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Classification of cubic forms with three variables

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Introduction

A degree 3 homogeneous polynomial, $\gamma=\sum_{1\leqq i,j,k\leqq n}a_{ijk}x_{i}x_{j}x_{k}$ is called a cubic form. Our objective is to classify the set of cubic forms by linear translations. Generally, let f be a singular germ with an isolated critical point at origin and corank n. Form the Thom's splitting lemma (D. Gromoll and W. Meyer [3]), f is right equivalent to $g+Q$ where $g(x_{1}, x_{2}, \cdots, x_{n})\in m^{3}$ and $Q(x_{n+1}, x_{n+2}, \cdots, x_{n+k})$ is a nondegenerate quadratic form. Therefore it is fundamental to give the information of canonical form of 3-jet of g, when we classify the finitely determined singular germ. Indeed, D. Siersma [6] classifies the singularities with the right codimension $\leqq 8$. In his paper, one of the difficulties of the classification is the canonical form of 3-jet g, though results of algebraic geometry and the work of Mather [4] (G. Wassermann [7]) are widely used.

In this paper, we will try to classify cubic forms with 3-variables. Our conclusion coincides with the work of van der Waerden [7] concerning with the surfaces represented by cubic forms that is the curves represented by cubic forms in the projective plane. The main result is theorem 4.1. We shall prove the theorem 4.1 in terms of the concepts of homology and intersection theory in manifolds. We give a proof in § 4, in which theorem 3.2 is crucial. It is very likely that the theorem also holds for $n>3$. At the end, I would like to thank Professor H. Suzuki and Professor Fukuda for their helpful advices.

§ 1. Preparation

Let $S(n)$ be the set of all $(n\times n)$-symmetric matrices and $SL(n)$ the special linear group. We define an $SL(n)$-action on $S(n)$ by setting $F_{P}A=PAP'$ for $A\in S(n)$, $P\in SL(n)$, where P' is the transposed matrix of P. Denote by $G_{k}(S(n))$ the set of k-dimensional linear subspaces of $S(n)$ when we view $S(n)$ as a vector space. We define the $SL(n)$-action on $G_{k}(S(n))$ by setting $F_{P}\gamma=\{F_{P}A|A\in\gamma\}$ for $P\in SL(n)$, $\gamma\in G_{k}(S(n))$. This is well defined, for F_{P} is a linear automorphism of $S(n)$ for each $P\in SL(n)$. Let γ

be a n-subspace of $S(n)$. If there exist symmetric matrices A_i, $i=1, 2, \cdots, n$ which span γ such that $A_i e_j = A_j e_i$ $(1 \leqq i < j \leqq n)$, then the set of A_i's is called a cubic basis of γ. Here e_i, $i=1, 2, \cdots, n$ is the standard basis of the n-dimensional euclidian space $\boldsymbol{R}^n$. Let CF_n be the subset of $G_n(S(n))$ each element of which has a cubic basis. From the observation in the following proof, we see that the classification of CF_n is equivalent to the classification non-degenerate cubic forms with n-variables.

LEMMA 1.1. *The subset CF_n in $G_n(S(n))$ is an invariant under the $SL(n)$-action.*

PROOF. It is the problem that n-subspace $F_P\gamma$ has a cubic basis, where $\gamma \in CF_n$. We will give one observation. For a cubic basis A_i of γ, we have a cubic form $\gamma(x)$ by taking $\gamma(x) = x'(x'A_1x, x'A_2x, \cdots, x'A_nx)'$ for $x \in \boldsymbol{R}^n$ (column vector). Conversely, given a cubic form $\gamma(x)$, we have symmetric matrices A_i, $(i=1, 2, \cdots, n)$ as follows:

$$\frac{1}{3} \frac{\partial}{\partial x_j} \gamma(x) \longrightarrow A_i$$

where A_i is naturally determined by quadratic form. Let γ be the subspace spanned by A_i's. be the subspace spanned by A_i's. When the dimension of γ is equal to n, we shall call the cubic form $\gamma(x)$ a nondegenerate cubic form. Then A_i is a cubic basis of n-subspace γ, because it is assured by symmetric properties of 2nd order derivatives of $\gamma(x)$. Under the observation, we can see that the cubic form $\gamma(P'x)$ determines a cubic basis of $F_P\gamma$ by straightformard calculations. q. e. d.

