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On the infinitesimal Blaschke conjecture

By Kazuyoshi KIYOHARA

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Introduction

Let M be a riemannian manifold and g its riemannian metric. Then M is called a C_l-manifold and g a C_l-metric if all of its geodesics are periodic and have the same length l. So far very few C_l-manifolds are known except for the following famous examples: The spheres S^n $(n\geqq 1)$ and the various projective spaces, *i. e.*, the real projective spaces $\boldsymbol{RP}^n$ $(n\geqq 2)$, the complex projective spaces $\boldsymbol{CP}^n$ $(n\geqq 2)$, the quaternion projective spaces $\boldsymbol{HP}^n$ $(n\geqq 2)$, and the Cayley projective plane $\boldsymbol{CaP}^2$, all of these being equipped with the standard metrics. In the case of S^n we know that there are non-standard C_l-metrics, which are given by Zoll and Weinstein (cf. [1]). On the other hand, for $\boldsymbol{RP}^n$, the non-existence of such metrics was proved by Berger (cf. [1] Appendix D). But it is not known whether there exist non-standard C_l-metrics on any other projective space. For a historical reason, the conjecture of non-existence of such metrics on the projective spaces is called the Blaschke conjecture.

The main purpose of the present paper is to study an infinitesimal version of the Blaschke conjecture, the infinitesimal Blaschke conjecture, and to give a partial affirmative answer to this conjecture.

Let M be one of the spaces $\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, and $\boldsymbol{CaP}^2$, and g_0 its standard C_π-metric. Let us consider a deformation g_t of the riemannian metric g_0 which satisfies the following conditions:

1) Each g_t is a C_π-metric;

2) Each g_t is semi-conformal to g_0, *i. e.*, for any projective line $N\subset M$ there is a function h_t on N such that $\iota^*g_t=h_t\iota^*g_0$, where ι denotes the inclusion $N\rightarrow M$.

Then we know that the linearization $f=\left.\dfrac{\partial g_t}{\partial t}\right|_{t=0}$ of g_t at $t=0$, being a symmetric 2-form on M, satisfies the following conditions:

a) $\displaystyle\int_0^\pi f\big(\dot\gamma(t),\dot\gamma(t)\big)\,dt=0$ for any geodesic $\gamma(t)$ with $\|\dot\gamma(t)\|=1$;

b) f is semi-conformal to g_0, *i. e.*, for any projective line $N\subset M$ there is a function h on N such that $\iota^*f=h\iota^*g_0$, where ι denotes the inclusion

$N \to M$.

Now we say that a deformation g_t of g_0 is a semi-conformal C_π-deformation of g_0 if it satisfies conditions 1) and 2), and correspondingly that a symmetric 2-form f on M is an infinitesimal semi-conformal C_π-deformation of g_0 if it satisfies conditions a) and b). Then our result may be stated as follows:

THEOREM A. *Any infinitesimal semi-conformal C_π-deformation f of g_0 is trivial, that is, there is a vector field X on M such that $f = \mathcal{L}_X g_0$.*

Here we make several remarks on the theorem:

(1) The theorem is not the case when M is $\boldsymbol{CP}^1$ or $\boldsymbol{HP}^1$ or $\boldsymbol{CaP}^1$ (cf. Lemma 1.2 and [5]).

(2) In case $M = \boldsymbol{CP}^n$, condition b) means that f is hermitian with respect to the standard complex structure I of $\boldsymbol{CP}^n$. Thus the adjective "semi-conformal" may be replaced by the adjective "hermitian", and the theorem asserts that any infinitesimal hermitian C_π-deformation of g_0 is trivial.

(3) In case $M = \boldsymbol{CP}^n$ or $\boldsymbol{HP}^n$, our theorem combined with the results of Tanaka [6] and Kaneda-Tanaka [3] yields the finite dimensionality of the space $\tilde{\mathcal{S}}'$ of infinitesimal semi-conformal C_π-deformations of g_0.

(4) Using these results, N. Tanaka has recently proved the following

THEOREM (Tanaka). *Let $M = \boldsymbol{CP}^n$, and let g_t be a kählerian C_π-deformation of g_0. If g_t depends real analytically on the parameter t, then there is a one-parameter family ϕ_t of holomorphic transformations of $\boldsymbol{CP}^n$ such that $\phi_0 =$ identity and $g_t = \phi_t^* g_0$.*

This paper consists of four sections and an appendix. In § 1 we reduce the proof of Theorem A to the case $M = \boldsymbol{CP}^2$. For this purpose we use Michel's result [5], which is also used in § 4. §§ 2–4 are devoted to the proof of the case $M = \boldsymbol{CP}^2$, *i.e.*, the following.

THEOREM A′. *On $\boldsymbol{CP}^2$, any infinitesimal hermitian C_π-deformation f of g_0 is trivial.*

In § 2 we first define a space $\mathcal{A}_0$ of hermitian 3-matrix valued functions on the unit sphere S^5 of $\boldsymbol{C}^3$, and then show that there is a one-to-one correspondence between the space $\mathcal{A}_0$ and the space $\tilde{\mathcal{S}}'$ of infinitesimal hermitian C_π-deformations of g_0 through the natural projection $\pi_0 : S^5 \to \boldsymbol{CP}^2$. In § 3 we define another space $\mathcal{A}_1$ and study the property of its elements in detail. Finally, in § 4 we relate the space $\mathcal{A}_0$ with the space $\mathcal{A}_1$ and, using the result in § 3, complete the proof. In Appendix we give a proof of Tanaka's theorem stated above.

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Notations and preliminary remarks

1) In this paper we always assume the differetiability of class C^∞ unless otherwise stated.

2) Let M be a compact symmetric space of rank one with the standard C_π-metric g_0, and $\mathcal{S}(M)$ be the space of symmetric 2-forms on M. Then we define a subspace $\mathcal{S}'(M)$ of $\mathcal{S}(M)$ by $\mathcal{S}'(M)=\{f\in\mathcal{S}(M)|f$ satisfies condition a) in Introduction$\}$. In case M is one of the spaces $\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, and $\boldsymbol{CaP}^2$, $\widetilde{\mathcal{S}}'(M)$ denotes the space of infinitesimal semi-conformal C_π-deformations.

§ 1. Reduction of Theorem A to the case $M=\boldsymbol{CP}^2$

In this section we shall show that we have only to prove Theorem A in the case $M=\boldsymbol{CP}^2$. For this purpose we need the following theorem due to Michel [5].

THEOREM 1.1. (Michel). *Let M be one of the spaces $\boldsymbol{RP}^n$, $\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, and $\boldsymbol{CaP}^2$, and let $h\in\mathcal{S}'(M)$. Assume that for each projective line $\boldsymbol{KP}^1$ in $M=\boldsymbol{KP}^n$, there is a vector field X on $\boldsymbol{KP}^1$ such that $\mathcal{L}_X(\iota^*g_0)=\iota^*h$, ι being the inclusion $\boldsymbol{KP}^1\to M$. Then there exists a vector field Y on M satisfying $\mathcal{L}_Y g_0=h$.*

We moreover need the following lemmas, which are well known (see [4] for the former, and [7] for the latter).

