Surfaces with vanishing normal curvature

Dedicate to Professor Yoshie Katsurada on her 60th birthday

By Yoshihiko TAZAWA

§1. Introduction.

The normal curvature of a submanifold is defined by the square of the length of the curvature form of the connection in the normal bundle (cf [6]). The minimal index (M-index) at a point of a submanifold is defined by the dimension of the linear space of all second fundamental forms with vanishing trace (cf [8]). In this paper we prove the following proposition:

PROPOSITION. *Let M be a compact connected surface with positive Gaussian curvature G isometrically immersed in a $(2+p)$-dimensional space form N of curvature c. If M is non-minimal and the mean curvature vector H is parallel in the normal bundle and the normal curvature vanishes identically, then M is a totally umbilical surface with M-index* 0. *Especially if N is euclidean then M is a sphere in a* 3*-dimensional linear subspace of N.*

Without the assumption that H is parallel the same result holds under the assumption that H never vanishes and $H/\|H\|$ is parallel, if G is constant and c is non-positive, or if the Lipschitz-Killing curvature corresponding to $H/\|H\|$ is constant.

The proof is based on the Laplacian of the length of the second fundamental form (cf [3]). In §2 we recall the connection in the normal bundle and obtain a formula similar to one essenlially used in [6] (cf REMARK 2). In §3 we prove that M is of M-index 0. In §4 we make use of a classical method in the theory of Weingarten surfaces and show that M is pseudo-umbilical and prove the proposition.

§2. Preliminaries.

Let ι be an isometric immersion of an n-dimensional Riemannian manifold M in an $(n+p)$-dimensional space form N with curvature c. We shall make use of the following convention of the range of indices:

$$1\leqq A, B, C, \cdots\leqq n+p;\ 1\leqq i,j,k,\cdots\leqq n;$$

$$n+1\leqq \alpha, \beta, \gamma, \cdots\leqq n+p;\ n+2\leqq r,s,t,\cdots\leqq n+p.$$

STRUCTURE EQUATIONS.

Let $O(N)$ and $O(N, M)$ be respectively the bundle of orthonormal frames of N and the bundle of adapted frames, and let $\tilde{\iota}$ be the injection from $O(N, M)$ into $O(N)$. We denote by (ω'^A) and (ω'^A_B) respectively the canonical form and the connection form of $O(N)$. ω'^A and ω'^A_B are 1-forms on $O(N)$. Let σ

$$\sigma:\ M \longrightarrow O(N, M), \quad \sigma(p) = (e_1, \cdots, e_{n+p})$$

be a local cross section and put

$$\omega^A = (\tilde{\iota}\circ\sigma)^*\omega'^A, \quad \omega^A_B = (\tilde{\iota}\circ\sigma)^*\omega'^A_B .$$

$$\begin{array}{ccc} O(N, M) & \xrightarrow{\ \tilde{\iota}\ } & O(N) \\ \sigma \uparrow & & \downarrow \pi \\ M & \xrightarrow{\ \iota\ } & N \end{array}$$

Then ω^A and ω^A_B are local 1-forms on M determined by σ and satisfy the following structure equations:

(ω^i) is the dual coframe field of (e_i)

(1) $$\omega^\alpha = 0$$

(2) $$\omega^\alpha_i = \sum h^\alpha_{ij}\omega^j , \quad h^\alpha_{ij} = h^\alpha_{ji} ,$$

(3) $$d\omega^i = -\sum \omega^i_j \wedge \omega^j , \quad \omega^i_j + \omega^j_i = 0 ,$$

(4) $$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \Omega^i_j ,$$

$$\Omega^i_j = \frac{1}{2} \sum R^i_{jkl}\omega^k \wedge \omega^l ,$$

$$R^i_{jkl} = \bar{R}^i_{jkl} + \sum (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}) ,$$

(5) $$d\omega^\alpha_\beta = -\sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta , \quad \omega^\alpha_\beta + \omega^\beta_\alpha = 0 ,$$

$$\Omega^\alpha_\beta = \frac{1}{2} \sum R^\alpha_{\beta kl}\omega^k \wedge \omega^l ,$$

$$R^\alpha_{\beta kl} = \bar{R}^\alpha_{\beta kl} + \sum (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}) ,$$

(6) $$\bar{R}^A_{BCD} = -\bar{R}^A_{BDC} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) .$$

CONNECTION IN $T^r_s(M)\otimes T^{\perp p}_q(M)$.

