let* f: $\mathfrak{o}(3)\rightarrow\mathfrak{g}=\mathfrak{sl}(4, \boldsymbol{R})$ *be an* (N)*-homomorphism. Then* f *is given by* (5.1). *In particular* $M=SO(3)$ *admits a unique IFPS.*

To prove this theorem we shall review the theory of Cartan-Iwahori concerning real irreducible representations of real semi-simple Lie algebras. For the notations and the terminology which we use in the following, see Iwahori [4].

The rank of $\mathfrak{o}(3)$ is 1. Let $\{\rho_1\}$ be the fundamental system of irreducible representations of $\mathfrak{o}(3)$ given by

$$(5.2)\quad \rho_1(X_1)=\frac{1}{2}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\quad \rho_1(X_2)=\frac{\sqrt{-1}}{2}\begin{pmatrix} & -1 \\ -1 & \end{pmatrix},\quad \rho_1(X_3)=\frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Let Λ be the highest weight of ρ_1 and let ρ_m (m is a non-negative integer) be the irreducible complex representation with highest weight $m\Lambda$. Note that the degree of ρ_m is $m+1$. By an easy calculation ρ_m is self-conjugate for all m and the index of ρ_m is $(-1)^m$. Hence if m is even, $[\rho_m]\in C^I_{m+1}(\mathfrak{l})\cong R^I_{m+1}(\mathfrak{l})$ and if m is odd, $[\rho_m]\in C^{II}_{m+1}(\mathfrak{l})=\hat{C}^{II}_{m+1}(\mathfrak{l})\cong R^{II}_{2m+2}(\mathfrak{l})$. Therefore the degrees of real irreducible representations of $\mathfrak{o}(3)$ are given by 1, 3, 4, 5, 7, 8, $\cdots$. By Proposition 5.1, any homomorphism f' : $\mathfrak{o}(3)\rightarrow\mathfrak{sl}(4, \boldsymbol{R})$ which satisfies condition (N) is necessarily equivalent (as a representation) to the real irreducible representation (5.1). By an easy calculation we have

LEMMA 5.4. *Let* f *be an* (N)*-homomorphism* (5.1) *and let* v *be an element of* $\boldsymbol{R}^4$. *We define a* (4, 4)*-matrix* P *by* $P=(f(X_1)v, f(X_2)v, f(X_3)v, v)$. *Then* $f(X_k)P=Pf(X_k)$ *for* $k=1, 2, 3$.

Therefore by Proposition 4.8, (5.1) is the unique (N)-homomorphism for $\mathfrak{l}=\mathfrak{o}(3)$ and we complete the proof of Theorem 5.3.

Next we shall study the case $M=SL(2, \boldsymbol{R})$. We fix the base $\{X_1, X_2, X_3\}$ of $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$ by

$$X_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad X_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\text{ and } X_3=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

We define a pseudo-Riemannian metric g by $g(X, Y)=\mathrm{Tr}\,XY$ for $X, Y\in \mathfrak{sl}(2, \boldsymbol{R})$. Then $(SL(2, \boldsymbol{R}), g)$ is a space of constant curvature and hence the Levi-Civita connection $\nabla_X Y=\frac{1}{2}[X, Y]$ $(X, Y\in\mathfrak{sl}(2, \boldsymbol{R}))$ is projectively flat. The corresponding Lie algebra homomorphism f: $\mathfrak{sl}(2, \boldsymbol{R})\rightarrow\mathfrak{g}=\mathfrak{sl}(4, \boldsymbol{R})$ is given by

$$f(X_1)=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},$$

$$(5.3)\qquad f(X_2)=\begin{pmatrix}0 & 0 & \frac{1}{2} & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0\end{pmatrix},$$

$$f(X_3)=\begin{pmatrix}0 & -\frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\\ 0 & \frac{1}{2} & 0 & 0\end{pmatrix}.$$

Obviously f satisfies condition (N). Let $\{a_1, a_2, a_3, a_4\}$ be the canonical base of $\boldsymbol{R}^4$. Then $\{a_1+a_4, a_3\}$ and $\{a_1-a_4, a_2\}$ are invariant subspaces of f and f is equivalent to the direct sum of two real irreducible representations with degree two.

The rank of $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$ is 1 and the standard inclusion ρ_1 : $\mathfrak{sl}(2, \boldsymbol{R})\rightarrow \mathfrak{gl}(2, \boldsymbol{C})$ forms the fundamental system of irreducible representations. Let Λ be the highest weight of ρ_1 and let ρ_m (m be a non-negative integer) be the irreducible complex representation of $\mathfrak{sl}(2, \boldsymbol{R})$ with highest weight $m\Lambda$. The degree of ρ_m is $m+1$. By an easy calculation ρ_m is self-conjugate and the index of ρ_m is 1 for all m. Therefore $[\rho_m]\in C_{m+1}^{I}(\mathfrak{l})\cong R_{m+1}^{I}(\mathfrak{l})$ for $m=0$, 1, 2, $\cdots$, and $\hat{C}_N^{II}(\mathfrak{l})\cong R_{2N}^{II}(\mathfrak{l})=\phi$ and hence for each positive integer k, $\mathfrak{sl}(2, \boldsymbol{R})$ has a unique equivalence class of real irreducible representations with degree k. If f: $\mathfrak{l}\rightarrow\mathfrak{g}=\mathfrak{sl}(4, \boldsymbol{R})$ satisfies condition (N), then by Proposition 5.1, f

is equivalent to the real irreducible representation or the direct sum of two real irreducible representations with degree two. If f is not real irreducible, f is equivalent (as a representation) to (5.3). In this case we have

LEMMA 5.5. *Let f be the homomorphism* (5.3). *For $v\in \boldsymbol{R}^4$, we set $P=(f(X_1)v, f(X_2)v, f(X_3)v, v)$. Then we have $f(X_k)P=Pf(X_k)$ for $k=1$, 2 and* 3.

The proof is easy. Therefore by Proposition 4.8, an (N)-homomorphism which is not real irreducible uniquely exists and it is the homomorphism (5.3).

Next we shall study the case where f is real irreducible. Let f be a real irreducible representation of $\mathfrak{sl}(2, \boldsymbol{R})$ with degree four given by

$$f(X_1)=\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix},$$

$$(5.4)\qquad f(X_2)=\begin{pmatrix} 0 & 3 & & \\ & 0 & 4 & \\ & & 0 & 3 \\ & & & 0 \end{pmatrix},$$

$$f(X_3)=\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix}.$$

In this case the similar result as in Lemma 5.5 does not hold. Indeed there are many (N)-homomorphisms which are equivalent to (5.4). In the following we shall identify two (N)-homomorphisms f_1 and f_2 which are equivalent to (5.4) if two invariant affine connections χ_1 and χ_2 corresponding to f_1 and f_2 are mapped to each other by an automorphism of $L=SL(2, \boldsymbol{R})$. Under this identification we shall determine the number of (N)-homomorphisms which are equivalent to (5.4). For this purpose we review the general theory.

Let $\tilde{\phi}: L\rightarrow L$ be an automorphism of L and $\phi: \mathfrak{l}\rightarrow\mathfrak{l}$ be an induced isomorphism. ϕ is expressed in the matrix form with respect to the base $\{X_1, \cdots, X_n\}$ of $\mathfrak{l}$. Let ∇ be a projectively flat invariant affine connection on L and $\tilde{\phi}^*\nabla$ be the induced connection by ϕ. Then $\tilde{\phi}^*\nabla$ is also invariant and projectively flat. Let f (resp. f_ϕ) be the (N)-homomorphism corresponding to ∇ (resp. $\tilde{\phi}^*\nabla$). Then by an easy calculation f_ϕ is expressed in the form

$$f_\phi(X)=\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix}^{-1}\left(f\circ\phi(X)\right)\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix} \quad \text{for} \quad X\in\mathfrak{l}. \tag{5.5}$$

Note that ϕ is expressed in the matrix form and hence

$$\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix}\in GL(n+1, \boldsymbol{R}).$$

We say that f is transformed into f_ϕ by ϕ and two (N)-homomorphisms f and g are called a-equivalent if there is an automorphism $\tilde{\phi}$ of L such that $f_\phi=g$. Obviously a-equivalence is an equivalence relation. Let f be the (N)-homomorphism (5.3). Then we have $f_\phi=f$ for all automorphisms ϕ of $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$.

In the case of $L=SL(2, \boldsymbol{R})$ it is known that any automorphism of $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$ is induced by an automorphism of $SL(2, \boldsymbol{R})$. We shall determine the number of a-equivalence classes of (N)-homomorphisms which are equivalent to (5.4).

We denote by $A(\mathfrak{l})$ and $I(\mathfrak{l})$ the automorphism group and the inner automorphism group of $\mathfrak{l}$ respectively. It is well known that for $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$ the order of the factor group $A(\mathfrak{l})/I(\mathfrak{l})$ is two (Murakami [10]). The typical outer automorphism ψ of $\mathfrak{l}=\mathfrak{sl}(2, \boldsymbol{R})$ is given by

$$\psi(X_1)=X_1, \qquad \psi(X_2)=-X_2 \quad \text{and} \quad \psi(X_3)=-X_3. \tag{5.6}$$

We define v_1 and $v_2\in\boldsymbol{R}^4$ by $v_1={}^t\!\left(1, 0, \frac{1}{2}, 0\right)$ and $v_2={}^t\!\left(1, 0, -\frac{1}{2}, 0\right)$, and we set $P_k\in GL(4, \boldsymbol{R})$ $(k=1, 2)$ by $P_k=(f(X_1)\,v_k, f(X_2)\,v_k, f(X_3)\,v_k, v_k)$ where f is the homomorphism (5.4). We define homomorphisms g_1, g_2 : $\mathfrak{sl}(2, \boldsymbol{R})\to\mathfrak{sl}(4, \boldsymbol{R})$ by $g_k(X)=P_k^{-1}f(X)\,P_k$ for $X\in\mathfrak{sl}(2, \boldsymbol{R})$, $k=1, 2$. Then g_1 and g_2 satisfy condition (N). The explicit form of g_1 and g_2 are given by

$$g_1(X_1)=\begin{pmatrix}2 & 0 & 0 & 1\\ 0 & 1 & 2 & 0\\ 0 & 0 & -3 & 0\\ 3 & 0 & 0 & 0\end{pmatrix}, \quad g_1(X_2)=\begin{pmatrix}0 & \frac{3}{2} & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & \frac{3}{2} & 3 & 0\end{pmatrix},$$

$$g_1(X_3)=\begin{pmatrix}0 & -1 & -\frac{1}{2} & 0\\ 2 & 0 & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & 3 & \frac{3}{2} & 0\end{pmatrix}. \tag{5.7}$$

$$g_2(X_1)=\begin{pmatrix}2&0&0&1\\0&1&-2&0\\0&0&-3&0\\3&0&0&0\end{pmatrix},\quad g_2(X_2)=\begin{pmatrix}0&-\frac{3}{2}&0&0\\-1&0&0&1\\0&0&0&0\\0&-\frac{3}{2}&3&0\end{pmatrix},$$