LEMMA 1.2. *Let $S^n=\left\{x\in\boldsymbol{R}^{n+1}|\ |x|=\frac{1}{2}\right\}$ and let g_0 be the riemannian metric on S^n induced from the standard metric on $\boldsymbol{R}^{n+1}$. Let X be a conformal vector field on S^n:*

$$\mathcal{L}_X g_0 = h g_0\,,$$

h being a function on S^n. Then h is a linear function on S^n, i.e., the restriction of a linear function on $\boldsymbol{R}^{n+1}$ to S^n. Conversely, if h is a linear function on S^n, there is a conformal vector field X on S^n such that $\mathcal{L}_X g_0= h g_0$.

LEMMA 1.3. *Let M be one of the spaces $\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, and $\boldsymbol{CaP}^2$*

with the standard C_π-metric g_0. Then for each totally geodesic submanifold S^2 of M which is isometric to (S^2, g_0), there is a totally geodesic submanifold $\boldsymbol{CP}^2$ of M which is isometric to $(\boldsymbol{CP}^2, g_0)$ and contains the S^2.

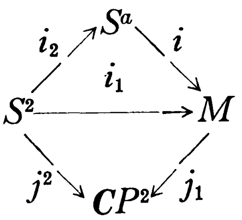
From the above facts we can obtain the following

PROPOSITION 1.4. *If Theorem* A *is true in the case $M=\boldsymbol{CP}^2$, then it is also true for any other case, i.e., $M=\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$ $(n\geqq 2)$, or $\boldsymbol{CaP}^2$.*

PROOF. Let M be as in Lemma 1.3. We put $a=2, 4$, or 8 accordingly as $M=\boldsymbol{CP}^n$, $\boldsymbol{HP}^n$, or $\boldsymbol{CaP}^2$ respectively. Let $f\in\tilde{\mathscr{S}}'(M)$ and fix a projective line S^a in M. From the very definition of $\tilde{\mathscr{S}}'(M)$ we can find a function h on S^a such that

$$i^*f=hi^*g_0\,,$$

where i being the inclusion $S^a\to M$. We now take a totally geodesic submanifold S^2 of S^a. Let i_1 and i_2 be the inclusions $S^2\to M$ and $S^2\to S^a$ respectively. By Lemma 1.3 there is a totally geodesic submanifold $\boldsymbol{CP}^2$ of M containing the S^2. Let j_1 and j_2 be the inclusions $\boldsymbol{CP}^2\to M$ and $S^2\to \boldsymbol{CP}^2$ respectively.



Since for any line $\boldsymbol{CP}^1$ in $\boldsymbol{CP}^2$ there is a line S^a in M which contains the $\boldsymbol{CP}^1$ (cf. [7]), we see $j_1^*f\in\tilde{\mathscr{S}}'(\boldsymbol{CP}^2)$. Hence there is a vector field Y on $\boldsymbol{CP}^2$ such that $\mathscr{L}_Y(j_1^*g_0)=j_1^*f$ by the assumption. We decompose Y on S^2 as follows: $Y=Z+W$, where Z is tangent to S^2 and W is normal to S^2. Then we can easily see that

$$\begin{aligned} i_1^*f &= j_2^*\left(\mathscr{L}_Y(j_1^*g_0)\right) \\ &= \mathscr{L}_Z(i_1^*g_0)\,. \end{aligned}$$

On the other hand, since $i^*f=hi^*g_0$, we obtain

$$i_1^*f=(i_2^*h)(i_1^*g_0)\,.$$

Hence by Lemma 1.2 we see that i_2^*h is a linear function on S^2. Since $S^2\to S^a$ is arbitrary, it follows that h is a linear function on S^a. Thus, again by Lemma 1.2, there is a vector field V on S^a such that $\mathscr{L}_V(i^*g_0)=hi^*g_0$. Therefore, by Theorem 1.1 there is a vector field X on M such that

$$\mathscr{L}_Xg_0=f\,. \qquad \text{q.e.d.}$$

§ 2. The spaces $\tilde{\mathscr{S}}'$ and $\mathscr{A}_0$

In this and subsequent sections we shall prove Theorem A′, which, combined with Proposition 1. 4, gives the proof of Theorem A.

Let S^5 be the unit sphere of $\boldsymbol{C}^3$, and $\pi_0: S^5 \to \boldsymbol{C}P^2$ be the natural projection. Let $\langle\ ,\ \rangle$ be the canonical hermitian inner product of $\boldsymbol{C}^3$:

$$\langle z, w\rangle = \sum_{i=1}^{3} z_i \bar{w}_i$$

where $z=(z_1, z_2, z_3)$, $w=(w_1, w_2, w_3) \in \boldsymbol{C}^3$. We define a submanifold T of $S^5 \times \boldsymbol{C}^3$ by

$$T = \left\{(z, w) \in S^5 \times \boldsymbol{C}^3 \mid \langle z, w\rangle = 0\right\},$$

and denote by π' the natural projection $T \to S^5$; $\pi'(z, w) = z$. Then we see that T is a complex vector bundle over S^5 with projection π' and that it can be naturally regarded as a subbundle of the tangent bundle TS^5 of S^5. Let π_1 be the restriction of π_{0*} to T. Then it is clear that $\pi_1: T \to T\boldsymbol{C}P^2$ is a homomorphism as complex vector bundles.

Now let $\tilde{\mathscr{S}}$ denote the space of hermitian (or semi-conformal) symmetric 2-forms on $\boldsymbol{C}P^2$, *i. e.*,

$$\tilde{\mathscr{S}} = \left\{f \in \mathscr{S}(\boldsymbol{C}P^2) \mid f(Iu, Iu) = f(u, u),\ u \in T\boldsymbol{C}P^2\right\},$$

where I denotes the complex structure on $\boldsymbol{C}P^2$. Let $f \in \tilde{\mathscr{S}}$. Then it is clear that $\pi_1{}^* f$ is a hermitian symmetric 2-form on each fiber $T_z (z \in S^5)$ of T, *i. e.*,

$$(\pi_1{}^* f)\big((z, iv), (z, iv)\big) = (\pi_1{}^* f)\big((z, v), (z, v)\big),\ (z, v) \in T.$$

Let $H(3)$ be the space of hermitian matrices of degree 3. We now define a vector space $\mathscr{A}$ as follows:

$H \in \mathscr{A}$ if and only if H is a $H(3)$-valued function on S^5 which satisfies the following conditions:

1) $H(z)z = 0$, where $z = (z_1, z_2, z_3) \in S^5$ should be considered as a column vector;

2) H is $U(1)$-invariant, *i. e.*, $H(\alpha z) = H(z)$ for each $\alpha \in \boldsymbol{C}$ with $|\alpha| = 1$.