(ω^i_j) defines the connection in the tangent bundle $T(M)$ and the cotangent bundle $T(M)^*$, and (ω^α_β) defines the connection in the normal bundle $T^\perp(M)$. We express this using ∇ as follows:

If we put

(7) $$\omega_j^i=\sum C_{jk}^i\omega^k\,,\quad \omega_\beta^\alpha=\sum C_{\beta k}^\alpha\omega^k\,,$$

then

(8) $$\nabla_{e_i}e_j=\sum C_{ji}^k e_k\,,$$

(9) $$\nabla_{e_i}\omega^j=-\sum C_{ki}^j\omega^k\,,$$

(10) $$\nabla_{e_i}e_\alpha=\sum C_{\alpha i}^\beta e_\beta\,.$$

We define the conormal bundle $T^\perp(M)^*$ by

(11) $$T^\perp(M)^*=\bigcup_{p\in M}\{T_p{}^\perp(M)^*\,|\,\text{the dual linear space of } T_p{}^\perp(M)\},$$

then (ω_β^α) defines a connection also in $T^\perp(M)^*$. If we denote by $(e^{*\alpha})$ the dual coframe field of (e_α), then

(12) $$\nabla_{e_i}e^{*\alpha}=-\sum C_{\beta i}^\alpha e^{*\beta}\,.$$

Hence (ω_j^i) and (ω_β^α) determine a connection in $T_s^r(M)\otimes T^\perp{}_q^p(M)$;

$$T_s^r(M)\otimes T^\perp{}_q^p(M)=\underbrace{T(M)\otimes\cdots\otimes T(M)}_{(r\text{ times})}\otimes\underbrace{T(M)^*\otimes\cdots\otimes T(M)^*}_{(s\text{ times})}\otimes\underbrace{T^\perp(M)\otimes\cdots\otimes T^\perp(M)}_{(p\text{ times})}\otimes\underbrace{T^\perp(M)^*\otimes\cdots\otimes T^\perp(M)^*}_{(q\text{ times})}$$

For a tensor field $K:M\to T_s^r(M)\otimes T^\perp{}_q^p(M)$ the covariant differential ∇K: $M\to T_{s+1}^r(M)\otimes T^\perp{}_q^p(M)$ is defined. For example a tensor field K

$$K:\ M\longrightarrow T(M)^*\otimes T(M)^*\otimes T^\perp(M)$$

can be considered as a bilinear mapping

$$K:\ T(M)\times T(M)\longrightarrow T^\perp(M)\,,$$

and ∇K

$$\nabla K:\ M\longrightarrow T(M)^*\otimes T(M)^*\otimes T(M)^*\otimes T^\perp(M)$$

regarded as a multilinear mapping

$$\nabla K:\ T(M)\times T(M)\times T(M)\longrightarrow T^\perp(M)$$

is given by

(13) $$\nabla K(X,Y;Z)=(\nabla_Z K)(X,Y)\,,\quad X,Y,Z\in T(M)\,.$$

If we express K and ∇K using σ as

$$K=\sum K_{ij}^\alpha\omega^i\otimes\omega^j\otimes e_\alpha\,,$$
$$\nabla K=\sum K_{ijk}^\alpha\omega^i\otimes\omega^j\otimes\omega^k\otimes e_\alpha\,,$$

then (K^{α}_{ijk}) satisfies the relation

(14) $$\sum K^{\alpha}_{ijk}\omega^k = dK^{\alpha}_{ij} - \sum K^{\alpha}_{kj}\omega^k_i - \sum K^{\alpha}_{ik}\omega^k_j + \sum K^{\beta}_{ij}\omega^{\alpha}_{\beta} \quad \text{(cf [3])}.$$

We can consider that $\nabla K = \sum(\nabla_{e_k}K)\otimes\omega^k$, hence

(15) $$\nabla_{e_k}K = \sum K^{\alpha}_{ijk}\omega^i\otimes\omega^j\otimes e_{\alpha}\,.$$

For $\nabla^2 K = \nabla(\nabla K)$ the similar formular to the Proposition 2.12, p 125, [1], holds;

