On minimal points of Riemann Surfaces, I.

Dedicated to Prof. Yoshie Katsurada on her 60th birthday

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Let R be a Riemann surface with positive boundary and let $\{R_n\}$ ($n=0,1,2,\cdots$) be its exhaustion with compact analytic relative boundary ∂R_n. Let G be a domain in R such that the relative boundary ∂G consists of at most an enumerable number of smooth curves clustering nowhere in R. In the present paper we consider only domains as mentioned above. If an open set G has a smooth relative boundary, we call it a regularly open set. Let G_1 be a domain in $R-R_0$ and G_2 be a regularly open set in G_1 such that $G_1\supset\bar{G}_2$, where $\bar{G}_2$ is the closure of G_2 in R. Let $U_n(z)$ be a harmonic function in $(G_1-\bar{G}_2)\cap R_n$ such that $U_n(z)=1$ on $\partial G_2\cap R_n$, $=0$ on $\partial G_1\cap R_n$ and $\frac{\partial}{\partial n}U_n(z)=0$ on $\partial R_n\cap(G_1-\bar{G}_2)$. If the Dirichlet integral $D(U_n(z))<M<\infty$ for any n, then $U_n(z)$ converges locally uniformly and in Dirichlet integral as $n\to\infty$ to a harmonic function which is denoted by $\omega(\bar{G}_2, z, G_1)$ and is called the Capacitary Potential[1] of $\bar{G}_2$ relative to G_1 (abbreviated by C. P.). Clearly $\omega(\bar{G}_2, z, G_1)$ is uniquely determined and has minimal Dirichlet integral (M. D. I.) over $(G_1-\bar{G}_2)$ among all harmonic functions with the same value as $\omega(\bar{G}_2, z, G_1)$ on $\partial G_1+\partial G_2$. Let $G_2\supset G_3\supset G_4,\cdots$ be a decreasing sequence of regularly open sets. Then $\omega(\bar{G}_n, z, G_1)$ converges locally uniformly and in Dirichlet integral as $n\to\infty$ to a harmonic function denoted by $\omega(\{\bar{G}_n\}, z, G_1)$. This is called C.P. of $\{\bar{G}_n\}$. Len F be a closed set in G_1, if there exists a harmonic function $U(z)$ in G_1-F such that $U(z)=0$ on ∂G_1, $U(z)=1$ on F and $D(U(z))<\infty$, there exists a uniquely determined harmonic function $\omega(F, z, G_1)$ such that $\omega(F, z, G)=1$ on ∂F except at most a set of capacity zero, $=0$ on ∂G_1 and has M. D. I. Let $\{F_n\}$ be a decreasing sequence in G_1. Then we can define C.P. of $\{F_n\}$ as above. Let $W(z)$ be the least positive harmonic function in G_1-F such that $W(z)=1$ on ∂F except at most a set of capacity zero. We call $W(z)$ the harmonic measure (abbreviated by H. M.) and denote it by $W(F, z, G_1)$. Similarly H. M. of a decreasing sequence $\{F_n\}$ can be defined also.

Let G be a domain in R such that $\bar{G}\cap\bar{R}_0=0$ and let $N_n(z,p)$ be a positive harmonic function in $(G-\{p\})\cap R_n$: $p\in R_n\cap G$ such that $N_n(z,p)=0$ on $\partial G\cap R_n$, $N_n(z,p)$ has a logarithmic singularity with coefficient 1 at p and

$\frac{\partial}{\partial n} N_n(z,p)=0$ on $\partial R_n \cap G$. Then $N_n(z,p) \to N(z,p)$ and $D(N_n(z,p)-N(z,p)) \to 0$ as $n \to \infty$. $N(z,p)$ has M. D. I., where the Dirichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p. Then we have N-Martin's topology[2] (N-top.) in $G+\varDelta(G,N)$ as usual manner with distance

$$\delta^N(p_1,p_2)=\sup_{z\in G_0}\left|\frac{N(z,p_1)}{1+N(z,p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right| \quad \text{for } p_1,p_2 \in G+\varDelta(G,N),$$

where G_0 is a disk in G and $\varDelta(G,N)$ is the ideal boundary of G obtained by completion of G with respect to the above metric.