PROPOSITION 2. 1. *There is a one-to-one correspondence* ($f \leftrightarrow H$) *between* $\tilde{\mathscr{S}}$ *and* $\mathscr{A}$, *where* f *and* H *are related by*

$$(*_1)\quad (\pi_1{}^* f)\big((z, v), (z, v)\big) = \big\langle H(z) v, v\big\rangle,\ (z, v) \in T.$$

PROOF. First suppose $f \in \tilde{\mathscr{S}}$. Fix $z \in S^5$. Since

$$(z, v)\to(\pi_1{}^*f)\Big((z, v),\ (z, v)\Big) = f\Big((\pi_1(z, v),\ \pi_1(z, v)\Big)$$

is a hermitian form on T_z, we see that there exists a unique element $H(z)\in H(3)$ satisfying $(*_1)$ and $H(z)z=0$. For each $\alpha\in U(1)$, we define a $H(3)$-valued function H^α by $H^\alpha(z)=H(\alpha z)$. Then we have $H^\alpha(z)z=0$ because $H(\alpha z)\alpha z=0$. Moreover, since $(z, v)\in T$ implies $(\alpha z, \alpha v)\in T$ and $\pi_1(\alpha z, \alpha v)=\pi_1(z, v)$, we have

$$\begin{aligned}\Big\langle H^\alpha(z)v, v\Big\rangle &= \Big\langle H(\alpha z)\alpha v, \alpha v\Big\rangle\\ &= f\Big(\pi_1(\alpha z, \alpha v),\ \pi_1(\alpha z, \alpha v)\Big)\\ &= f\Big(\pi_1(z, v),\ \pi_1(z, v)\Big).\end{aligned}$$

Therefore H^α also satisfies $(*_1)$ and $H^\alpha(z)z=0$. By the uniqueness it follows that $H=H^\alpha$, which implies that H is $U(1)$-invariant. Hence $H\in\mathscr{A}$. Conversely, suppose $H\in\mathscr{A}$. Then we have

$$\Big\langle H(\alpha z)\alpha v, \alpha v\Big\rangle = \Big\langle H(\alpha z)v, v\Big\rangle = \Big\langle H(z)v, v\Big\rangle$$

for any $(z, v)\in T$ and $\alpha\in U(1)$. Thus we see that there uniquely exists $f\in\widetilde{\mathscr{S}}$ satisfying $(*_1)$. q. e. d.

Let ζ_t be the geodesic flow on SCP^2, the unit tangent bundle of $\boldsymbol{C}P^2$. ζ_t is characterized by the following property:

Let $u\in SCP^2$ and $\gamma(t)$ be the geodesic on $\boldsymbol{C}P^2$ with $\dot\gamma(0)=u$. Then $\zeta_t u=\dot\gamma(t)$.

LEMMA 2.2. *Let $f\in\widetilde{\mathscr{S}}'=\widetilde{\mathscr{S}}'(\boldsymbol{C}P^2)$. Then we have*

$$f(\zeta_{\frac{\pi}{2}}u, \zeta_{\frac{\pi}{2}}u) = -f(u, u)$$

for any $u\in SCP^2$.

PROOF. Fix $u\in SCP^2$ and a projective line $\boldsymbol{C}P^1\subset\boldsymbol{C}P^2$ such that $u\in SCP^1$. Let ι: $\boldsymbol{C}P^1\to\boldsymbol{C}P^2$ be the inclusion. Since $f\in\widetilde{\mathscr{S}}'$, we see that there is a function h on $\boldsymbol{C}P^1$ such that

$$\iota^*f = h\iota^*g_0$$

and

$$\int_0^\pi h\Big(\gamma(t)\Big)dt = 0$$

for any (closed) geodesic $\gamma(t)$ in $\boldsymbol{C}P^1$ with $\|\dot\gamma(t)\|=1$. Let us now identify $\boldsymbol{C}P^1$ with the sphere $S^2\subset\boldsymbol{R}^3$ of radius $\frac{1}{2}$ as riemannian manifolds and let

τ be the antipodal map on S^2. Then it follows that h is an odd function, *i. e.*,

$$h\circ\tau=-h$$

(see [1] p. 123). Thus we have

$$\begin{aligned} f(\zeta_{\frac{\pi}{2}}u, \zeta_{\frac{\pi}{2}}u) &= h\left(\pi(\zeta_{\frac{\pi}{2}}u)\right)=h\left(\tau\left(\pi(u)\right)\right) \\ &= -h\left(\pi(u)\right)=-f(u, u)\,. \end{aligned}$$

q. e. d.

We define a submanifold S of T by

$$S=\left\{(z, w)\in \boldsymbol{C}^3\times \boldsymbol{C}^3 \,\middle|\, |z|=|w|=1,\ \langle z, w\rangle=0\right\},$$

which may be characterized as the unit sphere bundle of the vector bundle T over S^5. We also define a subspace $\mathscr{A}_0$ of $\mathscr{A}$ by

$$\mathscr{A}_0=\left\{H\in\mathscr{A} \,\middle|\, \left\langle H(z)v, v\right\rangle=-\left\langle H(v)z, z\right\rangle \text{ for any } (z, v)\in S\right\}.$$

Then we have

PROPOSITION 2.3. *The assignment $f\rightarrow H$ given in Proposition* 2.1 *also gives a one-to-one correspondence between $\widetilde{\mathscr{S}}'$ and $\mathscr{A}_0$.*

PROOF. Let $f\in\widetilde{\mathscr{S}}'$ and let H be the corresponding element of $\mathscr{A}$. Take any $(z, v)\in S$. Then we easily see that

$$\zeta_t\pi_1(z, v)=\pi_1(\cos t\cdot z+\sin t\cdot v,\ -\sin t\cdot z+\cos t\cdot v)\,,$$

especially,

$$\zeta_{\frac{\pi}{2}}\pi_1(z, v)=\pi_1(v, -z)\,.$$

Since $f(\zeta_{\frac{\pi}{2}}u, \zeta_{\frac{\pi}{2}}u)=-f(u, u)$ for any $u\in SCP^2$ and $\|\pi_1(z, v)\|=\|\pi_1(v, -z)\|=1$, we have

$$\begin{aligned} f\left(\pi_1(v, -z),\ \pi_1(v, -z)\right) &= f\left(\zeta_{\frac{\pi}{2}}\pi_1(z, v),\ \zeta_{\frac{\pi}{2}}\pi_1(z, v)\right) \\ &= -f\left(\pi_1(z, v),\ \pi_1(z, v)\right). \end{aligned}$$