(16) $$\nabla^2 K(\cdots;X;Y) = \nabla_Y(\nabla_X K) - \nabla_{\nabla_Y X}K\,.$$

We define now the canonical bundle isomorphism *

$$*:\ T^r_s(M)\otimes T^{\perp p}_{\ q}(M) \longrightarrow T^s_r(M)\otimes T^{\perp q}_{\ p}(M)\,,$$

for example, by

(17) $$\begin{aligned} &*(\sum K^{\alpha}_{ij}\omega^i\otimes\omega^j\otimes e_{\alpha}) \\ &= (\sum K^{\alpha}_{ij}\omega^i\otimes\omega^j\otimes e_{\alpha})^* \\ &= (\sum K^{\alpha}_{ij}e_i\otimes e_j\otimes e^{*\alpha})\,. \end{aligned}$$

By (8), (9), (10), and (12) we obtain

(18) $$(\nabla_X K)^* = \nabla_X K^* \qquad X\in T(M)\,.$$

RESTRICTED LAPLACIAN Δ'.

The "restricted" Laplacian (cf [4]) of a tensor field K is defined by

(19) $$\Delta' K = \sum(\nabla^2 K)\ (\cdots;e_i;e_i)\,.$$

This is independent of the choice of σ. If K is given by

$$K = \sum K^{\alpha}_{ij}\omega^i\otimes\omega^j\otimes e_{\alpha}\,,$$

and $\Delta' K$ and $\nabla^2 K$ is expressed by

$$\Delta' K = \sum(\Delta' K)^{\alpha}_{ij}\omega^i\otimes\omega^j\otimes e_{\alpha}$$
$$\nabla^2 K = \sum K^{\alpha}_{ijkl}\omega^i\otimes\omega^j\otimes\omega^k\otimes\omega^l\otimes e_{\alpha}\,,$$

then the relation between $((\Delta' K)^{\alpha}_{ij})$ and (K^{α}_{ijkl}) is expressed as

(20) $$(\Delta' K)^{\alpha}_{ij} = \sum_k K^{\alpha}_{ijkk} \qquad \text{(cf [3]))}\,.$$

When K is a function f, $\Delta' f$ coincides with the ordinary Laplacian Δf.

FIBRE METRIC.

We denote by g the fibre metric in $T^r_s(M)\otimes T^{\perp p}_{\ q}(M)$ induced by the Riemannian metric of N. Let C be the contraction

$$C:\ \left(T^r_s(M)\otimes T^{\perp p}_q(M)\right)\otimes\left(T^s_r(M)\otimes T^{\perp q}_p(M)\right)\longrightarrow T^0_0(M),$$

such that, for example,

$$C\left((e_i\otimes\omega^j\otimes e_\alpha\otimes e^{*\beta})\otimes(\omega^k\otimes e_l\otimes e^{*\lambda}\otimes e_\iota)\right)=\delta_{ik}\delta_{jl}\delta_{\alpha\lambda}\delta_{\beta\iota},$$

then

$$g(K,K')=C(K\otimes K'^*). \tag{21}$$

Since ∇_X is a type preserving derivation and commutes with every contraction, by (18) and (21)

$$\nabla_X g(K,K')=g(\nabla_X K,K')+g(K,\nabla_X K'). \tag{22}$$

SECOND FUNDAMENTAL FORM.

The second fundamental form $h:\ T(M)\times T(M)\to T^\perp(M)$, i.e., $h: M\to T(M)^*\otimes T(M)^*\otimes T^\perp(M)$ of the immersion ι is locally expressed, using (h^α_{ij}) of (2), by

$$h=\sum h^\alpha_{ij}\omega^i\otimes\omega^j\otimes e_\alpha. \tag{23}$$

The mean curvature vector H is given by

$$H=\frac{1}{n}\sum h^\alpha_{ii}e_\alpha. \tag{24}$$

If we put

$$\nabla h=\sum h^\alpha_{ijk}\omega^i\otimes\omega^j\otimes\omega^k\otimes e_\alpha$$
$$\nabla^2 h=\sum h^\alpha_{ijkl}\omega^i\otimes\omega^j\otimes\omega^k\otimes\omega^l\otimes e_\alpha,$$

then by (5) and (6) (cf [3])