(5. 8)

$$g_2(X_3)=\begin{pmatrix}0&-1&\frac{1}{2}&0\\-2&0&0&0\\-1&0&0&1\\0&3&-\frac{3}{2}&0\end{pmatrix}.$$

Then we have

PROPOSITION 5. 6. *Let g: $\mathfrak{sl}(2, \boldsymbol{R})\rightarrow\mathfrak{sl}(4, \boldsymbol{R})$ be a real irreducible representation satisfying condition (N). Then g is a-equivalent to g_1 or g_2. g_1 is not a-equivalent to g_2.*

For the proof of this proposition we prepare several lemmas. Let H be the set of all real irreducible representations f: $\mathfrak{sl}(2, \boldsymbol{R})\rightarrow\mathfrak{sl}(4, \boldsymbol{R})$ satisfying condition (N). For $v={}^t(x, y, z, u)\in\boldsymbol{R}^4$ we have $\det(f(X_1)\,v, f(X_2)\,v, f(X_3)\,v, v)$ $=2\,(18xyzu-9x^2u^2-8xz^3-6y^3u+3y^2z^2)$ where f is the homomorphism (5. 4). We denote by $2F(v)=2F(x, y, z, u)$ the right-hand side of the above equality and we set

$$W^0=\left\{v\in\boldsymbol{R}^4-\{0\}\,|\,F(v)=0\right\},$$

$$W^+=\left\{v\in\boldsymbol{R}^4|\,F(v)>0\right\},$$

$$W^-=\left\{v\in\boldsymbol{R}^4|\,F(v)<0\right\}$$

and $$W^{\pm}=W^+\cup W^-.$$

Then by Proposition 4. 8 (2) we can define a map $\tilde{\Phi}$: $W^{\pm}\rightarrow H$ by $\tilde{\Phi}(v)\,(X)=P^{-1}f(X)\,P$ for $v\in W^{\pm}$ and $X\in\mathfrak{sl}(2, \boldsymbol{R})$ where $P=(f(X_1)\,v, f(X_2)\,v, f(X_3)\,v, v)$. By Proposition 4. 8 (1) $\tilde{\Phi}$ is a surjective map. Let π : $\boldsymbol{R}^4-\{0\}\rightarrow P^3(\boldsymbol{R})$ be the natural projection. We set

$$P^3(\boldsymbol{R})^0=\pi(W^0),$$

$$P^3(\boldsymbol{R})^+=\pi(W^+),$$

$$P^3(\boldsymbol{R})^-=\pi(W^-)$$

and $$P^3(\boldsymbol{R})^{\pm}=P^3(\boldsymbol{R})^+\cup P^3(\boldsymbol{R})^-.$$

Then $P^3(\boldsymbol{R})^+$ and $P^3(\boldsymbol{R})^-$ are connected open subsets of $P^3(\boldsymbol{R})$ and $P^3(\boldsymbol{R})$ is a disjoint union of $P^3(\boldsymbol{R})^+$, $P^3(\boldsymbol{R})^-$ and $P^3(\boldsymbol{R})^0$. For $k\in\boldsymbol{R}-\{0\}$ we have $\tilde{\Phi}(kv)=\tilde{\Phi}(v)$ $(v\in W^{\pm})$ and hence $\tilde{\Phi}$ induces the map $\Phi: P^3(\boldsymbol{R})^{\pm}\to H$.

LEMMA 5.7. *The map Φ is bijective.*

PROOF. We have only to prove that Φ is injective. Let v, w be elements of $W^{\pm}$. We assume that $\tilde{\Phi}(v)=\tilde{\Phi}(w)$, i.e., $P_v^{-1}f(X)\,P_v=P_w^{-1}f(X)\,P_w$ for $X\in\mathfrak{l}$ where

$$P_v=\big(f(X_1)\,v, f(X_2)\,v, f(X_3)\,v,\, v\big)$$

and
$$P_w=\big(f(X_1)\,w, f(X_2)\,w, f(X_3)\,w,\, w\big).$$

Since the complexification f^c of f is a complex irreducible representation, we have by Schur's lemma $P_v=\alpha P_w$ for some $\alpha\in\boldsymbol{C}-\{0\}$. Since v and w are real vectors, we have $\alpha\in\boldsymbol{R}-\{0\}$ and $v=\alpha w$. Hence $\pi(v)=\pi(w)\in P^3(\boldsymbol{R})^{\pm}$, i.e., Φ is injective. q.e.d.

By definition we have $\tilde{\Phi}(v_k)=g_k$ for $k=1, 2$ and hence $\Phi^{-1}(g_1)\in P^3(\boldsymbol{R})^-$ and $\Phi^{-1}(g_2)\in P^3(\boldsymbol{R})^+$. By an easy calculation we have

LEMMA 5.8. *Let ψ be the outer automorphism* (5.6). *Then we have $(g_k)_\psi=g_k$ for $k=1, 2$. Accordingly we have $(g_k)_{\psi\phi}=(g_k)_\phi$ $(k=1, 2)$ for any inner automorphism ϕ of $\mathfrak{sl}(2, \boldsymbol{R})$.*

By this lemma any real irreducible (N)-homomorphism which is a-equivalent to g_k $(k=1, 2)$ can be obtained by transforming g_k by an inner automorphism of $\mathfrak{sl}(2, \boldsymbol{R})$.

For $v\in W^{\pm}$ we define a (4, 4)-matrix P_v by $P_v=(f(X_1)\,v, f(X_2)\,v, f(X_3)\,v, v)$ were f is the homomorphism (5.4). We transform the (N)-homomorphism $P_v^{-1}fP_v$ by the inner automorphism $\operatorname{Ad}A^{-1}: \mathfrak{sl}(2, \boldsymbol{R})\to\mathfrak{sl}(2, \boldsymbol{R})$ $(A\in SL(2, \boldsymbol{R}))$. We denote by $f_A: \mathfrak{sl}(2, \boldsymbol{R})\to\mathfrak{sl}(4, \boldsymbol{R})$ the transformed (N)-homomorphism. Let $\tilde{f}: SL(2, \boldsymbol{R})\to SL(4, \boldsymbol{R})$ be the Lie group homomorphism given by

$$\tilde{f}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a^3 & 3a^2b & 6ab^2 & 6b^3\\ a^2c & a^2d+2abc & 2b^2c+4abd & 6b^2d\\ \frac{1}{2}ac^2 & \frac{1}{2}bc^2+acd & ad^2+2bcd & 3bd^2\\ \frac{1}{6}c^3 & \frac{1}{2}c^2d & cd^2 & d^3\end{pmatrix} \tag{5.9}$$

$$\text{for } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2, \boldsymbol{R}).$$

The differential of $\tilde{f}$ is nothing but the Lie algebra homomorphism (5.4).

Obviously we have $f\circ\mathrm{Ad}\,A^{-1}=\mathrm{Ad}\tilde{f}(A)^{-1}f:\mathfrak{sl}(2,\boldsymbol{R})\to\mathfrak{sl}(4,\boldsymbol{R})$ for $A\in SL(2,\boldsymbol{R})$. Thus by the formula (5. 5) we have

$$\begin{aligned} f_A(X) &= \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix}^{-1} \left(P_v^{-1} f(\mathrm{Ad}\ A^{-1}\cdot X)\, P_v\right) \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix}^{-1} \left(P_v^{-1}\left(\mathrm{Ad}\tilde{f}(A)^{-1} f(X)\right) P_v\right) \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix}^{-1} \left(P_v^{-1} \tilde{f}(A)^{-1} f(X) \tilde{f}(A)\, P_v\right) \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix} \end{aligned}$$

for $X\in\mathfrak{sl}(2,\boldsymbol{R})$. Note that $\mathrm{Ad}\ A^{-1}$ is expressed in the matrix form with respect to the base $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}(2,\boldsymbol{R})$. We set

$$Q=\tilde{f}(A)\, P_v \begin{pmatrix} \mathrm{Ad}\ A^{-1} & \\ & 1 \end{pmatrix}.$$

Then $f_A=Q^{-1}fQ$ and by Proposition 4. 8 (1) Q is expressed in the form $Q=(f(X_1)\,w, f(X_2)\,w, f(X_3)\,w, w)$ for some $w\in\boldsymbol{R}^4$. Since the fourth column vector of Q is $\tilde{f}(A)\,v$, we have $w=\tilde{f}(A)\,v\in W^{\pm}$ and hence $\Phi([\tilde{f}(A)\,v])=f_A$ where $[\tilde{f}(A)\,v]$ is the element of $P^3(\boldsymbol{R})^{\pm}$ determined by $\tilde{f}(A)\,v\in W^{\pm}$. By Lemma 5. 7 and Lemma 5. 8, we have

LEMMA 5. 9. *Let v, $w\in W^{\pm}$. Then $\Phi([v])$ is a-equivalent to $\Phi([w])$ if and only if $[w]=[\tilde{f}(A)\,v]$ for some $A\in SL(2,\boldsymbol{R})$.*

We define a left action of $SL(2,\boldsymbol{R})$ on $P^3(\boldsymbol{R})$ by $\Psi(A)\,[v]=[\tilde{f}(A)v]\in P^3(\boldsymbol{R})$ for $A\in SL(2,\boldsymbol{R})$ and $[v]\in P^3(\boldsymbol{R})$. Then for any $A\in SL(2,\boldsymbol{R})$, $\Psi(A)$ preserves the connected open subsets $P^3(\boldsymbol{R})^+$ and $P^3(\boldsymbol{R})^-$. By Lemma 5. 7 and Lemma 5. 9 the number of a-equivalence classes is equal to the number of orbit spaces of $P^3(\boldsymbol{R})^{\pm}$. Since $\Phi^{-1}(g_1)=[v_1]\in P^3(\boldsymbol{R})^-$ and $\Phi^{-1}(g_2)=[v_2]\in P^3(\boldsymbol{R})^+$, g_1 is not a-equivalent to g_2 (Lemma 5. 9). Thus in order to prove Proposition 5. 6, we have only to show the following lemma.