Therefore

$$\left\langle H(v)z, z\right\rangle=-\left\langle H(z)v, v\right\rangle,$$

which implies $H\in\mathscr{A}_0$. Conversely, let $H\in\mathscr{A}_0$ and f be the corresponding element of $\widetilde{\mathscr{S}}$. For any $(z, v)\in S$, we have

$$
\begin{aligned}
f\left(\zeta_{\frac{\pi}{2}}\pi_1(z, v),\ \zeta_{\frac{\pi}{2}}\pi_1(z, v)\right) &= f\left(\pi_1(v, -z),\ \pi_1(v, -z)\right) \\
&= \left\langle H(v) z, z\right\rangle = -\left\langle H(z) v, v\right\rangle \\
&= -f\left(\pi_1(z, v),\ \pi_1(z, v)\right).
\end{aligned}
$$

Since tha map $\pi_1: S \to SCP^2$ is surjective, it follows that $f(\zeta_{\frac{\pi}{2}}u, \zeta_{\frac{\pi}{2}}u) = -f(u, u)$ for any $u \in SCP^2$. Thne we have

$$
\int_0^{\pi} f(\zeta_t u, \zeta_t u)\, dt = 0, \quad u \in SCP^2,
$$

and hence $f \in \widetilde{\mathscr{S}}'$. q. e. d.

§ 3. The space $\mathscr{A}_1$

First of all we define a subspace $\mathscr{A}_1$ of $\mathscr{A}$ as follows:

$H \in \mathscr{A}$ belongs to $\mathscr{A}_1$ if and only if for each $(z, v) \in S$, there exist real constants a, b, c, d such that

$$
\begin{aligned}
&\left\langle H(\alpha z + \beta v)(-\bar{\beta} z + \bar{\alpha} v),\ -\bar{\beta} z + \bar{\alpha} v\right\rangle \\
&\qquad = a|\alpha|^2 + b|\beta|^2 + c \operatorname{Re} \bar{\alpha}\beta + d \operatorname{Im} \bar{\alpha}\beta
\end{aligned}
$$

for any $(\alpha, \beta) \in S^3$, where S^3 stands for the unit sphere in C^2, *i.e.*, $S^3 = \{(\alpha, \beta) \in C^2 \mid |\alpha|^2 + |\beta|^2 = 1\}$.

Remark that the equality

$$
\left\langle \alpha z + \beta v,\ -\bar{\beta} z + \bar{\alpha} v\right\rangle = 0
$$

always holds, provided $(z, v) \in S$.

We now study the property of elements of $\mathscr{A}_1$ in detail. Consider the canonical basis $\{e_1, e_2, e_3\}$ of C^3:

$$
e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).
$$

Fix $h \in \mathscr{A}_1$ and define a function h_{11} on S^5 by

$$
h_{11}(z) = \left\langle H(z) e_1, e_1\right\rangle.
$$

Furthermore define functions a, b, c, d on

$$
S(e_1^{\perp}) = \left\{ v \in Ce_2 \oplus Ce_3 \mid |v| = 1 \right\}
$$

by

$$\left\langle H(\alpha e_1+\beta v)(-\bar{\beta}e_1+\bar{\alpha}v),\ -\bar{\beta}e_1+\bar{\alpha}v\right\rangle$$
$$=a(v)|\alpha|^2+b(v)|\beta|^2+c(v)\operatorname{Re}\bar{\alpha}\beta+d(v)\operatorname{In}\bar{\alpha}\beta\,,$$

where $(\alpha, \beta)\in S^3$. Then we have

LEMMA 3.1. *For any $v\in S(e_1^{\perp})$ and any $(\alpha, \beta)\in S^3$, we have*

$$h_{11}(\alpha e_1+\beta v)=|\beta|^2\{a(v)|\alpha|^2+b(v)|\beta|^2+c(v)\operatorname{Re}\bar{\alpha}\beta+d(v)\operatorname{Im}\bar{\alpha}\beta\}\,.$$

PROOF We have

$$\left\langle H(\alpha e_1+\beta v)(-\bar{\beta}e_1+\bar{\alpha}v),\ -\bar{\beta}e_1+\bar{\alpha}v\right\rangle$$
$$=a(v)|\alpha|^2+b(v)|\beta|^2+c(v)\operatorname{Re}\bar{\alpha}\beta+d(v)\operatorname{Im}\bar{\alpha}\beta$$

and

$$H(\alpha e_1+\beta v)(\alpha e_1+\beta v)=0\,,$$

from which follows easily the lemma. q. e. d.

Form the above lemma we know that the equality

$$(*_2)\quad h_{11}(\alpha e_1+\beta v)(|\alpha|^2+|\beta|^2)$$
$$=|\beta|^2\{a(v)|\alpha|^2+b(v)|\beta|^2+c(v)\operatorname{Re}\bar{\alpha}\beta+d(v)\operatorname{Im}\bar{\alpha}\beta\}$$

holds for any $(\alpha, \beta)\in S^3$. This being said, we extend h to a homogeneous function of degree 2 on $\boldsymbol{C}^3\backslash\{0\}$. Then we see that $(*_2)$ holds for any $(\alpha, \beta)\in \boldsymbol{C}^2\backslash\{0\}$, because both sides of $(*_2)$ are homogeneous functions of degree 4 in (α, β).

Let (z_1, z_2, z_3) be the natural complex coordinates of $\boldsymbol{C}^3$. We then put $z_j=x_j+\sqrt{-1}\,y_j$ $(j=1, 2, 3)$ and take $(x_1, y_1, x_2, y_2, x_3, y_3)$ as real coordinates of $\boldsymbol{C}^3$. We also put

$$\alpha=r_1+\sqrt{-1}\,s_1,\quad \beta=r_2+\sqrt{-1}\,s_2$$

for $(\alpha, \beta)\in\boldsymbol{C}^2$ and take (r_1, s_1, r_2, s_2) as real coordinates of $\boldsymbol{C}^2$.