$$h^\alpha_{ijk}-h^\alpha_{ikj}=\bar{R}^\alpha_{ijk}=0, \tag{25}$$

$$h^\alpha_{ijkl}-h^\alpha_{ijlk}=\sum h^\alpha_{im}R^m_{jkl}+\sum h^\alpha_{mj}R^m_{ikl}-\sum h^\beta_{ij}R^\alpha_{\beta kl}, \tag{26}$$

$$\begin{aligned}(\Delta' h)^\alpha_{ij}=\sum_k h^\alpha_{ijkk}&=\sum h^\alpha_{kkij}+nch^\alpha_{ij}-c\left(\sum h^\alpha_{kk}\right)\delta_{ij}\\&+\sum h^\alpha_{mi}h^\beta_{mj}h^\beta_{kk}+2\sum h^\alpha_{km}h^\beta_{mj}h^\beta_{ki}\\&-\sum\left(h^\alpha_{km}h^\beta_{km}h^\beta_{ij}+h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj}+h^\alpha_{mj}h^\beta_{ki}h^\beta_{mk}\right),\end{aligned} \tag{27}$$

$$\frac{1}{2}\Delta\left(\sum_{\alpha ij}(h^\alpha_{ij})^2\right)=\sum_{\alpha ijk}(h^\alpha_{ijk})^2+\sum_{\alpha ij}h^\alpha_{ij}(\Delta' h)^\alpha_{ij}. \tag{28}$$

(28) is the formula (3.12) of [3], but we give here a proof to compare it with the next lemma. By (15), (16), (19) and (22)

$$\begin{aligned}\varDelta\Big(\sum_{\alpha ij}(h^\alpha_{ij})^2\Big)&=\varDelta' g(h,h)=\sum_k \nabla_{e_k}\Big(\nabla_{e_k} g(h,h)\Big)-\sum_k \nabla_{\nabla_{e_k}e_k} g(h,h)\\&=2\sum_k g(\nabla_{e_k}\nabla_{e_k}h-\nabla_{\nabla_{e_k}e_k}h,h)+2\sum_k g(\nabla_{e_k}h,\nabla_{e_k}h)\\&=2g(\varDelta' h,h)+2\sum_k g(\nabla_{e_k}h,\nabla_{e_k}h)\\&=2\sum_{\alpha ij}h^\alpha_{ij}(\varDelta' h)^\alpha_{ij}+2\sum_{\alpha ijk}(h^\alpha_{ijk})^2.\end{aligned}$$

LEMMA 1. *If there exists a number α such e_α is parallel, then for this α*

$$\frac{1}{2}\Big(\sum_{ij}(h^\alpha_{ij})\Big)^2=\sum_{ijk}(h^\alpha_{ijk})^2+\sum_{ij}h^\alpha_{ij}(\varDelta' h)^\alpha_{ij}. \tag{29}$$

PROOF. Let C be a contraction

$$C:\ T(M)^*\otimes T(M)^*\otimes T^\perp(M)\otimes T^\perp(M)^*\longrightarrow T(M)^*\otimes T(M)^*.$$

We set

$$h^\alpha=C(h\otimes e^{*\alpha}),$$

then

$$h^\alpha=\sum_{ij}h^\alpha_{ij}\omega^i\otimes\omega^j.$$

Therefore

$$\begin{aligned}\varDelta\Big((\sum_{ij}(h^\alpha_{ij})^2\Big)&=\varDelta' g(h^\alpha,h^\alpha)=2g\Big(\sum_k(\nabla_{e_k}\nabla_{e_k}h^\alpha-\nabla_{\nabla_{e_k}ek}h^\alpha),h^\alpha\Big)\\&\quad+2\sum_k g(\nabla_{e_k}h^\alpha,\nabla_{e_k}h^\alpha).\end{aligned}$$