LEMMA 5. 10. *The orbital decomposition of $P^3(\boldsymbol{R})^{\pm}$ by $SL(2,\boldsymbol{R})$ is given by $P^3(\boldsymbol{R})^{\pm}=P^3(\boldsymbol{R})^+\cup P^3(\boldsymbol{R})^-$.*

PROOF. For $v\in\boldsymbol{R}^4-\{0\}$ we define a map ψ_v: $SL(2,\boldsymbol{R})\to P^3(\boldsymbol{R})$ by $\psi_v(A)=[\tilde{f}(A)\,v]$ for $A\in SL(2,\boldsymbol{R})$. Then it is easy to check that the rank of the map ψ_v at $e\in SL(2,\boldsymbol{R})$ is three if and only if $[v]\in P^3(\boldsymbol{R})^{\pm}$. Therefore the orbit space of $P^3(\boldsymbol{R})^{\pm}$ is open and connected. Since $P^3(\boldsymbol{R})^+$ and $P^3(\boldsymbol{R})^-$ are open and connected, the orbital decomposition of $P^3(\boldsymbol{R})^{\pm}$ is given by $P^3(\boldsymbol{R})^+\cup P^3(\boldsymbol{R})^-$. q. e. d.

By Lemma 5. 5 and Proposition 5. 6, we have

THEOREM 5.11. *We identify two left invariant flat projective structures* $[\nabla]$ *and* $[\nabla']$ *on* $SL(2, \boldsymbol{R})$ *if* $[\tilde{\phi}^*\nabla]=[\nabla']$ *for some automorphism* $\tilde{\phi}$ *of* $SL(2, \boldsymbol{R})$. *Then there exists three left invariant flat projective structures on* $SL(2, \boldsymbol{R})$. *The typical* (N)*-homomorphisms are given by* (5.3), (5.7) *and* (5.8).

§ 6. The case $\mathfrak{l}$ is a real form of B_m, C_m or D_m

In this section we shall study the case where $\mathfrak{l}$ is a real form of the classical complex simple Lie algebra $B_m=\mathfrak{o}(2m+1, \boldsymbol{C})$, $C_m=\mathfrak{sp}(m, \boldsymbol{C})$ or $D_m=\mathfrak{o}(2m, \boldsymbol{C})$. The main result in this section is Theorem 6.6 indicating that for most of such a real simple Lie algebra an (N)-homomorphism does not exist. Corollary 5.2 plays an important role in the following argument.

6.1. The case $\mathfrak{l}$ is a real form of $B_m=\mathfrak{o}(2m+1, \boldsymbol{C})$ $(m\geqq 2)$

$\mathfrak{l}^c\cong\mathfrak{o}(2m+1, \boldsymbol{C})$ is a complex simple Lie algebra of rank m. Let $\{\rho_1, \cdots, \rho_m\}$ be the fundamental system of irreducible representations of $\mathfrak{o}(2m+1, \boldsymbol{C})$ and let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the corresponding highest weights. (We take the standard numbering.) We denote by $d\left(\sum_{i=1}^m m_i\Lambda_i\right)$ the degree of the complex irreducible representation with highest weight $\sum_{i=1}^m m_i\Lambda_i$. Then it is well known that

$$d(\Lambda_k)=\binom{2m+1}{k} \qquad \text{for} \quad k=1, \cdots, m-1$$

and $d(\Lambda_m)=2^m$.

PROPOSITION 6.1. *Let* $\mathfrak{l}$ *be a real form of* $B_m=\mathfrak{o}(2m+1, \boldsymbol{C})$ $(m=2$ *or* $m\geqq 4)$. *Then there is no* (N)*-homomorphism* $f\colon \mathfrak{l}\to\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ *where* $n=\dim_{\boldsymbol{R}}\mathfrak{l}=m(2m+1)$.

PROOF. We shall show that a complex representation of $\mathfrak{l}^c$ with degree $n+1$ not containing a trivial representation does not exist for such a Lie algebra.

The case $m\geqq 7$. It is clear that $d(\Lambda_1)<d(\Lambda_2)<\cdots<d(\Lambda_{m-1})$ and $d(\Lambda_3)$, $d(\Lambda_m)>n+1$. By Weyl's formula we have $d(2\Lambda_1)=m(2m+3)>n+1$ and $d(2\Lambda_2)=\frac{1}{3}(m+1)(m-1)(2m+1)(2m+3)>n+1$. We have also $d(\Lambda_2)+1=n+1$. Therefore by Corollary 5.2, the complexification f^c of an (N)-homomorphism f is equivalent to the direct sum of ρ_1, i.e., $f^c=\rho_1\oplus\cdots\oplus\rho_1$. Hence if an (N)-homomorphism exists, there is a positive integer x such that $n+1=m(2m+1)+1=x(2m+1)$. But such an integer does not exist and therefore there is no (N)-homomorphism for $m\geqq 7$.

The case $m\leqq 6$. The degrees of irreducible representations of $\mathfrak{o}(2m+1, \boldsymbol{C})$ less than $n+1=m(2m+1)+1$ are given by 1, 4, 5, 10 (for $m=2$), 1, 9, 16, 36 (for $m=4$), 1, 11, 32, 55 (for $m=5$) and 1, 13, 64, 78 (for $m=6$). In each case there is no representation with degree $n+1$ not containing a 1-dimensional representation and hence an (N)-homomorphism does not exist for such a Lie algebra. q. e. d.

REMARK 6. 2. Let $\mathfrak{l}$ be a real form of $\mathfrak{o}(7, \boldsymbol{C})$ (i. e., the case $m=3$). Then the degrees of irreducible representations of $\mathfrak{o}(7, \boldsymbol{C})$ are given by 1, 7, 8, 21, 27, $\cdots$. In this case there exists a combination $n+1=22=7+7+8$ and hence in order to determine the existence or non-existence of (N)-homomorphisms, we have to decide the self-conjugateness and the index of ρ_1, ρ_2 and ρ_3 for each real form of $\mathfrak{o}(7, \boldsymbol{C})$. Next for each 22-dimensional real representation f of $\mathfrak{l}$ not containing a 1-dimensional representation, we construct a matrix $P=(f(X_1)\,v, \cdots, f(X_{22})\,v, v)$ for $v\in\boldsymbol{R}^{22}$ and determine whether $\det P\neq 0$ for some $v\in\boldsymbol{R}^{22}$, which requires a considerably complicated polynomial computations.

6. 2. The case $\mathfrak{l}$ is a real form of $D_m=\mathfrak{o}(2m, \boldsymbol{C})$ $(m\geqq 4)$

$\mathfrak{l}^c\cong\mathfrak{o}(2m, \boldsymbol{C})$ is a complex simple Lie algebra of rank m. Let $\{\rho_1, \cdots, \rho_m\}$ be the fundamental system of irreducible representations of $\mathfrak{o}(2m, \boldsymbol{C})$ and let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the corresponding highest weights. It is well known that

$$d(\Lambda_k)=\binom{2m}{k} \qquad \text{for} \quad k=1, \cdots, m-2$$

and $d(\Lambda_{m-1})=d(\Lambda_m)=2^{m-1}$.

PROPOSITION 6. 3. *Let $\mathfrak{l}$ be a real form of $D_m=\mathfrak{o}(2m, \boldsymbol{C})$ $(m=4, 6$ or $m\geqq 8)$. Then there is no (N)-homomorphism $f\colon \mathfrak{l}\to\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ where $n=\dim_{\boldsymbol{R}}\mathfrak{l}=m(2m-1)$.*

PROOF. The case $m\geqq 8$. We have $d(\Lambda_1)<d(\Lambda_2)<\cdots<d(\Lambda_{m-2})$ and $d(\Lambda_3)$, $d(\Lambda_{m-1})$, $d(\Lambda_m)>n+1$. By Weyl's formula we have $d(2\Lambda_1)=(m+1)(2m-1)>n+1$ and $d(2\Lambda_2)=\frac{1}{3}m(m+1)(2m+1)(2m-3)>n+1$. We have also $d(\Lambda_2)+1=n+1$. Thus if an (N)-homomorphism $f\colon \mathfrak{l}\to\mathfrak{sl}(n+1, \boldsymbol{R})$ exists, the complexification f^c is equivalent to the direct sum of ρ_1, i. e., $f^c=\rho_1\oplus\cdots\oplus\rho_1$ and hence there is a positive integer x such that $n+1=m(2m-1)+1=x\cdot 2m$. But such an integer does not exist and therefore there is no (N)-homomorphism for $m\geqq 8$.

The case $m=4$ and 6. The degrees of irreducible representations of $\mathfrak{o}(2m, \boldsymbol{C})$ less than $n+1=m(2m-1)+1$ are given by 1, 8, 28 (for $m=4$) and 1, 12, 32, 66 (for $m=6$). Hence in each case there is no representation with degree $n+1$ not containing a 1-dimensional representation and therefore an (N)-homomorphism does not exist. q. e. d.

REMARK 6. 4. For $m=3$, $\mathfrak{o}(6, \boldsymbol{C})$ is isomorphic to $\mathfrak{sl}(4, \boldsymbol{C})$ and in § 7 we shall show that a Lie group with Lie algebra $\mathfrak{l}=\mathfrak{sl}(4, \boldsymbol{R})$ or $\mathfrak{su}^*(4)$ admits a left invariant flat projective structure. For $m=5$, the degrees of irreducible representations of $\mathfrak{o}(10, \boldsymbol{C})$ are given by 1, 10, 16, 45, $\cdots$ and in this case there is a combination $n+1=46=10+10+10+16$. For $m=7$, the degrees of $\mathfrak{o}(14, \boldsymbol{C})$ are given by 1, 14, 64, 91, $\cdots$ and there is a combination $n+1=92=14+14+64$. Thus we can not determine the existence or non-existence of (N)-homomorphisms for the real form of $\mathfrak{o}(10, \boldsymbol{C})$ or $\mathfrak{o}(14, \boldsymbol{C})$ without complicated computations.

6. 3. The case $\mathfrak{l}$ is a real form of $C_m=\mathfrak{sp}(m, \boldsymbol{C})$ $(m\geqq 3)$

$\mathfrak{l}^c\cong\mathfrak{sp}(m, \boldsymbol{C})$ is a complex simple Lie algebra of rank m. Let $\{\rho_1, \cdots, \rho_m\}$ be the fundamental system of irreducible representations of $\mathfrak{sp}(m, \boldsymbol{C})$ and let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the corresponding highest weights. It is well known that $d(\Lambda_1)=2m$ and

$$d(\Lambda_k)=\binom{2m}{k}-\binom{2m}{k-2} \qquad \text{for} \quad k=2, \cdots, m\,.$$

PROPOSITION 6. 5. *Let $\mathfrak{l}$ be a real form of $C_m=\mathfrak{sp}(m, \boldsymbol{C})$ $(m\geqq 3)$. Then there is no (N)-homomorphism f: $\mathfrak{l}\rightarrow\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ where $n=\dim_{\boldsymbol{R}}\mathfrak{l}=2m^2+m$.*

PROOF. The case $m\geqq 4$. It is easy to check that $d(\Lambda_k)>n+1$ for $k=3, 4, \cdots, m$ and by Weyl's formula we have $d(2\Lambda_1)=2m^2+m=n$, $d(3\Lambda_1)=\frac{2}{3}m(m+1)(2m+1)>n+1$, $d(\Lambda_1+\Lambda_2)=\frac{8}{3}m(m+1)(m-1)>n+1$ and $d(2\Lambda_2)=\frac{1}{3}m(m-1)(2m-1)(2m+3)>n+1$. Hence the degrees of irreducible representations less than $n+1$ are given by $d(0)=1$, $d(\Lambda_1)=2m$, $d(\Lambda_2)=2m^2-m-1$ and $d(2\Lambda_1)=2m^2+m$. If an (N)-homomorphism f exists, f^c is equivalent to the direct sum of ρ_1 and ρ_2, i. e., $f^c=\rho_1\oplus\cdots\oplus\rho_1\oplus\rho_2\oplus\cdots\oplus\rho_2$. Therefore there are non-negative integers x and y such that $n+1=2m^2+m+1=2xm+y(2m^2-m-1)$. But such integers do not exist for $m\geqq 4$ and hence there is no (N)-homomorphism.