LEMMA 3.2. *Let $H\in\mathscr{A}_1$ and $v\in S(e_1^{\perp})$. Putting*

$$D_v=\operatorname{Re}v_2\frac{\partial}{\partial x_2}+\operatorname{Im}v_2\frac{\partial}{\partial y_2}+\operatorname{Re}v_3\frac{\partial}{\partial x_3}+\operatorname{Im}v_3\frac{\partial}{\partial y_3}\,,$$

we have

$$h_{11}(v)=\frac{1}{2}\left((D_v)^2h_{11}\right)(e_1)+\frac{1}{24}\left((D_v)^4h_{11}\right)(e_1)\,.$$

PROOF. We differentiate both sides of

$$(*_2)\quad h_{11}(\alpha e_1+\beta v)(|\alpha|^2+|\beta|^2) \\ =|\beta|^2\left\{a(v)|\alpha|^2+b(v)|\beta|^2+c(v)\mathrm{Re}\,\bar{\alpha}\beta+d(v)\mathrm{Im}\,\bar{\alpha}\beta\right\}$$

4 times with respect to the variable r_2. Since the map $(\alpha, \beta)\to\alpha e_1+\beta v$ $(\boldsymbol{C}^2\setminus\{0\}\to\boldsymbol{C}^3\setminus\{0\})$ transforms $\frac{\partial}{\partial r_2}$ to D_v, we obtain the following formula:

$$\left((D_v)^4 h_{11}\right)(\alpha e_1+\beta v)\cdot(|\alpha|^2+|\beta|^2)+8\left((D_v)^3 h_{11}\right)(\alpha e_1+\beta v)r_2 \\ +12\left((D_v)^2 h_{11}\right)(\alpha e_1+\beta v)=24\,b(v)\,.$$

Therefore putting $\alpha=1$ and $\beta=0$, we have

$$b(v)=\frac{1}{2}\left((D_v)^2 h_{11}\right)(e_1)+\frac{1}{24}\left((D_v)^4 h_{11}\right)(e_1)\,.$$

On the other hand, putting $\alpha=0$ and $\beta=1$ in $(*_2)$, we have $h_{11}(v)=b(v)$.
q. e. d.

Here we notice that the right-hand side of the equality in Lemma 3.2 is a sum of homogeneous polynomials in v of degrees 2 and 4. By exchanging the basis of $\boldsymbol{C}^3$ we thereby obtain the following proposition:

PROPOSITION 3.3. *Let $H\in\mathscr{A}_1$ and fix a unitary basis $\{v, w, z\}$ of $\boldsymbol{C}^3$. Then there exist $U(1)$-invariant homogeneous polynomials in $(\alpha, \beta)\in\boldsymbol{C}^2$ of degrees 2 and 4, f_2 and f_4, such that $f_2(\alpha, \beta)\in\boldsymbol{R}$, $f_4(\alpha, \beta)\in\boldsymbol{R}$ and*

$$\left\langle H(\alpha z+\beta w)v, v\right\rangle=f_2(\alpha, \beta)+f_4(\alpha, \beta)$$

for any $(\alpha, \beta)\in S^3$.

PROOF. The only part which is not clear is the $U(1)$-invariance of f_2 and f_4. To see this we first take f_2 and f_4 which are not necessarily $U(1)$-invariant. Since H is $U(1)$-invariant, it follows that

$$f_2(e^{it}\alpha, e^{it}\beta)+f_4(e^{it}\alpha, e^{it}\beta)=f_2(\alpha, \beta)+f_4(\alpha, \beta)$$

for any $t\in\boldsymbol{R}$. If we put

$$f_j'(\alpha, \beta)=\frac{1}{2\pi}\int_0^{2\pi}f_j(e^{it}\alpha, e^{it}\beta)\,dt \qquad (j=2, 4)\,,$$

we easily see that f_2' and f_4' are $U(1)$-invariant and satisfy

$$f_2'(\alpha, \beta)+f_4'(\alpha, \beta)=f_2(\alpha, \beta)+f_4(\alpha, \beta)\,.$$

Therefore we have seen that f_2' and f_4' satisfy all the conditions in the proposition. q. e. d.

LEMMA 3.4. *Let f_2 and f_4 be as in Proposition* 3.3. *Then $f_2(\alpha,\beta)-$ $-f_2(-\bar\beta,\bar\alpha)$ is a linear combination of $|\alpha|^2-|\beta|^2$,* Re$\bar\alpha\beta$, Im$\bar\alpha\beta$ *with real coeffecients. And $f_4(\alpha,\beta)-f_4(-\bar\beta,\bar\alpha)$ is of the form*:

$(|\alpha|^2+|\beta|^2)$ {*a linear combination of $|\alpha|^2-|\beta|^2$,* Re$\bar\alpha\beta$, Im$\bar\alpha\beta$ *with real coeffecients*}.

PROOF. f_2, being a $U(1)$-invariant quadratic form, is a hermitian form. Therefore there are real constants a, b, c, d such that

$$f_2(\alpha,\beta)=a|\alpha|^2+b|\beta|^2+c\mathrm{Re}\,\bar\alpha\beta+d\mathrm{Im}\,\bar\alpha\beta\,.$$

Then we have

$$f_2(-\bar\beta,\bar\alpha)=a|\beta|^2+b|\alpha|^2-c\mathrm{Re}\,\bar\alpha\beta-d\mathrm{Im}\,\bar\alpha\beta\,,$$

and hence

$$f_2(\alpha,\beta)-f_2(-\bar\beta,\bar\alpha)=(a-b)(|\alpha|^2-|\beta|^2)+2c\mathrm{Re}\,\bar\alpha\beta+2d\mathrm{Im}\,\bar\alpha\bar\beta\,.$$

For f_4 we first express it as

$$f_4(\alpha,\beta)=\sum a_{pqrs}\alpha^p\bar\alpha^q\beta^r\bar\beta^s,\quad a_{pqrs}\in\boldsymbol{C}\,,$$

where the sum is taken over all 4-tuples of non-negative integers (p,q,r,s) with $p+q+r+s=4$. We have

$$\frac{1}{2\pi}\int_0^{2\pi}f_4(e^{it}\alpha,e^{it}\beta)dt=f_4(\alpha,\beta)$$

and

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}(e^{it}\alpha)^p(\overline{e^{it}\alpha})^q(e^{it}\beta)^r(\overline{e^{it}\beta})^s\,dt\\&=\frac{1}{2\pi}\int_0^{2\pi}e^{it(p-q+r-s)}dt\cdot\alpha^p\bar\alpha^q\beta^r\bar\beta^s\\&=\begin{cases}\alpha^p\bar\alpha^q\beta^r\bar\beta^s, & p+r=q+s\ (=2)\\ 0, & p+r\neq q+s\,.\end{cases}\end{aligned}$$

Hence it follows that

$$f_4(\alpha,\beta)=\sum_{0\leqq p,\,q\leqq 2}b_{p,q}\alpha^p\bar\alpha^q\beta^{2-p}\bar\beta^{2-q}\,,$$

where $b_{p,q}=a_{p,q,2-p,2-q}$. Thus we have

$$\begin{aligned}&f_4(\alpha,\beta)-f_4(-\bar\beta,\bar\alpha)\\&=\sum_{0\leqq p,\,q\leqq 2}b_{p,q}\left\{\alpha^p\bar\alpha^q\beta^{2-p}\bar\beta^{2-q}-(-1)^{p+q}\bar\beta^p\beta^q\bar\alpha^{2-p}\alpha^{2-q}\right\}.\end{aligned}$$