Since e_α is parallel, $\nabla_X e^{*\alpha}=0$ for any $X\in T(M)$. Hence

$$\nabla_{e_k}h^\alpha=C\Big((\nabla_{e_k}h)\otimes e^{*\alpha}\Big)=\sum_{ij}h^\alpha_{ijk}\omega^i\otimes\omega^j,$$

and consequently

$$g(\nabla_{e_k}h^\alpha,\nabla_{e_k}h^\alpha)=\sum_{ij}(h^\alpha_{ijk})^2.$$

Similarly

$$\begin{aligned}\sum_k\nabla_{e_k}\nabla_{e_k}h^\alpha-\sum_k\nabla_{\nabla_{e_k}e_k}h^\alpha&=C\Big(\sum_k(\nabla_{e_k}\nabla_{e_k}h-\nabla_{\nabla_{e_k}ek}h)\otimes e^{*\alpha}\Big)\\&=C(\varDelta' h\otimes e^{*\alpha})\\&=\sum_{ij}(\varDelta' h)^\alpha_{ij}\omega^i\otimes\omega^j,\end{aligned}$$

$$g\Big(\sum_k(\nabla_{e_k}\nabla_{e_k}h^\alpha-\nabla_{\nabla_{e_k}e_k}h^\alpha),h^\alpha\Big)=\sum_{ij}h^\alpha_{ij}(\varDelta' h)^\alpha_{ij}.$$

Therefore we obtain

$$\frac{1}{2}\Delta\sum_{ij}(h^{\alpha}_{ij})^2=\sum_{ijk}(h^{\alpha}_{ijk})^2+\sum_{ij}h^{\alpha}_{ij}(\Delta' h)^{\alpha}_{ij}. \qquad \text{q. e. d.}$$

REMARK 1. If the mean curvature vector H never vanishes, we may choose e_{n+1} in the direction of H, i.e., $e_{n+1}=H/\|H\|$. If $H/\|H\|$ is parallel, then we get by LEMMA above

$$\frac{1}{2}\Delta\Big(\sum_{ij}(h^{n+1}_{ij})^2\Big)=\sum_{ijk}(h^{n+1}_{ijk})^2+\sum_{ij}h^{n+1}_{ij}(\Delta' h)^{n+1}_{ij}. \tag{30}$$

For (28) and (30)

$$\frac{1}{2}\Delta\sum_{rij}(h^{r}_{ij})^2=\sum_{rijk}(h^{r}_{ijk})^2+\sum_{rij}(h^{r}_{ij})(\Delta' h)^{r}_{ij} \qquad (r=n+2,\cdots,n+p). \tag{31}$$

REMARK 2. The formula (31) plays the essential role in [5] or [6], but it is not assumed there, that $H/\|H\|$ is parallel. It seems to the author that (31) does not hold without this assumption or other.

NORMAL CURVATURE (cf. [6]).

The normal curvature of the immersion ι is defined by

$$K_N=\sum_{\alpha\beta kl}(R^{\alpha}_{\beta kl})^2 \tag{32}$$

ince N is a space form,

$$K_N=\sum_{\alpha\beta kl}\Big(\sum_i(h^{\alpha}_{ik}h^{\beta}_{il}-h^{\alpha}_{il}h^{\beta}_{ik})\Big)^2.$$

Hence the normal curvature vanishes if and only if the p $n\times n$-matrices (h^{α}_{ij}) can be transformed simaltaneously in diagonalized forms.

§ 3. Surfaces with $K_N=0$.

In the following throughout this paper we assume;

M is a compact connected surface,

$K_N\equiv 0$,

the Gaussian curvature G of M is positive,

H never vanishes,

$H/\|H\|$ is parallel,

and we choose σ so that

$e_3=H/\|H\|$.

LEMMA 2.

(34) $$\sum_k h^r_{kkij}=0 \qquad \text{for} \quad r\geqq 4\,.$$

PROOF. We note

(35) $$\sum_k h^r_{kk}=0 \qquad \text{for} \quad r\geqq 4\,.$$

e_3 is parallel, hence by (7) and (10) we obtain

(36) $$\omega_3^\alpha=0\,.$$

By (14)

(37) $$\sum_k h^\alpha_{ijk}\omega^k=dh^\alpha_{ij}-\sum_l h^\alpha_{lj}\omega^l_i-\sum_l h^\alpha_{il}\omega^l_j+\sum_\beta h^\beta_{ij}\omega^\alpha_\beta\,,$$

hence, using (35) and (36), we have

$$\sum_k\Big(\sum_i h^r_{kki}\omega^i\Big)=0\,,$$

(38) $$\sum_k h^r_{kki}=0\,.$$

Similarly

(39) $$\sum_l h^\alpha_{ijkl}\omega^l=dh^\alpha_{ijk}-\sum_l h^\alpha_{ljk}\omega^l_i-\sum_l h^\alpha_{ilk}\omega^l_j-\sum_l h^\alpha_{ijl}\omega^l_k+\sum_\beta h^\beta_{ijk}\omega^\alpha_\beta\,,$$