Further let $L(z,p)$ be an N-Green's function of $R-R_0$ such that $L(z,p)=0$ on ∂R_0 with a logarithmic singulariy at p with coefficient 1 and $L(z,p)$ has M.D.I. over $R-R_0$. Then we have also an N-Martin's topology (L-top.) over $R-R_0+\varDelta(R,L)$ with distance

$$\delta^L(p_1,p_2)=\sup_{z\in D_0}\left|\frac{L(z,p_1)}{1+L(z,p_1)}-\frac{L(z,p_2)}{1+L(z,p_2)}\right|, \quad p_1,p_2 \in R-R_0+\varDelta(R,L)$$

where D_0 is a disk in $R-R_0$ and $\varDelta(R,L)$ is the ideal boundary.

Let G be a domain in $R-R_0$ with non compact relative boundary ∂G. We denote an N-Green's function of G by $N(z,p)$ (with letter N) vanishing on ∂G. On the other hand, when we regard G as a Riemann surface, we denote an N-Green's function in $G-G_0$ by $L'(z,p)$ (with letter L), where G_0 is a disk in G. Let D be a regularly open set in G. Let $U(z)$ be a non negative continuous superharmonic function in G such that $U(z)=0$ on ∂G and $D(\min(M, U(z)))<\infty$ for any M: $0<M<\infty$. Let ${}_{\bar D}U^M(z)$ be a function such that ${}_{\bar D}U^M(z)=\min(M, U(z))$ on $\bar D+\partial G$ and has M.D.I. over $G-\bar D$. Put ${}_{\bar D}U(z)=\lim\limits_{M=\infty} {}_{\bar D}U^M(z)$. If, for any relatirely compact regularly domain D, ${}_{\bar D}U(z)\leqq U(z)$, then $U(z)$ is called a full-superharmonic function in G.[3] A positive harmonic function $U(z)$ in G is called minimal, if $U(z)\geqq V(z)\geqq 0$ implies $V(z)=aU(z)$: $0\leqq a\leqq 1$ for any function $V(z)$ such that both $V(z)$ and $U(z)-V(z)$ are harmonic and full-superharmonic in G. Let ${}_1\varDelta(G,N)$ be the set of all minimal points.[3] Then it is known that $U(z)$ is minimal if and only if $U(z)=aN(z,p)$ for some $p\in {}_1\varDelta(G,N)$ and $0<a<\infty$. Let ${}_S\varDelta(G,N)$ be the set of all singular point.[4] Then

1). ${}_S\varDelta(G,N)\subset {}_1\varDelta(G,N)$.

2). Suppose $p\in {}_1\varDelta(G,N)$. Put ${}_{v_n(p)}N(z,p)=\lim\limits_{m=\infty} {}_{v_n(p)\cap R_m}N(z,p)$, where $v_n(p)=E\left[z\in G+\varDelta(G,N):\ \delta^N(z,p)<\frac{1}{n}\right]$. Then $N(z,p)={}_{v_n(p)}N(z,p)$, $N(z,p)$

$=\lim_{n=\infty} {}_{v_n(p)}N(z,p)$. $N(z,p)=M\ \omega(V_M(p),z,G)$ in $G-V_M(p)$ and $N(z,p)=\lim_{M\to M^*} {}_{V_M(p)}N(z,p)$, where $V_M(p)=E[z\in G: N(z,p)\geqq M]$ and $0<M<M^*=\sup N(z,p)$.

3). Suppose $p\in {}_1\Delta(G,N)$. For any $V_M(p)$: $M<M^*$, there exists a number n such that $(v_n(p)\cap G)\subset V_M(p)$.

4). $N(z,p)=M^*\lim_{M\to M^*}\omega(V_M(p),z,G)=M^*\omega(p,z,G)=M^*\lim_{n\to\infty}\omega(G\cap v_n(p),z,G)$ for $p\in {}_S\Delta(G,N)$: where $\omega(v_n(p),z,G)=\lim_{m=\infty}\omega(v_n(p)\cap R_m,z,G)$.