The case $m=3$. The degrees of irreducible representations of $\mathfrak{sp}(3, \boldsymbol{C})$ are given by 1, 6, 14, 21, $\cdots$, and hence there is no 22-dimensional representation not containing a 1-dimensional respresentation. Therefore an (N)-homomorphism does not exist. q. e. d.

Summarizing the above propositions, we have

THEOREM 6.6. *Let* $\mathfrak{l}$ *be a real form of one of the following classical simple complex Lie algebras:*

$$B_m=\mathfrak{o}(2m+1, \boldsymbol{C}) \quad (m=2 \text{ or } m\geqq 4),$$
$$C_m=\mathfrak{sp}(m, \boldsymbol{C}) \quad (m\geqq 3),$$
$$D_m=\mathfrak{o}(2m, \boldsymbol{C}) \quad (m=4,\ 6 \text{ or } m\geqq 8).$$

Then a Lie group L with Lie algebra $\mathfrak{l}$ *does not admit a left invariant flat projective structure.*

§ 7. The case $\mathfrak{l}$ is a real form of $\mathfrak{sl}(m, C)$

Different from a real form of B_m, C_m or D_m, some real form of $A_m=\mathfrak{sl}(m, \boldsymbol{C})$ admits an (N)-homomorphism. In this section we shall prove the following theorem.

THEOREM 7.1. *Let* $\mathfrak{l}=\mathfrak{sl}(m, \boldsymbol{R})$ $(m\geqq 2)$ *or* $\mathfrak{su}^*(2m)$ $(m\geqq 2)$. *Then a Lie group L with Lie algebra* $\mathfrak{l}$ *admits a left invariant flat projective structure.*

PROOF. By Corollary 4.3 we have only to construct an (N)-homomorphism for each $\mathfrak{l}$.

(1) The case $\mathfrak{l}=\mathfrak{sl}(m, \boldsymbol{R})$ $(m\geqq 2)$. We fix the base $\{X_1, X_2, \cdots, X_{m^2-1}\}$ of $\mathfrak{l}$ by

$$X_1=\begin{pmatrix} 1 & & \\ & 0 \ddots & \\ & & -1 \end{pmatrix},\quad X_2=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & 0 & & \\ & & & & \end{pmatrix},\quad \cdots,\quad X_m=\begin{pmatrix} 0 & \cdots & 0 & 1 \\ & 0 & & \\ & & & \end{pmatrix},$$

$$X_{m+1}=\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & 0 & & \end{pmatrix},\quad X_{m+2}=\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \\ \cdot & \cdots & \cdots & -1 \end{pmatrix},\quad \cdots,\quad X_{m^2-1}=\begin{pmatrix} 0 & & \\ & \ddots 1 & \\ & & -1 \end{pmatrix}.$$

We shall construct an (N)-homomorphism $f: \mathfrak{sl}(m, \boldsymbol{R})\rightarrow\mathfrak{sl}(m^2, \boldsymbol{R})$. Note that $\dim_{\boldsymbol{R}}\mathfrak{l}+1=m^2$. Let $g: \mathfrak{sl}(m, \boldsymbol{R})\rightarrow\mathfrak{sl}(m^2, \boldsymbol{R})$ be a homomorphism defined by

$$g(X)=\begin{pmatrix} X & & \\ & X & \\ & & \ddots \\ & & & X \end{pmatrix} \quad (m \text{ times}) \qquad \text{for} \quad X\in\mathfrak{sl}(m, \boldsymbol{R})$$

and we set $P=(g(X_1)v, \cdots, g(X_{m^2-1})v, v)$ for $v={}^t(a_{11}, a_{12}, \cdots, a_{1m}, a_{21}, \cdots, a_{mm})\in \boldsymbol{R}^{m^2}$. Then it is easy to see that $\det P=\pm m(\det(a_{ij}))^m$. If we choose $v\in\boldsymbol{R}^{m^2}$ such that $\det(a_{ij})\neq 0$, the homomorphism $f=P^{-1}gP: \mathfrak{l}\rightarrow\mathfrak{sl}(m^2, \boldsymbol{R})$ satisfies condition (N) and hence a left invariant flat projective structure exists on L.

(2) The case $\mathfrak{l}=\mathfrak{su}^*(2m)$ $(m\geqq 2)$. The Lie algebra $\mathfrak{l}=\mathfrak{su}^*(2m)$ is expressed in the form:

$$\left\{\begin{pmatrix} A & B \\ -\bar{B} & \bar{A}\end{pmatrix} \middle| \begin{matrix} A, B\in\mathfrak{gl}(m, \boldsymbol{C}) \\ \operatorname{Tr} A+\operatorname{Tr}\bar{A}=0\end{matrix}\right\}.$$

We define the elements X_i, Y_{ij}, Z_{ij}, S_{ij} and T_{ij} of $\mathfrak{l}$ by

$$X_i=E_{ii}-E_{mm}+E_{m+i,m+i}-E_{2m,2m} \qquad \text{for} \quad i=1, \cdots, m-1,$$

$$X_{m-1+j}=\sqrt{-1}\,E_{jj}-\sqrt{-1}\,E_{m+j,m+j} \qquad \text{for} \quad j=1, \cdots, m,$$

$$Y_{ij}=E_{ij}+E_{m+i,m+j} \qquad \text{for} \quad 1\leqq i, j\leqq m \ (i\neq j),$$

$$Z_{ij}=\sqrt{-1}\,E_{ij}-\sqrt{-1}\,E_{m+i,m+j} \qquad \text{for} \quad 1\leqq i, j\leqq m \ (i\neq j),$$

$$S_{ij}=E_{i,m+j}-E_{m+i,j} \qquad \text{for} \quad 1\leqq i, j\leqq m,$$

and

$$T_{ij}=\sqrt{-1}\,E_{i,m+j}+\sqrt{-1}\,E_{m+i,j} \qquad \text{for} \quad 1\leqq i, j\leqq m,$$

where E_{ij} is the $(2m, 2m)$-matrix such that the entry at i-th row and the j-th column is 1 and other entries are all zero. Then $\{X_i, Y_{ij}, Z_{ij}, S_{ij}, T_{ij}\}$ is a base of $\mathfrak{l}$. Note that $\dim_{\boldsymbol{R}}\mathfrak{l}=4m^2-1$. We define a Lie algebra homomorphism $g: \mathfrak{l}\rightarrow\mathfrak{sl}(4m^2, \boldsymbol{R})$ by

$$g(X)=\begin{pmatrix} \operatorname{Re} X & -\operatorname{Im} X & & & \\ \operatorname{Im} X & \operatorname{Re} X & & 0 & \\ & & \ddots & & \\ & 0 & & \operatorname{Re} X & -\operatorname{Im} X \\ & & & \operatorname{Im} X & \operatorname{Re} X \end{pmatrix}\in\mathfrak{sl}(4m^2, \boldsymbol{R})$$

for $X\in\mathfrak{l}$, where $\operatorname{Re} X$ (resp. $\operatorname{Im} X$) is a real part (resp. imaginary part) of X. We set $P=(f(X_1)v, \cdots, f(X_{2m-1})v, f(Y_{12})v, \cdots, f(Y_{m-1,m})v, f(Z_{12})v, \cdots, f(T_{mm})v, v)$ for $v\in\boldsymbol{R}^{4m^2}$. If v is given by

$$v={}^t(\underbrace{1, 0, \cdots, 0}_{4m}, \underbrace{0, 1, 0, \cdots, 0}_{4m}, \underbrace{0, 0, 1, 0, \cdots, 0}_{4m}, \cdots, \underbrace{\overbrace{0, \cdots, 0, 1}^{m}, 0, \cdots, 0}_{4m}),$$

then we have $\det P=\pm m$. Therefore $f=P^{-1}gP: \mathfrak{l}\rightarrow\mathfrak{sl}(4m^2, \boldsymbol{R})$ is an (N)-homomorphism and hence a left invariant flat projective structure exists on L. q. e. d.

REMARK 7.2. Let ∇ be the projectively flat invariant affine connection on $SL(m, \boldsymbol{R})$ corresponding to the above (N) (and hence (A))-homomorphism. If $m=2$, then ∇ is given by $\nabla_X Y=\frac{1}{2}[X, Y]$ for X, $Y\in\mathfrak{sl}(2, \boldsymbol{R})$, i.e., the corresponding homomorphism is (5.3). However for $m\geqq 3$, ∇ is not the bi-invariant connection. In fact it is easily checked that the bi-invariant affine connection given by $\nabla_X Y=\frac{1}{2}[X, Y]$ $(X, Y\in\mathfrak{sl}(m, \boldsymbol{R}))$ is not projectively flat for $m\geqq 3$.

Applying Theorem 4.7 we have

COROLLARY 7.3. *The Lie group $L=GL(m, \boldsymbol{R})$ $(m\geqq 2)$ admits a left invariant (affinely) flat affine connection.*

Applying the similar method as in § 6, we have

THEOREM 7.4. *Let L be a Lie group with Lie algebra $\mathfrak{l}=\mathfrak{su}(p, q)$ such that $m=p+q\geqq 3$, $p\geqq q\geqq 1$ and m is odd. Then L does not admit a left invariant flat projective structure.*

The proof of this theorem is quite analogous to that of Theorem 6.6 and we left it to the reader. As for the other real forms of $\mathfrak{sl}(m, \boldsymbol{C})$ it is hard to determine the existence or non-existence of (N)-homomorphisms by using Proposition 4.8 and Proposition 5.1. For example the Lie algebra $\mathfrak{l}=\mathfrak{su}(4)$ admits real irreducible representations with degree 1, 6, 8, 15, 20, $\cdots$, and there is a combination $n+1=16=8+8$. Thus in order to determine the existence or non-existence we have to calculate the determinant of (16, 16)-matrix P_v for each $v\in\boldsymbol{R}^{16}$. But if we use the following theorem we know that $\mathfrak{l}=\mathfrak{su}(m)$ $(m\geqq 3)$ does not admit an (N)-homomorphism since $SU(m)$ is compact, simply connected and is not diffeomorphic to the standard sphere for $m\geqq 3$.