therefore

$$\sum_k\Big(\sum_j h^r_{kkij}\omega^j\Big)=\sum_k dh^r_{kki}-\sum_{kl} h^r_{lki}\omega^l_k-\sum_{kl} h^r_{kli}\omega^l_k-\sum_{kl} h^r_{kkl}\omega^l_i+\sum_\alpha h^\alpha_{kki}\omega^r_\alpha\,.$$

Since $h^r_{lki}=h^r_{kli}$, the formula above is, using (36) and (38), reduced to

$$\sum_k\Big(\sum_j h^r_{kkij}\omega^j\Big)=0\,,$$

that is,

$$\sum_k h^r_{kkij}=0\,. \qquad \text{q. e. d.}$$

REMARK 3. In the case of minimal submanifolds

$$\sum_k h^\alpha_{kkij}=0 \qquad \text{for} \quad \alpha\geqq n+1\,,$$

as is seen in [3]. But in the case of nonminimal submanifolds, choosing e_{n+1} in the direction of H, we cannot obtain

$$\sum_k h^r_{kkij}=0 \qquad \text{for} \quad r\geqq n+2$$

without some additional condition.

For each α, we denote the symmetric 2×2-matrix (h^α_{ij}) by

(40) $$H_\alpha=(h^\alpha_{ij})\,,$$

and set

$$S_\alpha = \text{trace } H_\alpha \cdot {}^t H_\alpha = \sum_{ij} (h_{ij}^\alpha)^2, \tag{41}$$

$$S = \sum_\alpha S_\alpha, \tag{42}$$

$$\tilde{S} = \sum_r S_r = \sum_{rij} (h_{ij}^r)^2. \tag{43}$$

If we use the expression in the proof of LEMMA 1,

$$S = g(h, h),$$
$$\tilde{S} = g(h, h) - g(h^3, h^3).$$

Since e_3 is global, $\tilde{S}$ is a well defined function over M.

We now consider a decomposition of $T_p{}^\perp(M)$ (cf. [8]). We set

$$N_p = \left\{ e \in T_p{}^\perp(M) \mid g(e, e_3) = 0 \right\},$$

and define a linear mapping φ_σ from N_p into the set of 2×2-matrices by

$$\varphi_\sigma\Big(\sum_r \nu_r e_r\Big) = \Big(\sum_r \nu_r h_{ij}^r\Big).$$

Then the kernel of φ_σ, which we denote by O_p, is independent of the choice of σ, and $\dim O_p \geqq p-2$. In fact, if we put

$$\bar{e} = \sum_r h_{11}^r e_r, \quad \bar{\bar{e}} = \sum_r h_{12}^r e_r, \quad \bar{e}, \ \bar{\bar{e}} \in N_p,$$

$e \in N_p$ belongs to O_p when and only when

$$g(e, \bar{e}) = g(e, \bar{\bar{e}}) = 0.$$

Since $K_N = 0$, it follows that $\bar{e}$ and $\bar{\bar{e}}$ are linearly dependent in N_p. Therefore $\dim O_p \geqq \dim N_p - 1 = p-2$.

We call $\dim N_p - \dim O_p$ the minimal index (M-index) of M at p (cf. [8]).

LEMMA 3. *M-index of M is everywhere zero.*

PROOF. We define two subsets of M by

$$M_0 = \{ p \in M \mid \text{M-index} = 0 \text{ at } p \},$$
$$M_1 = \{ p \in M \mid \text{M-index} = 1 \text{ at } p \}.$$

Then $\tilde{S} = 0$ on M_0, $\tilde{S} > 0$ on M_1, M_1 is open and $M = M_0 \cup M_1$.