On the other hand, we can easily see that

$$\alpha^p\bar{\alpha}^q\beta^{2-p}\bar{\beta}^{2-q}-(-1)^{p+q}\bar{\beta}^p\beta^q\bar{\alpha}^{2-p}\alpha^{2-q}$$

is 0 or of one of the following forms for each (p, q) with $0\leqq p,\ q\leqq 2$:

$$\pm(|\alpha|^2+|\beta|^2)(|\alpha|^2-|\beta|^2),\ \pm(|\alpha|^2+|\beta|^2)\bar{\alpha}\beta,\ \pm(|\alpha|^2+|\beta|^2)\alpha\bar{\beta}.$$

Consequently we have seen that there are constants a, b, $c\in \boldsymbol{C}$ such that

$$\begin{aligned} &f_4(\alpha, \beta)-f_4(-\bar{\beta}, \bar{\alpha}) \\ &\quad=(|\alpha|^2+|\beta|^2)\{a(|\alpha|^2-|\beta|^2)+b\mathrm{Re}\bar{\alpha}\beta+c\mathrm{Im}\bar{\alpha}\beta\}\,. \end{aligned}$$

Since $f_4(\alpha, \beta)\in \boldsymbol{R}$, it is clear that a, b, c are real numbers. q. e. d.

By Proposition 3. 3 and Lemma 3. 4, we have proved the following

PROPOSITION 3. 5. *Let $H\in\mathscr{A}_1$. Fix a unitary basis $\{v, w, z\}$ of $\boldsymbol{C}^3$. Then there are real constants a, b, c such that*

$$\begin{aligned} &\langle H(\alpha z+\beta w)v, v\rangle-\langle H(-\bar{\beta}z+\bar{\alpha}w)v, v\rangle \\ &\quad=a(|\alpha|^2-|\beta|^2)+b\mathrm{Re}\,\bar{\alpha}\beta+c\mathrm{Im}\,\bar{\alpha}\beta \end{aligned}$$

for any $(\alpha, \beta)\in S^3$.

§ 4. Proof of Theorem A′

Let $f\in\widetilde{\mathscr{S}}'(\boldsymbol{C}P^2)$. By Proposition 2. 3 there uniquely exists $H'\in\mathscr{A}_0$ such that

$$\langle H'(z)w, w\rangle=f\big(\pi_1(z, w),\ \pi_1(z, w)\big),\quad (z, w)\in T\,.$$

LEMMA 4. 1. *There is $H\in\mathscr{A}_1$ such that*

$$\langle H'(z)w, w\rangle=\langle H(z)v, v\rangle-\langle H(w)v, v\rangle$$

for any unitary basis $\{v, w, z\}$ of $\boldsymbol{C}^3$.

PROOF. Fix $z\in S^5$ and a unitary basis $\{e_1, e_2\}$ of T_z. Then we define a $\boldsymbol{R}$-linear transformation σ of T_z by

$$\sigma(\alpha e_1+\beta e_2)=-\bar{\beta}e_1+\bar{\alpha}e_2,\quad (\alpha, \beta)\in\boldsymbol{C}^2\,.$$

Since the function

$$w\rightarrow\frac{1}{2}\langle H'(z)\sigma(w),\ \sigma(w)\rangle,\ w\in T_z$$

is a hermitian form on T_z, we see that there uniquely exists a hermitian matrix $H(z)$ of degree 3 such that $H(z)z=0$ and

$$\langle H(z)w, w\rangle=\frac{1}{2}\langle H'(z)\sigma(w),\ \sigma(w)\rangle,\quad w\in T_z\,.$$

Since $\{z, w, \sigma(w)\}$ is a unitary basis of C^3, it follows that

$$\left\langle H(z)w, w\right\rangle = \frac{1}{2}\left\langle H'(z)v, v\right\rangle$$

for any unitary basis $\{z, w, v\}$ of C^3. Clearly the map $z \to H(z)$ is $U(1)$-invariant, and hence $H \in \mathscr{A}$. Furthermore we have

$$\left\langle H(z)v, v\right\rangle = \frac{1}{2}\left\langle H'(z)w, w\right\rangle$$

and

$$\left\langle H(w)v, v\right\rangle = \frac{1}{2}\left\langle H'(w)z, z\right\rangle = -\frac{1}{2}\left\langle H'(z)w, w\right\rangle,$$

whence

$$\left\langle H(z)v, v\right\rangle - \left\langle H(w)v, v\right\rangle = \left\langle H'(z)w, w\right\rangle$$

for any unitary basis $\{z, v, w\}$ of C^3. On the other hand, since $\{\alpha z + \beta w, -\bar{\beta}z + \bar{\alpha}w, v\}$ is a unitary basis for any $(\alpha, \beta) \in S^3$, it follows that

$$\begin{aligned}
&\left\langle H(\alpha z+\beta w)(-\bar{\beta}z+\bar{\alpha}w),\ -\bar{\beta}z+\bar{\alpha}w\right\rangle \\
&= \frac{1}{2}\left\langle H'(\alpha z+\beta w)v, v\right\rangle \\
&= -\frac{1}{2}\left\langle H'(v)(\alpha z+\beta w),\ \alpha z+\beta w\right\rangle.
\end{aligned}$$

Clearly the last term can be expressed as a linear combination of $|\alpha|^2$, $|\beta|^2$, $\mathrm{Re}\,\bar{\alpha}\beta$, $\mathrm{Im}\,\bar{\alpha}\beta$ with real coefficients. Thus $H \in \mathscr{A}_1$. q. e. d.

We continue the proof of Theorem A′. We fix a unitary basis $\{v, w, z\}$ of C^3. Using the above lemma and Proposition 3.5 we have

$$\begin{aligned}
&f\left(\pi_1(\alpha z+\beta w,\ -\bar{\beta}z+\bar{\alpha}w),\ \pi_1(\alpha z+\beta w,\ -\bar{\beta}z+\bar{\alpha}w)\right) \\
&= \left\langle H(\alpha z+\beta w)v,\ v\right\rangle - \left\langle H(-\bar{\beta}z+\bar{\alpha}w)v, v\right\rangle \\
&= a(|\alpha|^2 - |\beta|^2) + b\,\mathrm{Re}\,\bar{\alpha}\beta + c\,\mathrm{Im}\,\bar{\alpha}\beta
\end{aligned}$$

for any $(\alpha, \beta) \in S^3$, where $H \in \mathscr{A}_1$, and a, b, c are real constants. On the other hand, the set

$$\left\{\pi_0(\alpha z+\beta w) \mid (\alpha, \beta) \in S^3\right\}$$

represents a projective line CP^1 in CP^2, and

$$\pi_1(\alpha z+\beta w,\ -\bar{\beta}z+\bar{\alpha}w) \in TCP^1 \subset TCP^2.$$