By the definition, F_P is a linear automorphism of $S(n)$ for each $P \in SL(n)$. The subset $\{F_P: P \in SL(n)\}$ is a Lie-subgroup in $\mathrm{Aut}(S(n))$. Let $\mathfrak{sl}(n)$ be the set of matrices with zero trace in $\mathfrak{gl}(n)$. For each $a \in \mathfrak{sl}(n)$, an endomorphism of $S(n)$ is defined by $f_a A = aA + Aa'$, $A \in S(n)$. The subset $\{f_a: a \in \mathfrak{sl}(n)\}$ of $\mathrm{End}(S(n))$ is a Lie-algebra of the above Lie-group.

LEMMA 1.2. *The following properties hold for f_a and F_P.* (1) $\exp f_a = F_{\exp a}$, (2) $f_{PaP^{-1}} = F_P f_a F_{P^{-1}}$.

The proof is assured directly.

From Lemma 1, 2, $F_{\exp ta}$ is a 1-parameter group for any real number t and it acts naturally on the Grassmanian manifold $G_k(S(n))$. Hence its derivative f_a induces a vector field on $G_k(S(n))$. We denote it by $*f_a$. Let $\mathrm{eq}(*f_a)$ be the set of $G_k(S(n))$ consisting of all equilibria of $*f_a$ for each $a \in \mathfrak{sl}(n)$. For $\alpha \in G(S(n))$, define $\mathrm{iso}(\alpha) \subset \mathfrak{sl}(n)$ in such a way that each element $a \in \mathrm{iso}(\alpha)$ satisfies that $*f_a$ has α as an equilibrium point. The $\mathrm{iso}(\alpha)$ is a Lie-algebra of Lie-group $I(\alpha) = \{P \in SL(n): F_P\alpha = \alpha\}$. We call the dimension of iso(α) the codimenstion of α. In § 2, when $n=3$, $k=2$, we will classify such α with codimension no less than 1 and this will be used in § 4. The following propotion will be used in § 2.

PROPOSITION 1.3. *Let* $C(a)=\{P\in SL(n):\ PaP^{-1}=a\}$. *If* $\alpha\in eq(^*f_a)$ *and* $P\in C(a)$, *then we have* $F_P\alpha\in eq(^*f_a)$.

This proposition follows from lemma 1.2, easily. Later, we must calculate $I(\alpha)$ for given α, however the computation of iso(α) is easier than that of $I(\alpha)$.

In order to represent a subspace or a vector in $S(n)$, we shall define a canonical basis of $S(n)$. Let u_i $(i=1,2,\cdots,n)$ be a basis of $\boldsymbol{R}^n$ (as column vector). Put $P_i=u_iu_i'$, $Q_{ij}=u_iu_j'+u_ju_i'$ $(1\leqq i<j\leqq n)$, then the set of these symmetric matrices becomes a basis of $S(n)$. We call this basis the canonical basis of $S(n)$ Induced by the basis u_i of $\boldsymbol{R}^n$.

§ 2. Classification of the orbit of $G_2(S(3))$ with codimension no less than 1.

If the orbit of $\alpha\in G_2(S(3))$ has codimension no less than 1, then there exists a non zero matrix $a\in$iso(α) such that α is equilibirum point of vector field $^*f_a$. It follows from lemma 1.2 that we may consider $^*f_a$ where the matrix a is a real Jordan normal form. The following matrices are all the possible cases:

$$(1)\begin{pmatrix}-2t&0&0\\0&t&1\\0&-1&t\end{pmatrix}\quad(2)\begin{pmatrix}-2t&0&0\\0&t&1\\0&0&t\end{pmatrix}\quad(3)\begin{pmatrix}t_1&0&0\\0&t_2&0\\0&0&t_3\end{pmatrix}\quad(4)\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

here $t_1+t_2+t_3=0$, $t_1<t_2\leqq t_3$ in (3).
We will represent the equilibrium point by using the canonical basis P_i, Q_{ij}, induced by the standard basis e_i of $\boldsymbol{R}^n$.

LEMMA 2.1. *For each above matrix a, the element of* $eq(^*f_a)$ *is transformed into one of following by using the group* $C(a)=\{P\in SL(3)|PaP^{-1}=a\}$ *without alternating the index of canonical basis*

(1) $[P_1,P_2+P_3]$, $[P_2-P_3,Q_{23}]$ *or* $[Q_{12},Q_{13}]$

(2) i) $t_1\neq0$, $[P_1,P_2]$, $[P_1,Q_{12}]$ *or* $[Q_{12},Q_{13}]$

ii) $t_1=0$, *the other of* i): $[Q_{12},Q_{13}+P_2]$, $[P_2,Q_{23}+P_1]$ *or* $[P_1-P_2,Q_{12}]$

(3) i) $t_3\neq t_2\neq0$, $[P_1,Q_{12}]$, $[P_1,P_2]$, $[P_1,Q_{23}]$ *or* $[Q_{12},Q_{13}]$

ii) $t_3=t_2\neq0$, *the other of* i): $[Q_{12},P_2+\varepsilon P_3]$, $[P_1,P_2+\varepsilon P_3]$ *or* $[P_2-P_3,Q_{23}]$

iii) $t_2=0$ *the other of* $t_2\neq 0$: $[P_2+Q_{13}, P_1]$ *or* $[P_2+Q_{13}, Q_{12}]$

(4) $[P_1, Q_{13}-P_2]$.