If $M_1 \neq \phi$, then N_p, $p \in M_1$, is decomposed as

$$N_p = N_p' \otimes O_p,$$

where N_p' is the 1-dimensional orthogonal complement of O_p in N_p. This decomposition is well-defined and smooth on M_1. Therefore we can choose

σ locally on M_1, so that $e_4 \in N'_p$. Since $K_N=0$ and H_α are simultaneously diagonalized, we may put

$$(44) \qquad H_3 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \; H_4 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \; a \neq 0,$$

$$H_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad r \geqq 5 .$$

Hence we obtain

$$(45) \qquad \tilde{S} = 2a^2 .$$

The Gaussian curvature G is given by

$$d\omega_2^1 = G(\omega^1 \wedge \omega^2) .$$

Accordingly by (4), (6) and (44)

$$G = c + k_1 k_2 - a^2 ,$$

that is, if we denote the Lipschitz-Killing curvature of the immersion ι by $G(p, e)$,

$$(46) \qquad G = c + G(p, e_3) - \frac{\tilde{S}}{2} .$$

Since $G>0$ and $\tilde{S} \geqq 0$, we obtain, using Lemma 2, (27) and (44),

$$(47) \qquad \sum_{rij} h^r_{ij}(\varDelta' h)^r_{ij} = \sum_{ij} h^4_{ij}(\varDelta' h)^4_{ij} = \sum_i h^4_{ii}(\varDelta' h)^4_{ii} \doteq 2\tilde{S}G \geqq 0 .$$

Therefore by (31)

$$(48) \qquad \frac{1}{2}\varDelta\tilde{S} = \frac{1}{2}\varDelta\Big(\sum_{rij}(h^r_{ij})^2\Big)$$
$$= \sum_{rijk}(h^r_{ijk})^2 + \sum_{rij} h^r_{ij}(\varDelta' h)^r_{ij} \geqq 0 ,$$

i.e., $\varDelta\tilde{S} \geqq 0$ on M_1.

At a boundary point of M_0, if any, $\varDelta\tilde{S} \geqq 0$ by continuity. At an inner point of M_0, if any, $\varDelta\tilde{S}=0$ clearly. Hence $\varDelta\tilde{S} \geqq 0$ on M_0. Therefore $\varDelta\tilde{S} \geqq 0$ over M and accordingly $\varDelta\tilde{S} \equiv 0$, because M is compact. Hence by (47) and (48) $\tilde{S}=0$ on M_1, which contradicts the construction of M_1. Therefore $M_1 = \phi$. q. e. d.

By LEMMA 3 4nd (31)

$$(49) \qquad h^r_{ijk} = 0 \qquad \text{for all} \quad r, i, j, k,$$

$$(50) \qquad \tilde{S} \equiv 0 .$$

§ 4. Proof of Proposition.

In this section we assume further that one of the following three conditions holds:

(51) $$G = const, \quad c \leqq 0,$$

(52) $$G(p, H/\|H\|) = const,$$

(53) $$\|H\| = const.$$

If we denote by k_1 and k_2 the principal curvatures corresponding to $H/\|H\|$, (51) or (52) means by (46) and (50)

(54) $$k_1 k_2 = const > 0,$$

and (53) means

(55) $$k_1 + k_2 = const.$$

LEMMA 4. *M is pseudo-umbilical, i.e., $k_1 = k_2$ every-where on M.*

PROOF. We shall use a well known method, for example, in [7].

We choose k_1 and k_2 so that $k_1 \geqq k_2$, then k_1 and k_2 are continuous functions on M and differentiable on the subset of M, where $k_1 > k_2$. Since M is compact, there exists a point p_0, by (54) or (55), where k_1 has a maximum and k_2 has a minimum.

If we assume $k_1(p_0) > k_2(p_0)$, then k_1 and k_2 are differentiable in a neighbourhood of p_0 and we can choose a local cross section σ around p_0 so that $e_3 = H/\|H\|$ and e_1 and e_2 are the principal directions corresponding respectively to k_1 and k_2. We have

(56) $$e_i k_1 = e_i k_2 = 0 \qquad \text{at} \quad p_0,$$

(57) $$e_i(e_i k_1) \leqq 0, \quad e_i(e_i k_2) \geqq 0 \qquad \text{at} \quad p_0.$$