Let ι denotes the inclusion $\boldsymbol{C}P^1 \to \boldsymbol{C}P^2$. Then, expressing f as

$$\iota^* f = h\iota^* g_0, \quad h \in C^\infty(\boldsymbol{C}P^1),$$

we have

$$f\left(\pi_1(\alpha z+\beta w, -\bar{\beta}z+\bar{\alpha}w), \pi_1(\alpha z+\beta w, -\bar{\beta}z+\bar{\alpha}w)\right)$$
$$= h\left(\pi_0(\alpha z+\beta w)\right)$$

and therefore

$$(*_3) \quad h\left(\pi_1(\alpha z+\beta w)\right) = a(|\alpha|^2-|\beta|^2)+b\mathrm{Re}\bar{\alpha}\beta+c\mathrm{Im}\bar{\alpha}\beta\,.$$

Remarking the fact that $(\alpha, \beta) \mapsto \pi_0(\alpha z+\beta w)$ gives the natural projection $\boldsymbol{C}^2 \supset S^3 \to \boldsymbol{C}P^1$, we can easily see that the map

$$\pi_0(\alpha z+\beta w) \mapsto \left(\frac{1}{2}(|\alpha|^2-|\beta|^2), \mathrm{Re}\,\bar{\alpha}\beta, \mathrm{Im}\,\bar{\alpha}\beta\right)$$

gives an isometry $\boldsymbol{C}P^1 \to S^2 = \left\{x \in \boldsymbol{R}^3 \mid |x| = \frac{1}{2}\right\}$. Under this identification we see by $(*_3)$ that h is a linear function on $\boldsymbol{C}P^1 = S^2$. Therefore by Lemma 1.2, it follows that there is a vector field Y on $\boldsymbol{C}P^1$ such that $\mathscr{L}_Y(\iota^* g_0) = \iota^* f$. If we vary $\{v, w, z\}$ over all unitary basis of $\boldsymbol{C}^3$, the set

$$\left\{\pi_0(\alpha z+\beta w) \mid (\alpha, \beta) \in S^3\right\}$$

can represent any projective line in $\boldsymbol{C}P^2$. Thus by Theorem 1.1 we have shown that there is a vector field X on $\boldsymbol{C}P^2$ such that $\mathscr{L}_X g_0 = f$.

q. e. d.

Appendix

In this appendix we give a proof of the following theorem due to Tanaka.

THEOREM (Tanaka). *Let $M = \boldsymbol{C}P^n$ $(n \geqq 2)$ and let $(g_t)_{t \in I}$ be a kählerian C_π-deformation of g_0, where I is an open interval containing* 0. *If g_t depends real analytically on the parameter t, then there is a one-parameter family $(\phi_t)_{t \in I}$ of holomorphic transformations of $\boldsymbol{C}P^n$ defined on the same interval I, such that $\phi_0 =$ identity and $g_t = \phi_t^* g_0$.*

PROOF. Let X be a vector field on $\boldsymbol{C}P^n$ and ϕ a 1-form dual to X. Let $D\phi$ be a symmetric 2-form defined by

$$(D\phi)(Y, Z) = (\nabla_Y \phi)(Z) + (\nabla_Z \phi)(Y)\,.$$

Then we can easily see that $\mathscr{L}_X g_0 = D\phi$. Let $L\phi$ be the anti-hermitian part of $D\phi$, *i.e.*,

$$(L\phi)(Y, Z) = \frac{1}{2}\left\{(D\phi)(Y, Z) - (D\phi)(IY, IZ)\right\}.$$

Then by Tanaka [6], we see that $L\phi = 0$ if and only if

$$\Delta\phi + d\delta\phi - \left((d\delta(\phi I)\right) I - 8(n+1)\phi = 0,$$

where ϕI is a 1-form defined by $(\phi I)(Y) = \phi(IY)$, and δ denotes the adjoint operator of d, and Δ denotes the usual Laplace operator. We remark that $\mathscr{L}_X g_0$ is hermitian if and only if $L\phi = 0$. Set

$$B = \{\phi | L\phi = 0\}$$
$$B_1 = \{df | f \text{ is a function on } CP^n,\ \Delta f = 4(n+1)f\}$$
$$B_2 = \{(df)I | f \text{ is a function on } CP^n,\ \Delta f = 4(n+1)f\}$$
$$B_3 = \{\phi | \delta\phi = \delta(\phi I) = 0,\ \Delta\phi = 8(n+1)\phi\}.$$

Remarking the fact that $\delta((df)I) = 0$ and $\Delta(\phi I) = (\Delta\phi)I$, we can easily see that

$$B = B_1 \oplus B_2 \oplus B_3 \text{ (orthogonal decomposition)}.$$

Moreover it is well known that X is an infinitesimal isometry if and only if $\phi \in B_2$ and that X is an infinitesimal holomorphic transformation if and only if $\phi \in B_1 + B_2$.

For any $\phi \in B$, we define a 2-form $P\phi$ by

$$(P\phi)(Y, Z) = (D\phi)(Y, IZ).$$

LEMMA 1. *If $\phi \in B_3$ and $dP\phi = 0$, then we have $\phi = 0$.*

PROOF. By calculating δdP, we have

$$0 = (\delta dP\phi)(Y, Z) = 4(n+1)\left\{(\nabla\phi)(IZ, Y) - (\nabla\phi)(IY, Z)\right\}.$$

From this we easily have $D\phi = 0$. Since $\phi \in B_3$, it follows that $\phi = 0$.

q. e. d.

By Lemma 1 we see that $\mathscr{L}_X g_0$ is hermitian and $dP\phi = 0$ if and only if X is an infinitesimal holomorphic transformation.

LEMMA 2. *There is a series of infinitesimal holomorphic transformations $X^{(i)}$ $(i \geqq 0)$ such that for any integer $l \geqq 0$ we have*

$$(*)_l \quad \mathscr{L}_{X_t} g_t \equiv \frac{\partial g_t}{\partial t} \pmod{t^{l+1}},$$

where $X_t=\sum_{i=0}^{l} t^i X^{(i)}$.