Here $\varepsilon=\pm 1$.

PROOF. The proof is a direct calculation. We only show the case (3), ii). The other cases are shown similarly. We note that an element of $C(a)$ has the following form,

$$\begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} \in SL(3) .$$

From the definition of f_a, we have $f_a P_1=-4P_1$, $f_a A=-2A$ for $A\in[Q_{12}, Q_{13}]$, $f_a A=A$ for $A\in[P_2, Q_{23}, P_3]$. $eq(*f_a)$ consists of:

a) $G_2([P_2, Q_{23}, P_3])$,

b) $[Q_{12}, Q_{13}]$,

c) $[x_1 Q_{12}+x_2 Q_{13}, x_3 P_2+x_4 Q_{23}+x_5 P_3]$,

d) $[P_1, x_3 P_2+x_4 Q_{23}+x_5 P_3]$,

e) $[P_1, x_1 Q_{12}+x_2 Q_{13}]$,

where $(x_1, x_2)\neq(0, 0)$, $(x_3, x_4, x_5)\neq(0, 0, 0)$. We have only to show that c) is transformed into $[Q_{12}, P_2+\varepsilon P_3]$, $[Q_{12}, P_3]$, $[Q_{13}, P_3]$ or $[Q_{12}, Q_{23}]$ and the rest follows earily. (we remark that $[Q_{12}, P_3]$, $[Q_{12}, Q_{23}]$, $[Q_{12}, P_3]$ are equivalent to $[P_1-P_2, P_3]$, $[Q_{12}, Q_{13}]$, $[P_1, Q_{12}]$ respectively.)

If the matrix $x_3 P_2+x_4 Q_{23}+x_5 P_3$ with rank 2 is semidefinite, there exists a matrix T in $C(a)$ such that we get $F_T(x_3 P_2+x_4 Q_{23}+x_5 P_3)=P_2+P_3$. $x_1 Q_{12}+x_2 Q_{13}$ is transformed into the matrix $y_1 Q_{12}+y_2 Q_{13}$ for some y_i by F_T. We choose a rotation matrix U with the vector e_1 as axis such that $F_U(y_1 Q_{12}+y_2 Q_{13})=y_3 Q_{12}$. Then we have $F_{UT}\alpha=[P_2+P_3, Q_{12}]$. We notice that the matrix $UT\in C(a)$.

Next if the matrix $x_3 P_2+x_4 Q_{23}+x_5 P_3$ with rank 2 is semi-indefinite, there exists a matrix T in $C(a)$ such that $F_T(x_3 P_2+x_4 Q_{23}+x_5 P_3)=Q_{23}$. We can put $F_T(x_1 Q_{12}+x_2 Q_{13})=y_1 Q_{12}+y_2 Q_{13}$ where $y_1\neq 0$. If $y_2=0$, we have $F_T\alpha=[Q_{12}, Q_{13}]$. If $y_2\neq 0$, there exists a diagonal matrix D such that $F_D(y_1 Q_{12}+y_2 Q_{13})=Q_{12}+Q_{13}$. Let U be the $\pi/4$-rotation matrix with the vector e_1 as axis, then we have $F_U(Q_{12}+Q_{13})=\sqrt{2}Q_{12}$ and $F_U Q_{23}=P_2-P_3$. Therefore $F_{UDT}\alpha=[Q_{12}, P_2-P_3]$, where $UDT\in C(a)$.

Finally if the matrix $x_3 P_2+x_4 Q_{23}+x_5 P_3$ has rank 1, there exists a matrix T in $C(a)$ such that $F_T(x_3 P_2+x_4 Q_{23}+x_5 P_2)=P_3$. We can put $F_T(x_1 Q_{12}+x_2 Q_{13})=y_1 Q_{12}+y_2 Q_{13}$. If $y_1=0$, we get $F_T\alpha=[Q_{13}, P_3]$. If $y_1\neq 0$, we choose a matrix U such that $Ue_1=e_1$, $Ue_2=e_2-\dfrac{y_2}{y_1}e_3$, $Ue_3=e_3$. Then we get

$F_U(y_1Q_{12}+y_2Q_{13})=y_1Q_{12}$. Therefore we obtain $F_{UT}\alpha=[Q_{12}, P_3]$ where $UT\in C(a)$. q. e. d.