Let $W(p,z,G)=\lim_{n=\infty}W(G\cap v_n(p),z,G)$ be H.M. of $p\in {}_S\Delta(G,N)$. We call p a singular point of first kind or second kind according as $W(p,z,G)=0$ or >0 respectively. We denote the set of all singular points of first kind and second kind by ${}_{S,1}\Delta(G,N)$ and ${}_{S,2}\Delta(G,N)$ respectively.

Let G be a regularly open set. Let $U(z)$ be a non negative harmonic function in R. We denote by $I_G[U]$ the upper envelope of all non negative continuous subharmonic functions in G vanishing on ∂G and smaller than $U(z)$ in G.[5] A non negative harmonic function $U(z)$ in G is called admissible, if $U(z)=0$ on ∂G and if there exists at least a superharmonic function in R larger than $U(z)$ in R. Let $E[U]$ be the lower envelope of continuous superharmonic functions in R larger than $U(z)$. Then we see [6]

1). E and I are positive linear operators.

2). If $U(z)$ is admissible, then $IE[U]=U$. For any $U(z)>0$ in R we have $IEI[U]=I[U]$.

3). If $G_i\cap G_j=0$, then $\underset{G_i\ G_j}{IEI}[U]=0$ for $i\neq j$.

Matsumoto proved the following

THEOREM 1.[7] *Let G_i $(i=1,2,3,\cdots)$ be domains such that $G_i\cap G_j=0$ for $i\neq j$. Put $G=\sum G_i$. If each G_i is of finite genus or more generally is representable as a covering surface of a finite number of sheets over the w-plane and if* 1). $EI[1]=1$ *and if* 2). $I[1]$ *has a finite Dirichlet integral for any i, then ${}_{S,2}\Delta(R,L)$ is empty.*

In the present paper we consider the relations between ${}_{S,i}\Delta(G,N)$ and ${}_{S,i}\Delta(R,L)$ $(i=1,2)$ and problems of types similar to Theorem 1.

Let G be a domain in R such that $G\cap \bar{R}_0=0$ and $L(z,p)$ and $N(z,p)$ be N-Green's functions of $R-R_0$ and G respectively. We suppose L and N topologies are defined on $R-R_0+\Delta(R,L)$ and $G+\Delta(G,N)$ respectively. If $p\in G$, then we have at once

$$L(z,p)-{}_{CG}L(z,p)=N(z,p).$$

Let $p\in {}_1\Delta(R,L)$. If $L(z,p)\neq {}_{CG}L(z,p)$, we write $G\overset{L}{\ni}p$ (this is equivalent to the thiness of CG at p). Let $G(L, \varDelta)$ be the set of all points p in ${}_1\varDelta(R, L)$ with $p \overset{L}{\in} G$. Let $p_i \overset{L}{\to} p^\alpha \in G(L, \varDelta)$ (this means that p_i tends to p^α relative to L-top.) and $p_i \overset{N}{\to} p^\beta$: $p_i \in G$. Then

$$N(z, p^\beta) = \gamma\Big((L(z, p^\alpha) - {}_{CG}L(z, p^\alpha)\Big): \quad 1 \geqq \gamma \geqq 0$$

and $p^\beta \in {}_1\varDelta(G, N)$ if and only if $\gamma = 1$.[8]

Let p^α be a point in $G(L, \varDelta)$. If there exists a sequence $\{p_i\}$ in G such that $p_i \overset{L}{\to} p^\alpha$ and $p_i \overset{N}{\to} p^\beta \in \varDelta_1(G, N)$. We say that p^β lies on p^α. Denote by $f(p^\alpha)$ the set of all points lying on p^α. Then $f(p^\alpha)$ contains one point p^β (denoted by $p^\beta = f(p^\alpha)$) in ${}_1\varDelta(G, N)$ and

$$L(z, p^\alpha) - {}_{CG}L(z, p^\alpha) = N(z, p^\beta).$$

Conversely let $p^\beta \in {}_1\varDelta(G, N)$. Then $f^{-1}(p^\alpha)$ consists of only one point p^α[8] in $G(L, \varDelta)$ and $L(z, p^\alpha) - {}_{CG}L(z, p^\alpha) = N(z, p^\beta)$. Hence the mapping f is one to one from $G(L, \varDelta)$ onto ${}_1\varDelta(G, N)$. We consider ${}_S\varDelta(R, L)$ and $\varDelta_S(G, N)$. Then we have