We have also

(58) $$\omega_2^1 = 0 \qquad \text{at} \quad p_0.$$

Indeed, setting $\alpha = 3$, $i = j = 1$ in (37) and using (36), we obtain

$$\sum h_{11l}^3 \omega^l = dh_{11}^3 = (e_l k_1)\omega^l,$$

$$h_{112}^3 = e_2 k_1.$$

Similarly by setting $\alpha = 3$, $i = j = 2$

$$h_{221}^3 = e_1 k_2.$$

By setting $\alpha = 3$, $i = 1$, $j = 2$

$$\sum_l h^3_{12l}\omega^l = -(k_1-k_2)\omega^1_2 .$$

Since $h^\alpha_{ijk}=h^\alpha_{jik}$, by (25) we see that

$$h^3_{121} = h^3_{211} = h^3_{112} = e_2k_1 , \tag{59}$$

$$h^3_{122} = h^3_{212} = h^3_{221} = e_1k_2 . \tag{60}$$

Therefore we obtain

$$(e_2k_1)\omega^1 + (e_1k_2)\omega^2 = -(k_1-k_2)\omega^1_2 .$$

At p_0 the left hand side equals zero and $k_1>k_2$, hence $\omega^1_2=0$. By (39) we obtain

$$h^3_{2112} = e_2(e_2k_1) \qquad \text{at } p_0 . \tag{61}$$

Indeed

$$\sum h^3_{211l}\omega^l = dh^3_{211} - h^3_{111}\omega^1_2 - h^3_{221}\omega^1_2 - h^3_{212}\omega^2_1 + \sum_\alpha h^\alpha_{211}\omega^3_\alpha ,$$

and using (36), (58) and (59), we reduce the formula above to

$$\sum h^3_{211l}\omega^l = dh^3_{211} = d(e_2k_1) = \sum e_l(e_lk_1)\omega^l , \quad h^3_{2112} = e_2(e_2k_1) .$$

Similarly by (60)

$$h^3_{2121} = e_1(e_1k_2) \qquad \text{at } p_0 . \tag{62}$$

Since $K_N=0$, $R^\alpha_{\beta ij}=0$. Hence setting $\alpha=3$, $i=l=2$, $j=k=1$ in (26), we obtain

$$h^3_{2112} - h^3_{2121} = (k_1-k_2)R^1_{212} .$$

From (4) and the formula $d\omega^1_2=G\omega^1\wedge\omega^2$, R^1_{212} is nothing but G. Hence

$$h^3_{2112} - h^3_{2121} = (k_1-k_2)G .$$

By (61) and (62) we have

$$e_2(e_2k_1) - e_1(e_1k_2) = (k_1-k_2)G \qquad \text{at } p_0 . \tag{63}$$

Since $G>0$, (63) contradicts, by (57), the assumption that $k_1>k_2$ at p_0. Therefore $k_1=k_2$ at p_0, which implies that $k_1=k_2$ everywhere on M. q.e.d.

By LEMMA 3 and LEMMA 4 we see that M is a pseudo-umbilical surface with M-index$=0$ (and hence totally umbilical) in N. Especially if N is the euclidean space E^{2+p}, then we may apply here Theorem 1 in [9] and see that M is a sphere in a linear subspace E^3 of E^{2+p}.

Department of Mathematics,
Hokkaido University

References

[1] S. KOBAYASHI and K. NOMIZU: Foundations of differential geometry, Vol. I, Interscience, New York, 1963.

[2] S, KOBAYASHI and K. NOMIZU: Foundations of differential geometry, Vol, II, Interscience, New York, 1969.

[3] S. S. CHERN, M. DO CARMO and S. KOBAYASHI: Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields, (1970), 59–75.

[4] K. NOMIZU and B. SMYTH: A formula of Simons' type and hypersurfaces with constant mean curvature, J. Differntial geometry, 3 (1969), 367–377.

[5] B.-Y. CHEN: Pseudo-umbilical Submanifolds of a Riemannian manifold of constant curvature, II, to appear.

[6] B.-Y. CHEN: Pseudo-umbilical Submanifolds of a Riemannian manifold of constant curvature, III, to appear.

[7] S. S. CHERN: Some new characterization of Euclidean sphere, Duke Math. J. 12 (1945), 279–290.

[8] T. ŌTSUKI: A theory of Riemannian submanifolds, Kōdai Math, Sem. Rep. 20 (1968), 282–295.

[9] T. ŌTSUKI: Pseudo-umbilical submanifolds with M-index≦1 in Euclidean spaces, Kōdai Math, Sem. Rep. 20 (1968), 296–304.

(Received July 28, 1972)