In view of lemma 2. 1 and by direct computation of f_α for each α, we obtain :

THEOREM 2. 2. *The following table is a classification of* $G_2(S(3))$ *with codimension no less than* 1.

Table 1.

codimension	*subspace*	*iso*
1	$[Q_{12}, P_2+\varepsilon P_3]$	$\begin{pmatrix} -2a_{22} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}$
2	$[P_1, P_2+\varepsilon P_3]$	$\begin{pmatrix} -2a_{22} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & -\varepsilon a_{23} & a_{22} \end{pmatrix}$
	$[Q_{12}, Q_{13}+P_2]$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & -a_{11} \end{pmatrix}$
3	$[P_1, Q_{13}+P_2]$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & -a_{12} \\ 0 & 0 & -a_{11} \end{pmatrix}$
4	$[Q_{12}, Q_{13}]$	$\begin{pmatrix} -a_{22}-a_{33}, & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$
	$[P_2+\varepsilon P_3, Q_{23}]$	$\begin{pmatrix} -2a_{22} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \varepsilon a_{23} & a_{22} \end{pmatrix}$
5	$[P_1, Q_{12}]$	$\begin{pmatrix} -a_{22}-a_{33}, & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

§ 3. Homological properties of the stratified set.

In this section, we consider homology groups of certain stratified set. Homology coefficients are assumed to be $\boldsymbol{Z}_2$ $(=\boldsymbol{Z}/2\boldsymbol{Z})$. Let $M=\{[Q_{12}, P_2+Q_{13}]\}$. We denote the closure of M in $G_2(S(3))$ by cl M.

THEOREM 3.1. *$H_6(cl\ M;\ \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$.*

Let [cl M] be the generator of H_6(cl M). It's image is a generator in $H_6(G_2(S(3)))$ $(\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2)$ by the inclusion i: cl $M\subset G_2(S(3))$. A linear inclusion map $\boldsymbol{R}^3\rightarrow S(3)$ induces a map $G_2(\boldsymbol{R}^3)\rightarrow G_2(S(3))$. Its image of the fundamental class $[G_2(\boldsymbol{R}^3)]$ is viewed as a generator of $H_2(G_2(S(3)))$. We denote also by [cl M], $[G_2(\boldsymbol{R}^3)]$ their images of the inclusions in $G_2(S(3))$. Then, the intersection pairing.: $H_6(G_2(S(3)))\times H_2(G_2(S(3)))\rightarrow\boldsymbol{Z}_2$ is defined.

THEOREM 3.2. *We have the intersection number* $[cl\ M]\cdot[G_2(\boldsymbol{R}^3)]\equiv 1$ $(mod\ 2)$.
Theorem 3.2 is proved in the last section.

A PROOF of THEOREM 3.1. We investigate the structure of cl M. The subset $\{[x_1P_1+Q_{12}, x_2P_1+x_3P_2+Q_{13}]: x_3\neq 0\}$ of $G_2(S(3))$ is contained in M. This is shown as follows: we take the basis u_i of $\boldsymbol{R}^3$ such that $u_1=e_1$, $u_2=y(e_2+x_1/2e_1)$, $u_3=1/y(e_3+(x_1/2+x_1^2x_3/8)e_1)$, where $y=x_1^{1/3}$, then we have $F_T[x_1P_1+Q_{12}, x_2P_1+x_3P_2+Q_{13}]=[Q_{12}, P_2+Q_{13}]$ where $T\in GL(3)$ and $Tu_i=e_i$, $i=1, 2, 3$. When we give the basis P_i, Q_{ij} an order like P_1, Q_{12} P_2, Q_{13}, $\cdots$, the subset $K=\{[x_1P_1+Q_{12}, x_2P_1+x_3P_2+Q_{13}]\}$ is regarded as a Schubert variety. It is easily checked that cl $K-K$ contains every 2-subspace with codimension no less than 3.

Let $T(3)$ be the upper triangular matrix with positive diagonal element. We have diffeomorphism φ: $SO(3)\times T(3)\rightarrow GL(3)$ such that $\varphi(P, T)=PT$ for $P\in SO(3)$, $T\in T(3)$, $T(3)$-orbit of cl K is cl K itself and therefore, we have cl $M=\{F_P\alpha: \alpha\in$ cl K, $P\in SO(3)\}$. From this structure of cl M, it is sufficient to prove the theorem 3.1 that we consider only the manifold structure of cl M at $[P_1, Q_{13}+P_2]$.