THEOREM D ([6]). *Let M be an n-dimensional compact, simply connected manifold with a flat projective structure $[\nabla]$. Then $(M, [\nabla])$ is projectively isomorphic to the standard flat projective structure on S^n.*

Note that $\mathfrak{l}=\mathfrak{su}(2)$ admits a unique (N)-homomorphism since $\mathfrak{su}(2)\cong\mathfrak{o}(3)$. For the Lie algebra $\mathfrak{l}=\mathfrak{su}(p, q)$ $(m=p+q\geqq 4$, $p\geqq q\geqq 1$ and m is even), we do not know whether $\mathfrak{l}$ admits an (N)-homomorphism or not.

§ 8. The case M is an irreducible Riemannian symmetric space of the classical type

In this section we shall determine the existence or non-existence of IFPS for each irreducible Riemannian symmetric space of the classical type. (We assume that M is not a Lie group.) The spaces $M=SO(n+1)/SO(n)$ and $SO_0(n,1)/SO(n)$ (type BD II) are the spaces of constant curvature and hence admit IFPS. We shall show that the spaces $M=SL(m,\boldsymbol{R})/SO(m)$ (the non-compact type of AI) and $M=SU^*(2m)/Sp(m)$ (the non-compact type of A II) also admit IFPS and the rest of (simply connected) irreducible Riemannian symmetric spaces of the classical type do not admit IFPS.

Let $M=L/K$ be an irreducible Riemannian symmetric space and let $\mathfrak{l}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition. We fix the base $\{X_1,\cdots,X_n\}$ ($n=\dim M$) of $\mathfrak{m}$. M satisfies the following conditions.

1) The linear isotropy representation $\rho_0:\mathfrak{k}\to\mathfrak{g}_0$ is irreducible. In particular $\mathfrak{m}=\{\sum_\alpha\rho_0(Y_\alpha)Z_\alpha \mid Y_\alpha\in\mathfrak{k}, Z_\alpha\in\mathfrak{m}\}$.

2) $\operatorname{Tr}\rho_0(Y)=0$ for all $Y\in\mathfrak{k}$.

Therefore by Proposition 4.6 any IFPS on M admits a unique invariant affine connection. In the following we shall decide all projectively flat invariant affine connections for each irreducible Riemannian symmetric space of the classical type. The following proposition plays an important role in our arguments.

PROPOSITION 8.1. *Let $M=L/K$ be an irreducible Riemannian symmetric space of dimension $n\geqq 3$. Let ∇ be a projectively flat invariant affine connection and $f:\mathfrak{l}\to\mathfrak{g}=\mathfrak{sl}(n+1,\boldsymbol{R})$ be the corresponding (N)-homomorphism. Then the complexification $f^c:\mathfrak{l}^c\to\mathfrak{g}^c=\mathfrak{sl}(n+1,\boldsymbol{C})$ is a complex irreducible representation of $\mathfrak{l}^c$. In particular f is a real irreducible representation of the 1-st class.*

To prove this proposition we use the following two lemmas.

LEMMA 8.2. *Let $\mathfrak{h}$ be a complex Lie algebra and let $g:\mathfrak{h}\to\mathfrak{gl}(m,\boldsymbol{C})$ ($m\geqq 2$) be a complex irreducible representation of $\mathfrak{h}$. Then invariant subspaces of the representation $g\oplus 1:\mathfrak{h}\to\mathfrak{gl}(m+1,\boldsymbol{C})$ (direct sum of g and a 1-dimensional trivial representation) are given by $\{0\}$, $\boldsymbol{C}^1$, $\boldsymbol{C}^m$ and $\boldsymbol{C}^{m+1}$.*

LEMMA 8.3. *Let $\mathfrak{h}$ be a complex Lie algebra and let $g_k:\mathfrak{h}\to\mathfrak{gl}(m,\boldsymbol{C})$ ($k=1,2,m\geqq 2$) be complex irreducible representations. Then the dimensions of invariant subspaces of $g_1\oplus g_2\oplus 1:\mathfrak{h}\to\mathfrak{gl}(2m+1,\boldsymbol{C})$ are given by 0, 1, m, $m+1$, $2m$ and $2m+1$. The subspace $\{a_{2m+1}\}$ (resp. $\{a_1,\cdots,a_{2m}\}$) is the unique*

1-dimensional (resp. 2m-dimensional) invariant subspace, where $\{a_1, \cdots, a_{2m+1}\}$ *is the standard base of* C^{2m+1}.

The proofs of these lemmas are easy and are left to the reader.

Proof of Proposition 8.1. We divide the proof according as the complexification $\rho_0{}^c: \mathfrak{k}^c \to \mathfrak{g}_0{}^c = \mathfrak{gl}(n, C)$ of ρ_0 is irreducible or reducible.

(1) The case $\rho_0{}^c$ is irreducible. We assume that f^c is reducible. Since $\mathfrak{l}^c$ is semi-simple, C^{n+1} is a direct sum of invariant subspaces V_1 and V_2 ($V_k \neq \{0\}$ for $k=1, 2$). The restriction of f^c to $\mathfrak{k}^c$ is a direct sum of $\rho_0{}^c$ and a 1-dimensional trivial representation. Note that $f(Y)$ is expressed in the form

$$f(Y) = \begin{pmatrix} \rho_0(Y) & 0 \\ 0 & 0 \end{pmatrix}$$

since $\mathrm{Tr}\,\rho_0(Y)=0$ for $Y \in \mathfrak{k}$. Thus V_1 and V_2 are also invariant subspaces of $\rho_0{}^c \oplus 1$ and hence by Lemma 8.2 we have $\dim_C V_1 = n$ and $\dim_C V_2 = 1$. The space $\{a_{n+1}\}$ is obviously an invariant subspace of $\rho_0{}^c \oplus 1$ and by the uniqueness of a 1-dimensional invariant subspace we have $V_2 = \{a_{n+1}\}$. Since f is an (N)-homomorphism we have $f^c(X_k)\, a_{n+1} = a_k \in V_2 = \{a_{n+1}\}$ for $k=1, \cdots, n$, which is a contradiction. Therefore f^c is irreducible.

(2) The case $\rho_0{}^c$ is reducible. It is well known that $\rho_0{}^c$ is a direct sum of two complex irreducible representations of the same degree (cf. [4]). We set $n=2m$ ($m \geqq 2$) and $\rho_0{}^c = g_1 \oplus g_2$. Now we assume that f^c is reducible. Then we have $C^{n+1} = V_1 \oplus V_2$ as before. The restriction of f^c to $\mathfrak{k}^c$ is $\rho_0{}^c \oplus 1 = g_1 \oplus g_2 \oplus 1$ and V_k ($k=1, 2$) are invariant subspaces of $\rho_0{}^c \oplus 1$. By Lemma 8.3 we have $\dim_C V_k = 1$, m, $m+1$ or $2m$. If $\dim_C V_2 = 1$, we have $V_2 = \{a_{n+1}\}$ and a contradiction follows in the same way as above. Therefore $\dim_C V_1 = m+1$ and $\dim_C V_2 = m$. We assume that $\{a_{n+1}\} \not\subset V_1$. Then $\{a_{n+1}\} \oplus V_1$ is an $m+2$-dimensional invariant subspace of $g_1 \oplus g_2 \oplus 1$ which is impossible for $m \geqq 3$. If $m=2$, $\{a_{n+1}\} \oplus V_1$ is a 4-dimensional invariant subspace and by Lemma 8.3 $\{a_{n+1}\} \oplus V_1 = \{a_5\} \oplus V_1 = \{a_1, a_2, a_3, a_4\}$ which is also impossible. Thus we have $\{a_{n+1}\} \subset V_1$. Since f is an (N)-homomorphism, $f^c(X_k) a_{n+1} = a_k \in V_1$ for $k=1, \cdots, n$ and hence $\{a_1, \cdots, a_n, a_{n+1}\} \subset V_1$. It is a contradiction and therefore f^c is irreducible. q. e. d.

Applying Proposition 8.1 we can prove the non-existence of IFPS on many irreducible Riemannian symmetric spaces.

PROPOSITION 8.4. *Let* ∇ *be an invariant projectively flat affine connection on an irreducible Riemannian symmetric space* M *and let* $f: \mathfrak{l} \to \mathfrak{g}$ *be the corresponding* (N)*-homomorphism. Then the induced connection* $\sigma^* \nabla$ *by the symmetry* σ *at* $o \in M$ *is also invariant and projectively flat.*

The corresponding homomorphism f' is given by $f'(X)=f_{-1}(X)-f_0(X)+f_1(X)$ for $X\in\mathfrak{m}$ and $f'(Y)=f(Y)$ for $Y\in\mathfrak{k}$, where f_p is the $\mathfrak{g}_p$-component of f. If ∇ is identical to the canonical (Riemannian) connection on M, we have $f'=f$ (i. e., $f_0(X)=0$ for $X\in\mathfrak{m}$).

The proof of this proposition is easy and is left to the reader.

In the following we shall determine the existence or non-existence of IFPS for each irreducible Riemannian symmetric space of the classical type using Proposition 8. 1. By Theorem D we have already known the non-existence of IFPS on compact simply connected symmetric spaces which are not diffeomorphic to the standard sphere (for example $M=SO(p+q)/SO(p)\times SO(q)$ $(p\geqq q\geqq 2)$, $SU(m)/SO(m)$ $(m\geqq 3)$). But we shall prove the non-existence of IFPS on these spaces using the representation theory for the sake of completeness.