PROOF. We shall define $X^{(i)}$ inductively. Let Ω_t be the 2-form associated with g_t, *i. e.*, $\Omega_t(Y, Z)=g_t(Y, IZ)$. Since g_t is kählerian, we have $d\Omega_t=0$. We describe g_t as

$$(*)_0 \quad g_t \equiv g_0+th \pmod{t^2}.$$

Then we see that $h\in\widetilde{\mathscr{S}}'$. Hence by Theorem A there is a vector field X such that $\mathscr{L}_X g_0=h$. Let Θ be the 2-form defined by

$$\Theta(Y, Z)=h(Y, IZ).$$

Then we have

$$\Omega_t \equiv \Omega_0+t\Theta \pmod{t^2}$$

and $d\Theta=0$. Hence by Lemma 1 we see that X is an infinitesimal holomorphic transformation. We put $X^{(0)}=X$. Now we assume that there are infinitesimal holomorphic transformations $X^{(0)}, \cdots, X^{(l)}$ such that $X_t=\sum_{i=0}^{l} t^i X^{(i)}$ satisfies

$$(*)_l \quad \mathscr{L}_{X_t} g_t \equiv \frac{\partial g_t}{\partial t} \pmod{t^{l+1}}.$$

Let Φ_t be the one-parameter family of holomorphic transformations generated by X_t, *i. e.*, Φ_0=identity and

$$(\Phi)_t^{-1}*\left\{\frac{\partial}{\partial t}\Phi_t(x)\right\}=(X_t)_x,\ x\in CP^n.$$

Putting $\bar{g}_t=(\Phi_t^{-1})^* g_t$, we have

$$\mathscr{L}_{X_t} g_t+\Phi_t{}^*\frac{\partial \bar{g}_t}{\partial t}=\frac{\partial g_t}{\partial t}.$$

Thus we obtain

$$\Phi_t{}^*\frac{\partial \bar{g}_t}{\partial t}\equiv 0 \pmod{t^{l+1}},$$

and hence

$$\frac{\partial \bar{g}_t}{\partial t}\equiv 0 \pmod{t^{l+1}}.$$

Therefore we have

$$\bar{g}_t\equiv g_0+\frac{1}{l+2}t^{l+2}\bar{h} \pmod{t^{l+3}},$$

$\bar{h}$ being a symmetric 2-form. Since $\bar{g}_t$ is kählerian and C_π, it follows easily that $\bar{h}\in\widetilde{\mathscr{S}}'$. Hence by Theorem A there is a vector field $\bar{X}$ such that $\mathscr{L}_{\bar{X}}g_0=\bar{h}$. Let $\bar{\Omega}_t$ and $\bar{\Theta}$ be the 2-forms defined respectively by

$$\bar{\Omega}_t(Y, Z)=\bar{g}_t(Y, IZ),\ \bar{\Theta}(Y, Z)=\bar{h}(Y, IZ).$$

Then we have

$$\bar{\Omega}_t\equiv\Omega_0+\frac{1}{l+2}t^{l+2}\bar{\Theta}\quad(\mathrm{mod}\ t^{l+3})$$

and $d\bar{\Theta}=0$. Thus by Lemma 1 we see that $\bar{X}$ is an infinitesimal holomorphic transformation. Therefore, putting

$$Y_t=X_t+t^{l+1}\bar{X},$$

we have

$$\begin{aligned}\mathscr{L}_{Y_t}g_t&=\mathscr{L}_{X_t}g_t+t^{l+1}\mathscr{L}_{\bar{X}}g_t\\&=\frac{\partial g_t}{\partial t}-\Phi_t{}^*\frac{\partial\bar{g}_t}{\partial t}+t^{l+1}\mathscr{L}_{\bar{X}}g_t\\&\equiv\frac{\partial g_t}{\partial t}-t^{l+1}\bar{h}+t^{l+1}\mathscr{L}_{\bar{X}}g_t\quad(\mathrm{mod}\ t^{l+2})\\&\equiv\frac{\partial g_t}{\partial t}\quad(\mathrm{mod}\ t^{l+2})\end{aligned}$$

Thus, putting $X^{(l+1)}=\bar{X}$, we see that $(*)_{l+1}$ holds. q. e. d.

LEMMA 3. *The holomorphic sectional curvature of* (CP^n, g_t) *is constant, and this constant does not depend on the parameter t.*

PROOF. Let ∇_t, R_t, c_t be the connection, the curvature and the holomorphic sectional curvature of g_t respectively. Here we note that c_t is a function on the grassmann bundle of 1-dimensional complex contact elements to CP^n. Let $X_t=\sum_{i=0}^{l}t^iX^{(i)}$ be as in Lemma 2 and Φ_t be the one-parameter family of holomorphic transformations generated by X_t. Put $\bar{g}_t=(\Phi^{-1})^*g_t$. Let $\bar{\nabla}_t$, $\bar{R}_t$, $\bar{c}_t$ be the connection, the curvature, and the holomorphic sectional curvature of $\bar{g}_t$ respectively. Then we have

$$\bar{g}_t\equiv g_0\quad(\mathrm{mod}\ t^{l+2})$$

and

$$\begin{aligned}\bar{\nabla}_t&\equiv\nabla_0\quad(\mathrm{mod}\ t^{l+2})\\\bar{R}_t&\equiv R_0\quad(\mathrm{mod}\ t^{l+2})\\\bar{c}_t&\equiv c_0\quad(\mathrm{mod}\ t^{l+2}).\end{aligned}$$

Since c_0 is constant, we have

$$d\bar{c}_t \equiv 0 \pmod{t^{l+2}}.$$

Since $c_t = \Phi_t{}^* \bar{c}_t$, it follows $dc_t = \Phi_t{}^* d\bar{c}_t$, and hence

$$dc_t \equiv 0 \pmod{t^{l+2}}.$$

Since this is true for any $l \geqq 0$, and since c_t is analytic in t, we have $dc_t = 0$. Thus c_t is constant. On the other hand, from $\Phi_t{}^* \bar{c}_t = c_t$ we have

$$\mathscr{L}_{X_t} c_t + \Phi_t{}^* \frac{\partial \bar{c}_t}{\partial t} = \frac{\partial c_t}{\partial t}.$$

Since $\mathscr{L}_{X_t} c_t = 0$ and $\frac{\partial \bar{c}_t}{\partial t} \equiv 0 \pmod{t^{l+1}}$, we obtain

$$\frac{\partial c_t}{\partial t} \equiv 0 \pmod{t^{l+1}}.$$

Since this is true for any $l \geqq 0$, and since c_t is analytic in the parameter t, it follows that $\frac{\partial c_t}{\partial t} = 0$. Therefore we have $c_t = c_0 = \text{constant}$. q. e. d.

By Lemma 3 and [2] II Theorem 7. 9, we see that each $(\boldsymbol{CP}^n, g_t)$ is holomorphically isometric to $(\boldsymbol{CP}^n, g_0)$. The construction of ψ_t is then trivial.

q. e. d. of Theorem.

Finally we make a remark. If the infinitesimal Blaschke conjecture is true for a projective space M, then we have the following:

Let $(g_t)_{t \in I}$ be a one-parameter family of C_π-metrics on M such that g_0 is the standard C_π-metric. If g_t depends real analytically on the parameter t. then there is a one-parameter family $(\psi_t)_{t \in I}$ of diffeomorphisms of M such that $\psi_0 = \text{identity}$ and $\psi_t{}^* g_0 = g_t$.

The proof is completely analogous to the above. We use $\nabla_t R_t$ instead of c_t, and prove $\nabla_t R_t = 0$, which implies that (M, g_t) is a symmetric space.

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Department of Mathematics
Hokkaido University