THEOREM 2. *The mapping f is one to one from $G(L, \varDelta) \cap {}_S\varDelta(R, L)$ onto ${}_S\varDelta(G, N)$.*

PROOF. Let $p \in G(L, \varDelta) \cap {}_S\varDelta(R, L)$ and put $q = f(p)$. Then $N(z, q) = L(z, p) - {}_{CG}L(z, p)$ and $\sup L(z, p) < \infty$. Hence we have at once $\sup N(z, q) < \infty$ and $q \in {}_S\varDelta(G, N)$. Let $q \in {}_S\varDelta(G, N)$. Then $L(z, p) - {}_{CG}L(z, p) = N(z, q)$ and $\sup N(z, q) = M < \infty$.

$${}_{CG}L(z, p) + M \geqq L(z, p) \ \textit{in} \ R - R_0.$$

Assume $p \in {}_1\varDelta(R, L) - {}_S\varDelta(R, L)$. Then $\omega(p, z, R - R_0) = 0$ and

$${}_{v_n(p)}\Big({}_{CG}L(z, p) + M\Big) \geqq {}_{v_n(p)}L(z, p) = L(z, p) = \lim_{n=\infty} {}_{v_n(p)}L(z, p).$$

Now by $G \overset{L}{\ni} p$ and by $\omega(p, z, R - R_0) = 0$ we have ${}_p({}_{CG}L(z, p)) = \lim_{n=\infty} {}_{v_n(p)}({}_{CG}L(z, p)) = 0$.[9] Also ${}_{v_n(p)}M = M\ \omega(\bar{v}_n(p) \cap R, z, R - R_0) \to 0$ as $n \to \infty$, whence $\lim_{n=\infty} {}_{v_n(p)}L(z, p) = 0$. This is a contradiction. Hence $p \in G(L, \varDelta) \cap {}_S\varDelta(R, L)$.

LEMMA 1. *Let G_1 be a domain in $R - R_0$ and G_2 be a regularly open set such that $G_1 \supset \bar{G}_2$ and there exists a harmonic function $V(z)$ in $G_1 - G_2$ with the property: $V(z) = 0$ on ∂G_1, $V(z) = 1$ on ∂G_2 and $D(V(z)) < \infty$. Let $\{G^n\}$ $n \geqq 3$ be a decreasing sequence of regularly open sets: $G_2 \supset G_n$. Then $\omega(\{\bar{G}_n\}, z, G_1)$ can be defined. Suppose $\omega(\{\bar{G}_n\}, z, G_1) > 0$. Let $U(z)$ be a non constant positive harmonic function in $R - R_0$ with finite Dirichlet integral such that sup $U(z) < \infty$ and $U(z) = {}_{CG_1}U(z)$ in G_1. Then there exists at least* *a point* z_0 *in* $G_1-\cap\bar{G}_n$ *such that*