(a) $M=SO(n+1)/SO(n)$, $SO_0(n, 1)/SO(n)$ (type BD II)

Since these spaces are the spaces of constant curvature, the canonical affine connection (i. e., Riemannian connection) is projectively flat. We shall show that this connection is the unique projectively flat invariant affine connection on M. Let $M=SO(n+1)/SO(n)$ $(n\geqq 2)$ and let $\mathfrak{l}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition:

$$\mathfrak{k}=\left\{\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}\middle| A\in\mathfrak{o}(n)\right\},$$

$$\mathfrak{m}=\left\{\begin{pmatrix} 0 & v \\ -{}^t v & 0\end{pmatrix}\middle| v\in\boldsymbol{R}^n\right\}.$$

We define the base $\{X_1, \cdots, X_n\}$ of $\mathfrak{m}$ by $X_k=E_{k,n+1}-E_{n+1,k}$ for $k=1, \cdots, n$, where E_{ij} is the $(n+1, n+1)$-matrix such that the entry at the i-th row and the j-th column is 1 and other entries are all zero. Then the natural inclusion ι: $\mathfrak{o}(n+1)\rightarrow\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ satisfies condition (N). This homomorphism corresponds to the canonical affine connection on M. In the case of $M=SO_0(n, 1)/SO(n)$ the natural inclusion ι: $\mathfrak{o}(n, 1)\rightarrow\mathfrak{sl}(n+1, \boldsymbol{R})$ is an (N)-homomorphism corresponding to the canonical affine connection if we set $X_k=E_{k,n+1}+E_{n+1,k}$ for $k=1, \cdots, n$ as a base of $\mathfrak{m}$. We shall show the uniqueness of (N)-homomorphism. Let f: $\mathfrak{l}\rightarrow\mathfrak{g}$ be an (N)-homomorphism ($\mathfrak{l}=\mathfrak{o}(n+1)$ or $\mathfrak{o}(n, 1)$). If $n\geqq 3$, then by Proposition 8. 1 f^c is a complex irreducible representation of $\mathfrak{o}(n+1, \boldsymbol{C})$ with degree $n+1$. We first show that f^c is equivalent to the standard inclusion ι: $\mathfrak{o}(n+1, \boldsymbol{C})\rightarrow\mathfrak{sl}(n+1, \boldsymbol{C})$. We divide the proof according as n is even or odd.

(a-1) $n=2m$ $(m\geqq 2)$. The rank of $\mathfrak{o}(2m+1, \boldsymbol{C})$ is m. Let $\{\Lambda_1, \cdots, \Lambda_m\}$

be the highest weights of fundamental system of irreducible representations. In the following we denote an irreducible representation by its highest weight. It is well known that

$$d(\Lambda_k) = \binom{2m+1}{k} \qquad \text{for} \quad k=1, \cdots, m-1$$

and $d(\Lambda_m)=2^m$. If $m \geqq 3$, we have $d(\Lambda_m)=2^m > n+1=2m+1$. Hence the irreducible representation of $\mathfrak{o}(2m+1, \boldsymbol{C})$ with degree $2m+1$ is equivalent to Λ_1. In the case $m=2$ $\mathfrak{o}(5, \boldsymbol{C})$ admits a unique representation with degree $n+1=5$ and it is the standard inclusion.

(a-2) $n=2m-1$ $(m \geqq 2)$. The rank of $\mathfrak{o}(2m, \boldsymbol{C})$ is m. Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the highest weights of fundamental system of irreducible representations. Then

$$d(\Lambda_k) = \binom{2m}{k} \qquad \text{for} \quad k=1, \cdots, m-2$$

and $d(\Lambda_{m-1})=d(\Lambda_m)=2^{m-1}$. If $m \geqq 5$, we have $d(\Lambda_{m-1})=d(\Lambda_m)=2^{m-1}>n+1=2m$. Thus f^c must be equivalent to Λ_1. For $m=2$, there are three irreducible representations of $\mathfrak{o}(4, \boldsymbol{C})$ with degree 4: $3\Lambda_1$, $3\Lambda_2$ and $\Lambda_1+\Lambda_2$. But for $\mathfrak{l}=\mathfrak{o}(4)$ and $\mathfrak{o}(3, 1)$, the complex representations $3\Lambda_1$ and $3\Lambda_2$ are both of the 2-nd class and hence f^c is equivalent to $\Lambda_1+\Lambda_2$ which is the standard inclusion. For $m=3$, the irreducible representation of $\mathfrak{o}(6, \boldsymbol{C})$ with degree 6 is Λ_1 only. For $m=4$, there are three irreducible representations of $\mathfrak{o}(8, \boldsymbol{C})$ with degree 8: Λ_1, Λ_3 and Λ_4. Λ_3 and Λ_4 are the spin representations and the restrictions of Λ_3 and Λ_4 to $\mathfrak{o}(7, \boldsymbol{C})$ are also the spin representations of $\mathfrak{o}(7, \boldsymbol{C})$. In particular they are irreducible. But the restriction of f^c to $\mathfrak{k}^c$ must be a direct sum of $\rho_0{}^c$ and a 1-dimensional trivial representation. Thus f^c is equivalent to Λ_1. Therefore the complexification f^c of an (N)-homomorphism is equivalent to the standard inclusion for $n \geqq 3$. It is easy to see that for both Lie algebras $\mathfrak{l}=\mathfrak{o}(n+1)$ and $\mathfrak{o}(n, 1)$, the standard inclusion $\iota: \mathfrak{l} \to \mathfrak{sl}(n+1, \boldsymbol{R})$ is the unique equivalence class of irreducible representation of $\mathfrak{l}$ whose complexification is equivalent to $\iota: \mathfrak{o}(n+1, \boldsymbol{C}) \to \mathfrak{sl}(n+1, \boldsymbol{C})$. Hence by Proposition 4.9 the standard inclusion is the unique (N)-homomorphism for both Lie algebras $\mathfrak{l}=\mathfrak{o}(n+1)$ and $\mathfrak{o}(n, 1)$ $(n \geqq 3)$. In the case $n=2$ since we know all the real irreducible representations of $\mathfrak{l}=\mathfrak{o}(3)$ and $\mathfrak{o}(2, 1) \cong \mathfrak{sl}(2, \boldsymbol{R})$, we can prove the uniqueness of an (N)-homomorphism using Proposition 4.8. We omit the details.

(b) $M=SO(p+q)/SO(p)\times SO(q)$, $SO_0(p, q)/SO(p)\times SO(q)$
$(p \geqq q \geqq 2$, $(p, q) \neq (2, 2)$: type BD I)

In this case an IFPS does not exist on M, except the case $M=SO_0(3, 3)/$

$SO(3)\times SO(3)$, which is isomorphic to $SL(4, \boldsymbol{R})/SO(4)$ (type AI). The dimension of M is pq and $\mathfrak{l}^c \cong \mathfrak{o}(p+q, \boldsymbol{C})$. We show that for $(p, q)\neq(3, 3)$ there is no irreducible representation of $\mathfrak{l}^c$ with degree $pq+1$. Then by Proposition 8.1 an IFPS does not exist on M. We divide the proof into two cases.

(b-1) $p+q=2m+1$ $(m\geqq 2)$. The maximum of pq is m^2+m $(p=m+1$ and $q=m)$. Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the highest weights of fundamental system of irreducible representations. Then $d(\Lambda_1)<d(\Lambda_2)<\cdots<d(\Lambda_{m-1})$, $d(\Lambda_1)=2m+1$ and $d(\Lambda_2)=2m^2+m>m^2+m+1$. By Weyl's formula we have $d(2\Lambda_1)=2m^2+3m>m^2+m+1$. If $m\geqq 5$, then $d(\Lambda_m)=2^m>m^2+m+1$ and hence Λ_1 is the unique (non-trivial) irreducible representation with degree less than m^2+m+1. But there are no integers p and q $(p\geqq q\geqq 2)$ satisfying $p+q=2m+1$ and $pq+1=2m+1$. Therefore, by Proposition 8.1, an (N)-homomorphism does not exist. For $m=2$, 3 and 4 we can easily prove the non-existence of an irreducible representation with degree $pq+1$, calculating the degrees of irreducible representations by Weyl's formula.

(b-2) $p+q=2m$ $(m\geqq 3)$. The maximum of pq is m^2 $(p=q=m)$. If $m\geqq 7$, we can prove in the same way as above that Λ_1 is the unique (non-trivial) irreducible representation with degree less than m^2+1. But there are no integers p and q satisfying $p+q=2m$ and $pq+1=d(\Lambda_1)=2m$. For $m=4, 5$ and 6 we can prove the non-existence of an (N)-homomorphism in the same manner. We omit the details. For $m=3$, there exists a possible combination $(p, q)=(3, 3)$. For the compact type, the degrees of real irreducible representations of $\mathfrak{l}=\mathfrak{o}(6)$ are given by 1, 6, 8, 15, $\cdots$. Hence there is no (N)-homomorphism. (Note that $\dim M+1=10$.) For the non-compact type, $SO_0(3, 3)/SO(3)\times SO(3)$ is isomorphic to $SL(4, \boldsymbol{R})/SO(4)$ (type AI) and we shall show in (c) that $SL(m, \boldsymbol{R})/SO(m)$ $(m\geqq 2)$ adimts an IFPS.

(c) $M=SU(m)/SO(m)$, $SL(m, \boldsymbol{R})/SO(m)$ $(m\geqq 2$: type AI)

If $m=2$, $SU(2)/SO(2)\cong SO(3)/SO(2)$ and $SL(2, \boldsymbol{R})/SO(2)\cong SO_0(2, 1)/SO(2)$ (local isomorphism) and we have already proved the uniqueness of an IFPS on these spaces in (a). Now we assume that $m\geqq 3$. The dimension of M is $\frac{1}{2}(m-1)(m+2)$. First we shall determine the complex irreducible representations of $\mathfrak{l}^c=\mathfrak{sl}(m, \boldsymbol{C})$ with degree $\frac{1}{2}(m-1)(m+2)+1=\frac{1}{2}m(m+1)$. Let $\{\Lambda_1, \cdots, \Lambda_{m-1}\}$ be the highest weights of fundamental system of irreducible representations. Note that the rank of $\mathfrak{sl}(m, \boldsymbol{C})$ is $m-1$. If $m\geqq 7$, we have

$$d(\Lambda_3)=d(\Lambda_{m-3})=\frac{1}{6}m(m-1)(m-2)>\frac{1}{2}m(m+1),$$

$$d(\Lambda_1+\Lambda_2)=d(\Lambda_{m-2}+\Lambda_{m-1})=\frac{1}{3}m(m-1)(m+1)>\frac{1}{2}m(m+1),$$

$$d(2\Lambda_2)=d(2\Lambda_{m-2})=\frac{1}{12}m^2(m-1)(m+1)>\frac{1}{2}m(m+1)$$

and

$$d(\Lambda_1+\Lambda_{m-1})=(m-1)(m+1)>\frac{1}{2}m(m+1).$$

We have also

$$d(\Lambda_1)=d(\Lambda_{m-1})=m\neq\frac{1}{2}m(m+1),$$

$$d(\Lambda_2)=d(\Lambda_{m-2})=\frac{1}{2}m(m-1)\neq\frac{1}{2}m(m+1)$$

and

$$d(2\Lambda_1)=d(2\Lambda_{m-1})=\frac{1}{2}m(m+1).$$

Hence irreducible representations with degree $\frac{1}{2}m(m+1)$ are $2\Lambda_1$ and $2\Lambda_{m-1}$. For $m=3$, 4, 5 and 6 we can easily prove that the same result holds as above.