Let D^3 be the 3-disc $\{[P_1+x_1Q_{13}+x_2Q_{23}+x_3P_3, P_2+Q_{13}]: x_i\in\boldsymbol{R}\}$ with the center $\alpha_0=[P_1, P_2+Q_{13}]$ in $G_2(S(3))$. D^3 intersects transversally with $\{[P_1, P_2+Q_{13}]\}$ at $[P_1, P_2+Q_{13}]$. This is shown by the following considerations and some computations. We can identify the tangent space $T_{\alpha_0}G_2(S(3))$ with hom $(\alpha_0, \alpha_0^{\perp})$. Then we define the local homeomorphism φ of hom $(\alpha_0, \alpha_0^{\perp})$ to $G_2(S(3))$ by $\varphi(V)=\{A+VA: A\in\alpha_0\}$.

We obtain $D^3\cap\text{cl}\ M=\left\{\left[P_1+\frac{3}{2}t^2Q_{13}+t^3Q_{23}-\frac{3}{4}P_3, P_2+Q_{13}\right]: t\in\boldsymbol{R}\right\}$ by the computation of iso (β), $\beta\in D^3$. This intersection is homeomorphic (not diffeomorphic) to a 1-disc and is denoted by D^1.

Let α_t a point of D^1 for $t \in \boldsymbol{R}$. $T_{\alpha_t}M$ is identified with $\mathrm{iso}(\alpha_t{}^{\perp})$ by the following correspondence; $\mathrm{iso}(\alpha_t)^{\perp} \cong \mathfrak{sl}(3)/\mathrm{iso}(\alpha_t) \overset{h}{\to} \mathrm{End}(S(3))/\{A: A\alpha_t \subset \alpha_t\} \cong \mathrm{hom}(\alpha_t, \alpha_t^{\perp})$, where $h([a]) = [f_a]$, $a \in \mathfrak{sl}(3)$. Under this identificantion, we obtain $\lim_{t\to 0}(T_{\alpha_t}D^1)^{\perp} \cap \mathrm{iso}(\alpha_t)^{\perp} = \mathrm{iso}(\alpha_0)^{\perp}$, in $G_5(\mathfrak{sl}(3))$. Therefore the orthogonal projection of $\mathfrak{sl}(3)$ to $\mathrm{iso}(\alpha_0)^{\perp}$ induces the linear isomorphism $\varphi_t: (T_{\alpha_t}D^1)^{\perp} \cap \mathrm{iso}(\alpha_t)^{\perp} \leftarrow \mathrm{iso}(\alpha_0)^{\perp}$, for $|t| < \varepsilon$ and sufficiently small $\varepsilon > 0$. We define a map $g: D^1 \times \mathrm{iso}(\alpha_0)^{\perp} \to \mathrm{cl}\, M$ by $g(\alpha_t, a) = F_{\exp\varphi_t(a)}\alpha_t$. From the definition, $\lim_{t\to 0} T_{\alpha_t}D^1 + \lim_{t\to 0} \mathrm{Im}\, dg|_{\alpha_t \times \mathrm{iso}(\alpha_0)^{\perp}} = \lim_{t\to 0} T_{\alpha_t}M$. We notice that $\lim_{t\to 0} \mathrm{Im}\, dg|_{\alpha_t \times \mathrm{iso}(\alpha_0)^{\perp}}$ is $T_{\alpha_0}\{[P_1, P_2 + Q_{13}]\}$. This propertry implies that $T_{\alpha_t}D^1 + \mathrm{Im}\, dg|_{\alpha_t \times \mathrm{iso}(\alpha_0)^{\perp}} = T_{\alpha_t}M$, for $|t| < \varepsilon$, and sufficiently small $\varepsilon > 0$, then a local homeomorphism of g at α_0 is assured. q. e. d.