$$U(z_0)<\sup_{z\in G_1} U(z)\Big(\omega(\{\bar{G}_n\}, z, G_1)\Big).$$

PROOF. Put $\omega(z)=\omega(\{\bar{G}_n\}, z, G_1)$. Then $\omega(z)=M\ \omega(V_M, z, G_1)$ on G_1-V_M: $V_M=E[z\in G_1:\ \omega(z)\geqq M]$: $0<M<1$ and $\omega(z)$ has M.D.I. over $V_{M_1}-\text{int } V_{M_2}$: $M_2>M_1$ among all harmonic functions with the same value as $\omega(z)$ on $\partial V_{M_1}+\partial V_{M_2}$. Whence $\omega_n(z)\to\omega(z)$ as $n\to\infty$, where $\omega_n(z)$ is a harmonic function in $(V_{M_1}-\text{int } V_{M_2})\cap R_n$ such that $\omega_n(z)=\omega(z)$ on $(\partial V_{M_1}+\partial V_{M_2})\cap R_n$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\partial R_n\cap(V_{M_1}-\text{int } V_{M_2})$. Let $\{f_n(z)\}$: $n=1,2,\cdots$ be a sequence of continuous functions on ∂V_M such that $|f_n(z)|<K<\infty$ and $f_n(z)\to f(z)$ as $n\to\infty$. Then

$$\int_{\partial V_M}\frac{\partial}{\partial n}\omega(z)ds=D\big(\omega(z)\big)=\lim_{n=\infty}\int_{\partial V_M\cap R_n}\frac{\partial}{\partial n}\omega_n(z)ds, \quad \text{and}$$

$$\int_{\partial V_M} f(z)\ \frac{\partial}{\partial n}\omega(z)ds=\lim_{n=\infty}\int_{\partial V_M\cap R_n} f_n(z)\frac{\partial}{\partial n}\omega_n(z)ds^{[10]} \quad \text{for almost}$$

all M: $0\leqq M\leqq 1$. Such ∂V_M is called a regular niveau curve. Now $U(z)={}_{CG}U(z)=\lim_{n=\infty} U_n(z)$, where $U_n(z)$ is a harmonic function in $G_1\cap R_n$ such that $U_n(z)=U(z)$ on $\partial G_1\cap R_n$ and $\frac{\partial}{\partial n}U_n(z)=0$ on $G_1\cap\partial R_n$. Hence by Green's formula

$$\int_{(\partial V_{M_1}+\partial V_{M_2})\cap R_n+\partial R_n\cap(V_{M_1}-V_{M_2})} U_n(z)\frac{\partial}{\partial n}\omega_n(z)ds = \int_{(\partial V_{M_1}+\partial V_{M_2})\cap R_n+\partial R_n\cap(V_{M_1}-V_{M_2})}\omega_n(z)\frac{\partial}{\partial n}U_n(z)ds$$

where ∂V_{M_1} and ∂V_{M_2} are regular niveau curves.

By $\displaystyle\int_{\partial V_{M_1}\cap R_n}\frac{\partial}{\partial n}U_n(z)ds=\int_{V_{M_1}\cap\partial R_n}\frac{\partial}{\partial n}U_n(z)ds=0=\int_{V_{M_2}\cap\partial R_n}\frac{\partial}{\partial n}U_n(z)ds=$

$\displaystyle\int_{\partial V_{M_2}\cap R_n}\frac{\partial}{\partial n}U_n(z)ds$ *we have*

$$\int_{\partial V_{M_1}\cap R_n} U_n(z)\frac{\partial}{\partial n}\omega_n(z)ds=\int_{\partial V_{M_2}\cap R_n} U_n(z)\frac{\partial}{\partial n}\omega_n(z)ds.$$

Let $n\to\infty$. Then

$$\int_{\partial V_{M_1}} U(z)\frac{\partial}{\partial n}\omega(z)ds=\int_{\partial V_{M_2}} U(z)\frac{\partial}{\partial n}\omega(z)ds.$$

Since $l=\sup_{z\in G_1} U(z)$, we can find a constant ε_0: $0<\varepsilon_0<1$ such that

$$\int_{\partial V_{M_1}} U(z)\frac{\partial}{\partial n}\omega(z)\,ds \leqq (1-\varepsilon_0)l \int_{\partial V_{M_1}} \frac{\partial}{\partial n}\omega(z)ds. \qquad (1)$$

Assume $U(z)\geqq l\omega(z)$ in G_1. Then $\int_{\partial V_{M_2}} U(z)\frac{\partial}{\partial n}\omega(z)ds \geqq \int_{\partial V_{M_2}} l\omega(z)\frac{\partial}{\partial n}\omega(z)ds=$ $lM_2\int_{\partial V_{M_2}} \frac{\partial}{\partial n}\omega(z)ds$.