(c-1) The case $M=SU(m)/SO(m)$ $(m\geqq 3)$. In this case an IFPS does not exist on M. In general, for the Lie algebra $\mathfrak{su}(n+1)$ the complex irreducible representation with highest weight $m_1\Lambda_1+\cdots+m_n\Lambda_n$ (m_k are non-negative integers) is of the 1-st class (cf. [4]) if and only if $m_k=m_{n+1-k}$ for $k=1,\cdots,n$ (the case $n=2i$ or $4i+3$), $m_k=m_{n+1-k}$ for $k=1,\cdots,n$ and m_{2i+1} is even (the case $n=4i+1$). Thus if $m\geqq 3$, both $2\Lambda_1$ and $2\Lambda_{m-1}$ are of the 2-nd class and hence there is no real irreducible representation of $\mathfrak{su}(m)$ with degree $\frac{1}{2}m(m+1)$. Therefore M does not admit an IFPS for $m\geqq 3$.

(c-2) The case $M=SL(m,\boldsymbol{R})/SO(m)$ $(m\geqq 3)$. In this case it is easy to check that both representations $2\Lambda_1$ and $2\Lambda_{m-1}$ are of the 1-st class and hence $\mathfrak{l}=\mathfrak{sl}(m,\boldsymbol{R})$ admits two (inequivalent) real irreducible representations with degree $\frac{1}{2}m(m+1)$. We show that each equivalence class of representations contains a unique (N)-homomorphism. Let $\mathfrak{l}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition: $\mathfrak{k}=\mathfrak{o}(m)$ and $\mathfrak{m}=\{E_{ij}+E_{ji}\ (1\leqq i<j\leqq m),\ E_{ii}-E_{mm}\ (1\leqq i\leqq m-1)\}$. Then the linear isotropy representation $\rho_0:\mathfrak{k}\rightarrow\mathfrak{gl}(\mathfrak{m})$ is given by

$$(8.1)\quad \begin{aligned}&\rho_0(E_{ij}-E_{ji})(E_{pq}+E_{qp})=\delta_{jp}(E_{iq}+E_{qi})+\delta_{jq}(E_{ip}+E_{pi})\\ &\qquad -\delta_{ip}(E_{jq}+E_{qj})-\delta_{iq}(E_{jp}+E_{pj})\quad \text{for } 1\leqq i<j\leqq m,\ 1\leqq p<q\leqq m,\\ &\rho_0(E_{ij}-E_{ji})(E_{kk}-E_{mm})=\delta_{jk}(E_{ik}+E_{ki})-\delta_{jm}(E_{im}+E_{mi})\\ &\qquad -\delta_{ik}(E_{jk}+E_{kj})\qquad\qquad \text{for } 1\leqq i<j\leqq m,\ 1\leqq k\leqq m-1.\end{aligned}$$

It is well known that the representation space V of $2\Lambda_1$ is the space of polynomials of degree two with variables $\{x_1, \cdots, x_m\}$. We define the base of V by $\{4x_i x_j\ (1 \leqq i < j \leqq m),\ 2x_1^2 - 2x_m^2, \cdots, 2x_{m-1}^2 - 2x_m^2,\ x_1^2 + \cdots + x_m^2\}$ and express the real irreducible representation $f\colon \mathfrak{l} \to \mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ corresponding to $2\Lambda_1$ in the matrix form with respect to this base. Then for $E_{ij} - E_{ji} \in \mathfrak{k}$ $(1 \leqq i < j \leqq m)$, we have

$$f(E_{ij} - E_{ji})(4x_p x_q) = 4(\delta_{jp} x_i x_q + \delta_{jq} x_i x_p - \delta_{ip} x_j x_q - \delta_{iq} x_j x_p)$$

for $1 \leqq p < q \leqq m$,

$$f(E_{ij} - E_{ji})(2x_k^2 - 2x_m^2) = 4(\delta_{jk} x_i x_k - \delta_{jm} x_i x_m - \delta_{ik} x_j x_k)$$

for $1 \leqq k \leqq m-1$ and $f(E_{ij} - E_{ji})(x_1^2 + \cdots + x_m^2) = 0$. Therefore the restriction of f to $\mathfrak{k}$ is $\rho_0 \oplus 1$ (compare with (8. 1)). Next for the elements $E_{ij} + E_{ji}$ $(1 \leqq i < j \leqq m)$ and $E_{ii} - E_{mm}$ $(1 \leqq i \leqq m-1)$ of $\mathfrak{m}$, we have

$$f(E_{ij} + E_{ji})(x_1^2 + \cdots + x_m^2) = 4x_i x_j$$

and

$$f(E_{ii} - E_{mm})(x_1^2 + \cdots + x_m^2) = 2(x_i^2 - x_m^2)\,.$$

Thus $f\colon \mathfrak{l} \to \mathfrak{sl}(V)$ satisfies condition (N). Note that we identify two spaces $\mathfrak{m} \oplus \boldsymbol{R}$ and V by the correspondence:

$$E_{ij} + E_{ji} \longleftrightarrow 4x_i x_j\,,$$

$$E_{ii} - E_{mm} \longleftrightarrow 2x_i^2 - 2x_m^2$$

and $$1 \longleftrightarrow x_1^2 + \cdots + x_m^2\,.$$

By Proposition 4. 9 f is the unique (N)-homomorphism which is equivalent to $2\Lambda_1$.

Next we consider the representation $2\Lambda_{m-1}$. We define a linear map $g\colon \mathfrak{l} \to \mathfrak{sl}(V)$ by

$$g(X) = f_{-1}(X) - f_0(X) + f_1(X) \qquad \text{for} \quad X \in \mathfrak{m}\,,$$

$$g(Y) = f(Y) \qquad \text{for} \quad Y \in \mathfrak{k}\,,$$

where $f\colon \mathfrak{l} \to \mathfrak{sl}(V)$ is the homomorphism constructed as above. Then by Proposition 8. 4 g is an (N)-homomorphism and it is easy to see that g is a real irreducible representation corresponding to $2\Lambda_{m-1}$. By Proposition 4. 9 g is the unique (N)-homomorphism which is equivalent to $2\Lambda_{m-1}$. Since $f \neq g$, f and g are not the canonical (Riemannian) connection on $M = SL(m, \boldsymbol{R})/SO(m)$.

(d) $M = Sp(m)/U(m),\ Sp(m, \boldsymbol{R})/U(m)$ $(m \geqq 3$: type CI)

In this case an IFPS does not exist on M. The dimension of M is

m^2+m. We show that there is no complex irreducible representation of $\mathfrak{l}^c \cong \mathfrak{sp}(m, \boldsymbol{C})$ with degree m^2+m+1. Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the highest weights of fundamental system of irreducible representations of $\mathfrak{sp}(m, \boldsymbol{C})$. Then $d(\Lambda_1)=2m$ and

$$d(\Lambda_k) = \binom{2m}{k} - \binom{2m}{k-2} \qquad \text{for} \quad k=2, \cdots, m\,.$$

By Weyl's formula $d(2\Lambda_1)=2m^2+m>m^2+m+1$. It is easy to see that $d(\Lambda_k)>m^2+m+1$ for $k \geqq 2$ and there is no integer m such that $2m=m^2+m+1$. Hence $M=Sp(m)/U(m)$ and $Sp(m, \boldsymbol{R})/U(m)$ do not admit an IFPS for $m \geqq 3$.

(e) $M=Sp(p+q)/Sp(p)\times Sp(q)$, $Sp(p, q)/Sp(p)\times Sp(q)$ ($p \geqq q \geqq 1$: type CII)

If $p=q=1$, then $Sp(2)/Sp(1)\times Sp(1) \cong SO(5)/SO(4)$ and $Sp(1, 1)/Sp(1)\times Sp(1) \cong SO_0(4, 1)/SO(4)$ and in this case we have already proved the uniqueness of an IFPS on M. Now assume that $(p, q) \neq (1, 1)$. Then an IFPS does not exist on M. The dimension of M is $4pq$ and $4pq+1 \leqq m^2+1$ if we set $m=p+q (\geqq 3)$. Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the highest weights of fundamental system of irreducible representations of $\mathfrak{sp}(m, \boldsymbol{C})$. Then $d(\Lambda_k)>m^2+1$ for $k=2, \cdots, m$ and $d(2\Lambda_1)=2m^2+m>m^2+1$. Hence Λ_1 is the unique (non-trivial) irreducible representation of $\mathfrak{l}^c \cong \mathfrak{sp}(m, \boldsymbol{C})$ with degree less than m^2+1. But there are no integers p and q such that $4pq+1=2m=2(p+q)$. Therefore $M=Sp(p+q)/Sp(p)\times Sp(q)$ and $Sp(p, q)/Sp(p)\times Sp(q)$ do not admit an IFPS.

(f) $M=SO(2m)/U(m)$, $SO^*(2m)/U(m)$ ($m \geqq 4$: type DIII)

In this case an IFPS does not exist on M. The dimension of M is m^2-m and we show that there is no irreducible representation of $\mathfrak{l}^c \cong \mathfrak{o}(2m, \boldsymbol{C})$ with degree m^2-m+1. Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be the highest weights of fundamental system of irreducible representations of $\mathfrak{o}(2m, \boldsymbol{C})$. If $m \geqq 6$, we have $d(\Lambda_{m-1})=d(\Lambda_m)=2^{m-1}>m^2-m+1$ and

$$d(\Lambda_k) = \binom{2m}{k} > m^2-m+1 \qquad \text{for} \quad k=2, \cdots, m-2\,.$$

By Weyl's formula $d(2\Lambda_1)=2m^2+m-1>m^2-m+1$. Since there is no integer m such that $d(\Lambda_1)=2m=m^2-m+1$, an IFPS does not exist on M for $m \geqq 6$. For $m=4$ and 5 we can prove that there is not an irreducible representation of $\mathfrak{o}(2m, \boldsymbol{C})$ with degree m^2-m+1 using Weyl's formula. We omit the details.

(g) $M=SU(2m)/Sp(m)$, $SU^*(2m)/Sp(m)$ ($m\geqq 2$: type AII)

If $m=2$, we have $SU(4)/Sp(2)\cong SO(6)/SO(5)$ and $SU^*(4)/Sp(2)\cong SO_0(5,1)/SO(5)$ and both spaces admit a unique IFPS. We assume that $m\geqq 3$. The dimension of M is $(2m+1)(m-1)$. Let $\{\Lambda_1, \cdots, \Lambda_{2m-1}\}$ be the highest weights of fundamental system of irreducible representations of $\mathfrak{l}^c\cong\mathfrak{sl}(2m, \boldsymbol{C})$. Then $d(2\Lambda_1)=d(2\Lambda_{2m-1})=2m^2+m>2m^2-m=(2m+1)(m-1)+1$, $d(\Lambda_1+\Lambda_{2m-1})=4m^2-1>2m^2-m$ and there is no positive integer m such that $d(\Lambda_1)=d(\Lambda_{2m-1})=2m=2m^2-m$. Therefore irreducible representations of $\mathfrak{sl}(2m, \boldsymbol{C})$ with degree $2m^2-m$ are given by Λ_2 and Λ_{2m-2}.