§ 4. The classification of CF_3

THEOREM 4. 1. *The classification of CF_3 by $SL(3)$ action is as follows.*

Table 2.

codimension	*subspace*	*iso*
0	$[\varepsilon P_1 + P_3, Q_{23}, P_2 + Q_{13} + tP_3]$ $(\varepsilon t^2 + 1 \neq 0)$ $[Q_{23}, Q_{13} + P_2, Q_{12} + P_3]$	0 *matrix*
1	$[\varepsilon_1 P_1 + \varepsilon_2 P_2 + P_3, \varepsilon_2 Q_{12}, Q_{13}]$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & \varepsilon_2 a_{23} & 0 \end{bmatrix}$
2	$[P_2 - \varepsilon P_3, Q_{12}, -\varepsilon Q_{13}]$ $[P_1, Q_{23}, P_2 + \varepsilon P_3]$	$\begin{bmatrix} -2a_{22} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & \varepsilon a_{23} & a_{22} \end{bmatrix}$
3	$[P_3, P_2, Q_{13}]$	$\begin{bmatrix} -a_{22}-a_{33} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$
4	$[P_3, Q_{23}, Q_{13} + P_2]$	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & -a_{11} \end{bmatrix}$

$(\varepsilon_i = \pm 1)$

PROOF. For any $\gamma \in CF_3$, we denote by $G_2(\gamma^{\perp})$ the Grassmanian manifold which consists of all the 2-subspaces in $\gamma^{\perp}$. From theorem 3. 2, we have $[G_2(\gamma^{\perp})]\cdot[\mathrm{cl}\ M]\equiv 1 \pmod 2$. Then there exists $\alpha \in \mathrm{cl}\ M \cap G_2(\gamma^{\perp})$. We may assume that α is one of the elements in Table 1 of theorem 2. 2. Then we show that for each α, γ can be transformed into one of the above table. To do so, we start with the following assertion.

Assertion. If $\alpha \subset \gamma^{\perp}$ and $A \in I(\alpha)$, then we have $\alpha \subset (F_{A'}\gamma)^{\perp}$. The proof is easy, so we will omit it.

If $\alpha=[Q_{12}, P_2+Q_{13}]$, then the cubic basis representation (we write c. b. r. for convenience.) of γ is $[x_1P_1+x_2P_3, -2x_2Q_{23}+x_3P_3, x_2(Q_{13}-2P_2)+x_3Q_{23}+x_4P_3]$. If $x_2 \neq 0$, we can choose a matrix $A' \in I(\alpha)$ such that $Ae_1=e_1$, $Ae_2=e_2+x_3/4x_2e_3$, $Ae_3=e_3$. In this case, we may assume $x_2=1$. Now, since we get $F_A(-2Q_{23}+x_3P_2)=-1/2Q_{23}$, $F_A(Q_{13}-2P_2+x_3Q_{23}+x_4P_3)=Q_{13}-2P_2+x_3/2Q_{23}+(3x_3^2/8+x_4)P_3$, then the c. b. r. of $F_A\gamma$ is $[y_1P_1+P_3, -2Q_{23}, Q_{13}-2P_2+y_2P_3]$ for some y_i. Using a diagonal matrix B in $SL(3)$, we can find the more simplified form $[\varepsilon P_1+P_3, Q_{23}, Q_{13}+P_2+tP_3]$ as the c. b. r. of $F_{BA}\gamma$, other than $[\varepsilon P_1+P_3, -2Q_{23}, Q_{13}-2P_2+tP_3]$. If $x_2=0$, γ includes a 2-subspace with codimension larger than 2. So, we deal with this case later.

If $\alpha=[P_1, P_2+Q_{13}]$, the c. b. r. of γ is $[x_1Q_{23}+x_2P_3, x_1(Q_{13}-2P_2)-2x_2Q_{23}+x_3P_3, x_1Q_{12}+x_2(Q_{13}-2P_2)+x_3Q_{23}+x_4P_3]$. If $x_1 \neq 0$, we choose $A' \in I(\alpha)$ such that $Ae_1=e_1+x_2/2x_1e_2$, $Ae_2=e_2-x_2/2x_1e_3$, $Ae_3=e_3$. Similarly as above, we assume $x_1=1$. Since we have that $F_A(Q_{23}+x_2P_3)=Q_{23}$, $F_A(Q_{13}-2P_2-2x_2Q_{23}+x_3P_3)=Q_{13}-2P_2-x_2/2Q_{23}+(3x_2^2/2+x_3)P_3$, $F_A(Q_{12}+x_2(Q_{13}-2P_2)+x_3Q_{23}+x_4P_3=Q_{12}+x_2/2(Q_{13}-2P_2)+(5x_2^2/4+x_3)Q_{23}+(-x_2^2/2-x_2x_3+x_4)P_3$, then the c. b. r. of $F_A\gamma$ is $[Q_{23}, Q_{13}-2P_2+y_1P_3, Q_{12}+y_2Q_{23}+y_3P_3]$ for some y_i. If we choose $B' \in I(\alpha)$ such that $Be_1=e_1-y_1/2e_3$, $Be_2=e_2$, $Be_3=e_3$, then we have that $F_B(Q_{13}-2P_2+y_1P_3)=Q_{13}-2P_2$, $F_B(Q_{12}+y_2Q_{23}+y_3P_3)=Q_{12}+(-y_1/2+y_2)Q_{23}+y_3P_3$. The c. b. r. of $F_{BA}\gamma$ is $[Q_{23}, Q_{13}-2P_2, Q_{12}+y_3P_3]$. Finally we use a diagonal matrix C in $SL(3)$ to yield that $F_{CBA}\gamma=[Q_{23}, Q_{13}+P_2, Q_{12}+P_3]$.