Let $M_2\to 1$. Then

$$\int_{\partial V_{M_1}} U(z)\frac{\partial}{\partial n}\omega(z)ds \geqq \lim_{M_2\uparrow 1} lM_2 \int_{\partial V_{M_2}} \frac{\partial}{\partial n}\omega(z)ds = l\int_{\partial V_{M_1}} \frac{\partial}{\partial n}\omega(z)ds. \qquad (2)$$

(1) contradicts (2). Hence we have the lemma.

THEOEREM 3. *The mapping f in Theorem 2 is one to one from* $_{S,i}\Delta(R, L)\cap G(L, \Delta)$ *onto* $_{S,i}\Delta(G, N)$ $(i=1, 2)$.

PROOF. Let $p\in {}_S\Delta(R, L)\cap G(L, \Delta)$. Then $q=f(p)\in {}_S\Delta(G, N)$. Let $v_n(q)$ $=E[z\in G+\Delta(G, N)\colon \overset{N}{\delta}(z, q)<\frac{1}{n}]$. Put $\overset{*}{\omega}(q, z, R-R_0)=\lim_{n=\infty}\omega(\overline{v_n(q)}\cap R, z, R-R_0)$. Then $\overset{*}{\omega}(q, z, R-R_0)\geqq \omega(q, z, G)>0$. Since $\overset{*}{\omega}(q, z, R-R_0)$ is C.P. determined by a sequence $\{\overline{v_n(q)}\cap R\}$, $\overset{*}{\omega}(q, z, R-R_0)={}_{v_n(q)\cap R}\overset{*}{\omega}(q, z, R-R_0)$.[11] Hence $\overset{*}{\omega}(q, z, R-R_0)$ is represented by a canonical measure on $_1\Delta(R, L)\cap$ $(\bigcap_{n=1}^{\infty}\overline{R\cap v_n(q)}^{L})$, where $\overline{}^{L}$ means the closure of $R\cap \overline{v_n(q)}$ relative to L-top. Let $p'\in {}_1\Delta(R, L)\cap(\bigcap_{n=1}^{\infty}(\overline{R\cap v_n(q)}^{L}))$. Then there exists a sequence $\{p_i\}$ $(p_i\in G)$ such that $p_i\overset{L}{\to}p'$ and $p_i\overset{N}{\to}q$ and $L(z, p_i)-{}_{CG}L(z, p_i)=N(z, p_i)$. Let $i\to\infty$. Then

$$L(z, p')-{}_{CG}L(z, p')=N(z, q).$$

Since such p' is uniquely determined, $p'=p$. Hence $_1\Delta(R, L)\cap(\bigcap_{n=1}^{\infty}(\overline{R\cap v_n(q)}^{L}))$ $=p$ and $\overset{*}{\omega}(q, z, R-R_0)=a\ L(z, p)$: $0<a<\infty$. By $\sup\omega(p, z, R-R_0)=1=$ $\sup\overset{*}{\omega}(q, z, R-R_0)$[12] we have

$$\overset{*}{\omega}(q, z, R-R_0)=\omega(p, z, R-R_0). \qquad (3)$$

Hence

$$\omega(p, z, R-R_0)=\overset{*}{\omega}(q, z, R-R_0)\geqq\omega(q, z, G)=\lim_{n=\infty}\omega\Big(\bar{v}_n(q)\cap G, z, G\Big). \quad (4)$$

Pun $V'_{1-\varepsilon'}=E[z\in G: \omega(q, z, G)\geqq 1-\varepsilon'](0<\varepsilon'<1)$. Then

$$\lim_{\varepsilon'\to 0}\omega\Big(V'_{1-\varepsilon'}\cap(G-v_n(q)), z, G\Big)=0 \quad \text{for any given } v_n(q).^{[13]}$$