(g-1) The case $M=SU(2m)/Sp(m)$ ($m\geqq 3$). Let ρ_k ($k=1, \cdots, 2m-1$) be the complex irreducible representations of $\mathfrak{l}=\mathfrak{su}(2m)$ corresponding to Λ_k. Then it is easy to see that ρ_k is conjugate to ρ_{2m-k} for $k=1, \cdots, 2m-1$ and the index of ρ_m is $(-1)^m$. In particular ρ_2 and ρ_{2m-2} are of the 2-nd class. Hence there is no real irreducible representation of $\mathfrak{l}=\mathfrak{su}(2m)$ with degree $2m^2-m$ and an IFPS does not exist on M.

(g-2) The case $M=SU^*(2m)/Sp(m)$ ($m\geqq 3$). Let ρ_k ($k=1, \cdots. 2m-1$) be the complex irreducible representations of $\mathfrak{l}=\mathfrak{su}^*(2m)$ corresponding to Λ_k. In this case ρ_k is self-conjugate for $k=1, \cdots, 2m-1$ and the index of ρ_k is $(-1)^k$. In particular ρ_2 and ρ_{2m-2} are of the 1-st class. First we shall construct an (N)-homomorphism f: $\mathfrak{l}\rightarrow\mathfrak{g}$ which is equivalent to Λ_2.

Let $\mathfrak{l}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition:

$\mathfrak{k}=\mathfrak{sp}(m)$,

$$\mathfrak{m}=\left\{\begin{pmatrix} A & B \\ -\bar{B} & \bar{A}\end{pmatrix}\middle| A, B\in\mathfrak{gl}(m, \boldsymbol{C}).\ {}^tA=\bar{A}, \operatorname{Tr}A=0 \text{ and } {}^tB+B=0\right\}.$$

We set $W=\boldsymbol{C}^{2m}$ and fix a base $\{a_1, \cdots, a_m, b_1, \cdots, b_m\}$ of W. Let $\tilde{f}$: $\mathfrak{l}=\mathfrak{su}^*(2m)\hookrightarrow\mathfrak{sl}(2m, \boldsymbol{C})\rightarrow\mathfrak{gl}(\wedge^2 W)$ be the composite of the natural inclusion and the irreducible representation of $\mathfrak{sl}(2m, \boldsymbol{C})$ corresponding to Λ_2. We express $\tilde{f}$ in the matrix form with respect to the above base. We define the $2m^2-m$ dimensional real subspace V of $\wedge^2 W$ by

$$V=\Big\{\sqrt{-1}\,(a_i\wedge b_j-a_j\wedge b_i),\ a_i\wedge b_j+a_j\wedge b_i,\ a_i\wedge a_j+b_i\wedge b_j,\ \sqrt{-1}\,(a_i\wedge a_j-b_i\wedge b_j)$$
$$(1\leqq i<j\leqq m),\ a_k\wedge b_k\ (1\leqq k\leqq m)\Big\}.$$

It is easy to see that V^c is isomorphic to $\wedge^2 W$ and for each $X\in\mathfrak{l}$, $\tilde{f}(X)$ preserves the real subspace V. Therefore we obtain the homomorphism ρ_2: $\mathfrak{l}\rightarrow\mathfrak{gl}(V)$ corresponding to Λ_2. We consider W to be a real vector space of dimension $4m$ and define a linear map ϕ: $\mathfrak{m}\oplus\boldsymbol{R}\rightarrow\wedge^2 W$ by

$$\phi(X, r)=\rho_2(X)\,c+rc \qquad \text{for} \quad X\in\mathfrak{l} \text{ and } r\in\boldsymbol{R},$$

where $c=a_1\wedge b_1+\cdots+a_m\wedge b_m\in V$. Then ϕ is an injective linear map over $\boldsymbol{R}$ and the image of ϕ is V. We identify V and $\mathfrak{m}\oplus\boldsymbol{R}$ by this map and obtain the homomorphism $f:\mathfrak{l}\to\mathfrak{gl}(\mathfrak{m}\oplus\boldsymbol{R})$, i. e., $f(X)=\phi^{-1}\circ\rho_2(X)\circ\phi:\mathfrak{m}\oplus\boldsymbol{R}\to\mathfrak{m}\oplus\boldsymbol{R}$ for $X\in\mathfrak{l}$. Since $\mathfrak{l}$ is simple the image of f is contained in $\mathfrak{sl}(\mathfrak{m}\oplus\boldsymbol{R})$. We show that f is an (N)-homomorphism. First we observe that $\rho_2(Y)c=0$ for all $Y\in\mathfrak{k}$. Then for $X\in\mathfrak{m}$, $Y\in\mathfrak{k}$ and $r\in\boldsymbol{R}$, we have $\rho_2(Y)\ \phi(X,r)=\rho_2(Y)(\rho_2(X)c+rc)=\rho_2[Y,X]c+\rho_2(X)\rho_2(Y)c+r\rho_2(Y)c=\rho_2[Y,X]c=\phi([Y,X],0)$. (Note that $[Y,X]\in\mathfrak{m}$.) Hence by definition we have $f(Y)(X,r)=([Y,X],0)$, which implies that the restriction of f to $\mathfrak{k}$ is a direct sum of $\rho_0=\mathrm{ad}:\mathfrak{k}\to\mathfrak{gl}(\mathfrak{m})$ and a 1-dimensional trivial representation. Next for $X\in\mathfrak{m}$, we have $\rho_2(X)\ \phi(0,1)=\rho_2(X)c=\phi(X,0)$, i. e., $f(X)(0,1)=(X,0)$ which implies that f satisfies condition (N). By Proposition 4. 9 f is the unique (N)-homomorphism which is equivalent to Λ_2.

Using Proposition 8. 4 we obtain another (N)-homomorphism f'. It is easy to see that f' is equivalent to Λ_{2m-2}. f and f' are not the canonical (Riemannian) connection on M since $f\neq f'$.

(h) $M=SU(p+q)/S(U_p\times U_q)$, $SU(p,q)/S(U_p\times U_q)$ $(p\geqq q\geqq 1$: type AIII)

If $p=q=1$, we have $SU(2)/S(U_1\times U_1)\cong SO(3)/SO(2)$ and $SU(1,1)/S(U_1\times U_1)\cong SO_0(2,1)/SO(2)$ and we have already proved the uniqueness of an IFPS on M. Now we assume that $(p,q)\neq(1,1)$. The dimension of M is $2pq$. We consider the complex irreducible representation of $\mathfrak{l}^c\cong\mathfrak{sl}(n,\boldsymbol{C})$ with degree $2pq+1$ where $n=p+q\geqq 3$.

(h-1) The case $n=2m$ $(m\geqq 2)$. The maximum of $2pq$ is $2m^2$ $(p=q=m)$. Let $\{\Lambda_1,\cdots,\Lambda_{2m-1}\}$ be the highest weights of fundamental system of irreducible representations of $\mathfrak{sl}(2m,\boldsymbol{C})$. Then $d(\Lambda_3)=\frac{2}{3}m(m-1)(2m-1)>2m^2+1$ for $m\geqq 3$, $d(2\Lambda_1)=d(2\Lambda_{2m-1})=2m^2+m>2m^2+1$ and $d(\Lambda_1+\Lambda_{2m-1})=4m^2-1>2m^2+1$. Thus the irreducible representations of $\mathfrak{sl}(2m,\boldsymbol{C})$ with degree less than $2m^2+1$ are Λ_1, Λ_2, Λ_{2m-2} and Λ_{2m-1}. Let ρ_k $(k=1,\cdots,2m-1)$ be the complex irreducible representations of $\mathfrak{l}$ ($\mathfrak{l}=\mathfrak{su}(p+q)$ or $\mathfrak{su}(p,q)$) corresponding to Λ_k. Then for both Lie algebras $\mathfrak{su}(p+q)$ and $\mathfrak{su}(p,q)$ ρ_k is conjugate to ρ_{p+q-k} for $k=1,\cdots,2m-1$. Hence if $m\geqq 3$, then ρ_1, ρ_2, ρ_{2m-2} and ρ_{2m-1} are of the 2-nd class and there is no real irreducible representation of $\mathfrak{l}=\mathfrak{su}(p+q)$, $\mathfrak{su}(p,q)$ with degree less than $2m^2+1$ whose complexification is complex irreducible. For $m=2$, ρ_1 and ρ_3 are of the 2-nd class for both Lie algebras. Therefore if an (N)-homomorphism f exists, f^c must be equivalent to ρ_2. But it is easily checked that positive integers p and q satisfying $p+q=4$ and $2pq+1=d(\Lambda_2)=6$ do not exist. Hence an IFPS does not exist on M.

(h-2) The case $n=2m+1$ $(m\geqq 1)$. The maximum of $2pq$ is $2m^2+2m$ $(p=m+1$ and $q=m)$. As in the case $n=2m$, it can be proved that the irreducible representations of $\mathfrak{sl}(2m+1, \boldsymbol{C})$ with degree less than $2m^2+2m+1$ are Λ_1, Λ_2, Λ_{2m-1} and Λ_{2m}. In this case ρ_1, ρ_2, ρ_{2m-1} and ρ_{2m} are of the 2-nd class and hence there is no real irreducible representation of $\mathfrak{l}=\mathfrak{su}(p+q)$, $\mathfrak{su}(p, q)$ with degree less than $2m^2+2m+1$ whose complexification is complex irreducible. Therefore an IFPS does not exist on M.

Summarizing the above results, we obtain

THEOREM 8.5. *Let $M=L/K$ be a simply connected irreducible Riemannian symmetric space of the classical type. If M admits an IFPS, then M must be one of the following spaces:*

(1) $M_1=SL(m, \boldsymbol{R})/SO(m)$ *for $m\geqq 3$ (the non-compact type of AI),*

(2) $M_2=SU^*(2m)/Sp(m)$ *for $m\geqq 3$ (the non-compact type of AII),*

(3) $M_3=SO(n+1)/SO(n)$ *for $n\geqq 2$ (the compact type of BD II),*

(4) $M_4=SO_0(n, 1)/SO(n)$ *for $n\geqq 2$ (the non-compact type of BD II).*

M_1 and M_2 admit two IFPS, while M_3 and M_4 admit a unique IFPS. Each IFPS admits a unique invariant affine connection, which is the canonical affine connection in the cases (3) *and* (4). *Two projectively flat invariant affine connections on M_1 and M_2 are not the canonical affine connections and they are mapped to each other by the symmetry σ at $o\in M_k$* $(k=1, 2)$.

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