If $\alpha=[P_1, Q_{12}]$, the c. b. r. of γ is $[P_3, x_1P_2+x_2Q_{23}+x_3P_3, Q_{13}+x_2P_2+x_3Q_{23}+x_4P_3]$. If $x_1 \neq 0$, we can choose $A' \in I(\alpha)$ such theat $Ae_1=e_1$, $Ae_2=e_2-x_2/x_1e_3$, $Ae_3=e_3$. Since we have: $F_A(x_1P_2+x_2Q_{23}+x_3P_3)=x_1P_2+(-x_2^2/x_1+x_3)P_3$, $F_A(Q_{13}+x_2P_2+x_3Q_{23}+x_4P_3)=Q_{13}+x_2P_2+(-x_2^2/x_1+x_3)Q_{23}+(-x_2^3/x_1^2-2x_2x_3/x_1+x_4)P_3$, it follows that the c. b. r. of $F_A\gamma$ is $[P_3, P_2+y_2P_3, Q_{13}+y_1Q_{23}+y_2P_3]$ for some y_i. We choose $B' \in I(\alpha)$ such that $Be_1=e_1-y_1e_2-y_2/2e_3$, $Be_2=e_2$, $Be_3=e_3$, and then we have $F_B(Q_{13}+y_1Q_{23}+y_2P_3)=Q_{13}$. Hence the c. b. r. of $F_{BA}\gamma$ is $[P_3, P_2, Q_{13}]$. If $x_1=0$, (we may assume $x_2 \neq 0$, for a cubic basis must exist on γ) we can choose $A' \in I(\alpha)$ such that $Ae_1=e_1$, $Ae_2=e_2-x_3/2x_2e_3$, $Ae_3=e_3$. Then we have $F_A(x_2Q_{23}+x_3P_3)=x_2Q_{23}$, $F_A(Q_{13}+x_2P_2+x_3Q_{23}+x_4P_3)=$

$Q_{13}+x_2P_2+x_3/2Q_{23}+(-x_3^2/4x_2+x_4)P_3$. The c. b. r. of $F_A\gamma$ is $[P_3, Q_{23}, Q_{13}+y_1P_2+y_2P_3]$ for some y_i. Nextly, we choose $B'\in I(\alpha)$ such that $Be_1=e_1-y_2/2e_3$, $Be_2=e_2$, $Be_3=e_3$, and then we have $F_B\gamma(Q_{13}+y_1P_2+y_2P_3)=Q_{13}+y_1P_2$. The c. b. r. of $F_{BA}\gamma$ is $[P_3, Q_{23}, Q_{13}+y_1P_2]$. Finally, using a diagonal matrix $C\in SL(3)$, we see that the c. b. r. of $F_{CBA}\gamma$ is $[P_3, Q_{23}, Q_{13}+P_2]$.

Let α is $[P_2+\varepsilon P_3, Q_{23}]$. The c. b. r. of $\gamma=[x_1P_1+x_2Q_{12}+x_3Q_{13}+P_2-\varepsilon P_3, x_2P_1+Q_{12}, x_3P_1-\varepsilon Q_{13}]$. We choose $A'\in I(\alpha)$ as follow: $Ae_1=e_1$, $Ae_2=e_2-x_2/2e_1$, $Ae_3=e_3-\varepsilon x_3/2e_1$, then we have $F_A(x_1P_1+x_2Q_{12}+x_3Q_{13}+P_2-\varepsilon P_3)=(x_1-(3x_2^2-5\varepsilon x_3^2)/4)P_1+x_2/2Q_{12}+3x_3/2Q_{13}+P_2-\varepsilon P_3$, $F_A(x_2P_1+Q_{12})=Q_{12}$, $F_A(x_3P_1-\varepsilon Q_{13})=-\varepsilon Q_{13}$. Hence the c. b. r. of $F_A\gamma$ is $[y_1P_1+P_2-\varepsilon P_3, Q_{12}, -\varepsilon Q_{13}]$. By using the diagonal $B\in SL(3)$, we see that c. b. r. of $F_{BA}\gamma$ is $[\varepsilon_1P_1+\varepsilon_2P_2+P_3, \varepsilon_2Q_{12}, Q_{13}]$ or $[\varepsilon P_2+P_3, \varepsilon Q_{12}, Q_{13}]$ where $\varepsilon_i=1$.