Hence

$$0=\lim_{\varepsilon'\to 0}\omega\Big(V'_{1-\varepsilon'}\cap(G-v_n(q)), z, G\Big)\geqq\lim_{\varepsilon'\to 0}W\Big(V'_{1-\varepsilon'}\cap(G-v_n(q)), z, G\Big), \quad (5)$$

where $W(V'_{1-\varepsilon'}\cap(G-v_n(q)), z, G)$ is H.M. of $V'_{1-\varepsilon'}\cap(G-v_n(q))$ relative to G. Since $W(V'_{1-\varepsilon'}, z, G)\leqq W(V'_{1-\varepsilon'}\cap\bar{v}_n(q), z, G)+W(V'_{1-\varepsilon'}\cap(G-v_n(q)), z, G)$, we have $\lim_{\varepsilon'\to 0}W(V'_{1-\varepsilon'}, z, G)\leqq W(\overline{v_n(q)}, z, G)$. Let $n\to\infty$. Then $\lim_{\varepsilon'\to 0}W(V'_{1-\varepsilon'}, z, G)\leqq W(q, z, G)$. On the other hand, for any $V'_{1-\varepsilon'}$, there exists a neighbourhood $v_n(q)$ such that $V'_{1-\varepsilon'}\supset(G\cap\bar{v}_n(q))$. Hence

$$\lim_{\varepsilon'\to 0}W(V'_{1-\varepsilon'}, z, G)=W(q, z, G). \quad (6)$$

Let $p\in_{S,2}\Delta(R, L)\cap G(L, \Delta)$. We shall show $f(p)=q\in_{S,2}\Delta(G, N)$. By (6) it is sufficient to show $\lim_{\varepsilon'\to 0}W(V'_{1-\varepsilon'}, z, G)>0$. Let $V_{1-\varepsilon}=E[z\in R-R_0: \omega(p, z, R-R_0)\geqq 1-\varepsilon]$: $0<\varepsilon<1$. Put $H_1(z)=\dfrac{\omega(p, z, R-R_0)}{1-\varepsilon}$. Then $H_1(z)\geqq 1$ on $V_{1-\varepsilon}$, $H_1(z)=0$ on ∂R_0 and $D(H_1(z))<\infty$. Let $V'_{1-\varepsilon'}=E[z\in G: \omega(q, z, G)\geqq 1-\varepsilon']$. Then for any given $\varepsilon'>0$ there exists a neighbourhood $v_n(q)$ such that $(\bar{v}_n(q)\cap G)\subset V'_{1-\frac{\varepsilon'}{2}}$. Put $H_2(z)=\dfrac{\left(1-\frac{\varepsilon'}{2}\right)-\omega(q, z, G)}{\frac{\varepsilon'}{2}}$. Then $H_2(z)\leqq 0$ on $V'_{1-\frac{\varepsilon'}{2}}$, $H_2(z)\geqq 1$ on $G-\text{int}\, V'_{1-\varepsilon'}$ and $D(H_2(z))<\infty$. Hence there exists at least a piecewise smooth function $H_3(z)$ in $R-R_0-v_n(q)$ such that $H_3(z)=0$ on $R\cap v_n(q)$, $H_3(z)=1$ on $R-R_0-\text{int}\, V'_{1-\varepsilon'}$ and $D(H_3(z))<\infty$. Put $U_{\varepsilon,\varepsilon',n}(z)=\min(H_2(z), H_3(z))$. Then $U_{\varepsilon,\varepsilon',n}(z)$ is a piecewise smooth function in $R-R_0-v_n(q)$ such that $U_{\varepsilon,\varepsilon',n}(z)=0$ on $\partial R_0+\partial(v_n(q)\cap R)$ and $=1$ on $V_{1-\varepsilon}-\text{int}\, V'_{1-\varepsilon'}$ and $D(U_{\varepsilon,\varepsilon',n}(z))<\infty$. Hence we can define C.P. $\omega(\{V_{1-\varepsilon}-\text{int}\, V'_{1-\varepsilon'}\}, z, R-R_0-v_n(q))$ determined by a sequence $\{(V_{1-\varepsilon}-\text{int}\, V'_{1-\varepsilon'})\}$ relative to $R-R_0-v_n(q)$, where $\varepsilon\to 0$ and ε' is fixed. Put $\overset{*}{\omega}_{\varepsilon',n}(z)=\omega(\{V_{1-\varepsilon}\,\text{int}\, V'_{1-\varepsilon'}\}, z, R-R_0-v_n(q))$. Assume $\overset{*}{\omega}_{\varepsilon',n}(z)>0$. Since $\overset{*}{\omega}(p, z, R-R_0)=\lim_{n=\infty}\omega(R\cap\bar{v}_n(q), z, R-R_0)$ by (3), $\omega(p, z, R-R_0)$ has M.D.I. over $R-R_0-v_n(q)$ with the same value as $\omega(p, z, R-R_0)$ on $\partial R_0+\partial(v_n(q)\cap R)$. Hence by Lemma 1 there exists a point z_0 such that