Let $\alpha=[Q_{12}, Q_{13}]$. The c. b. r. of γ is $[P_1, x_1P_2+x_2Q_{23}+x_3P_3, x_2P_2+x_3Q_{23}+x_4P_3]$. This case is equivalent to the classification of two variables cubic from (the reference of [1]). Therefore we only show the result, $F_A\gamma=[P_1, Q_{23}, P_2+\varepsilon P_3]$ or $[P_1, Q_{23}, P_2]$ where $A'\in I(\alpha)$. q. e. d.

PROOF of theorem 3. 2.

Let $\gamma=[P_1+P_3, Q_{23}, P_2+Q_{13}+P_3]$. We will show that $G_2(\gamma^{\perp})$ has a transversal intersection in M, and then count of its number. If $G_2(\gamma^{\perp})\cap(\mathrm{cl}\,M-M)\neq\phi$, then by the argument of theorem 4. 1, we see that iso$(\gamma)=\{0\}$. This is impossible by choosing γ. Then $G_2(\gamma)\cap\mathrm{cl}\,M=G_2(\gamma^{\perp})\cap M$. Let $\alpha\in G_2(\gamma)\cap M$. The transeversality at α can be shown by the direct computation of the tangent space like the result of [2]. Since this is not difficult, we omit it. We need the following assertion to count the interection numbers.

ASSERTION. *Let $\gamma\in CF_3$ and A_i $(i=1, 2, 3)$ be a cubic basis of γ and let P_i, Q_{ij} $(1\leqq i<j\leqq 3)$ be a canonical basis induced by u_i. If $[Q_{12}, P_2+Q_{13}]\subset\gamma^{\perp}$, then u_i satisfy the following equations:*

(1) i) $u_2'(u_2'A_1u_2,\ u_2'A_2u_2,\ u_2'A_2u_2)=0$

ii) $\det\left(\sum_{i=1}^{3}u_{2i}A_i\right)=0$ *where* $u_2=(u_{21}, u_{22}, u_{23})'$.

(2) $\left(\sum_{i=1}^{3}u_{2i}A_i\right)u_i=0$

(3) $\left(\sum_{i=1}^{3}u_{1i}A_i\right)u_3=-(u_2'A_1u_2,\ u_2'A_2u_2,\ u_2'A_3u_2)'$.

PROOF of assertion. If Q_{12}, $P_2+Q_{13}\in\gamma^{\perp}$ then we have $\mathrm{tr}\,Q_{12}A_i=0$ and $\mathrm{tr}\,(P_2+Q_{13})A_i=0$ $(i=1, 2, 3)$. Using the relations: $\mathrm{tr}\,u_iu_j'A_k=u_i'A_ku_j$ or $u_j'A_ku_i$ and $A_ie_j=A_je_i$, the former equation is reduced to $\left(\sum_{i=1}^{3}u_{1i}A_i\right)u_2=0$, while the latter is $\left(\sum_{i=1}^{3} u_{1i} A_i\right) u_3 = -(u_2' A_1 u_2, u_2' A_2 u_2, u_2' A_3 u_2)$. $\sum_{i=1}^{3} u_{1i} A_i$ is the symmetric linear map, then the kernel is orthgonal to the image. Therefore we obtain $u_2'(u_2' A_1 u_2, u_2' A_2 u_2, u_2' A_3 u_2)' = 0$. Using $A_i e_j = A_j e_i$, the above former equation can be reduced to $\left(\sum_{i=1}^{3} u_{2i} A_i\right) u_1 = 0$. Then we see that $\det\left(\sum_{i=1}^{3} u_{2i} A_i\right) = 0$. *We finish the proof of assertion.*

We are now in the position to prove the theorem 3. 2. The number of intersections is equal to the number of solution of equations (1) by the assertion. For given γ, (1) is as follow:

$x^3 + 3xz^2 + 3y^2 z + z^3 = 0$, $x^2 z + xz^2 - xy^2 - z^3 = 0$. where we put $u_2 = (x, y, z)'$. Except the trivial solution $(0, y, 0)$, we have two solution by a simple calculation. Therefore this show theorem 3. 2.

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