$$\omega(p, z_0, R-R_0)<\overset{*}{\omega}_{\varepsilon,n}(z_0).$$

On the other hand, $\omega(p, z, R-R_0)=\lim_{\varepsilon\to 0}\dfrac{\omega(V_{1-\varepsilon}, z, R-R_0)}{1-\varepsilon}\geqq\overset{*}{\omega}_{\varepsilon',n}(z)$ in $R-R_0-v_n(q)$. This is a contradiction. Hence $\overset{*}{\omega}_{\varepsilon',n}(z)=0$.

By the Dirichlet principle $D(\omega(V_{1-\varepsilon}-\text{int } V'_{1-\varepsilon'}, z, R-R_0))\leqq D(\omega(V_{1-\varepsilon}\text{ int } V'_{1-\varepsilon'}, z, R-R_0-v_n(q)))\to 0$ as $\varepsilon\to 0$, whence

$$0=\lim_{\varepsilon\to 0}\omega(V_{1-\varepsilon}-\text{int } V'_{1-\varepsilon'}, z, R-R_0)\geqq\lim_{\varepsilon\to 0} W(V_{1-\varepsilon}-\text{int } V'_{1-\varepsilon'}, z, R-R_0)$$
$$\geqq\lim_{\varepsilon\to 0} W((V_{1-\varepsilon}-\text{int } V'_{1-\varepsilon'})\cap G, z, G)\geqq 0.$$

On the other hand, we proved, if $p\in_{S,2}\Delta(R, L)\cap G(L, \Delta)$, then

$$W(p, z, G)=\lim_{n=\infty} W(G\cap\overline{v_n(p)}, z, G)^{[14]}>0,$$

where $v_n(p)=E[z\in R-R_0+\Delta(L, R)\colon\ \delta^L(z, p)<\frac{1}{n}]$.

Since for any $V_{1-\varepsilon}$ there exists a $v_n(p)$ such that $(R-R_0)\cap\overline{v_n(p)}\subset V_{1-\varepsilon}$,

$$\lim_{\varepsilon\to 0} W(V_{1-\varepsilon}\cap G, z, G)>0. \qquad (7)$$

Now $0<\lim_{\varepsilon\to 0} W(G\cap V_{1-\varepsilon}, z, G)\leqq W(G\cap V'_{1-\varepsilon'}, z, G)+\lim_{\varepsilon\to 0} W((V_{1-\varepsilon}-\text{int } V'_{1-\varepsilon'})\cap G, z, G)$. Hence by (6) and (7) we have

$$W(q, z, G)=\lim_{\varepsilon\to 0} W(V'_{1-\varepsilon'}\cap G, z, G)\geqq 0 \text{ and } q\in_{S,2}\Delta(G, N).$$

Next suppose $q\in_{S,2}\Delta(G, N)$ and let $p=f^{-1}(q)$. Then by (4) $\omega(p, z, R-R_0)\geqq\omega(q, z, G)$.

By (6) we have

$$W(p, z, R-R_0)=\lim_{\varepsilon\to 0} W(V_{1-\varepsilon}, z, R-R_0)\geqq\lim_{\varepsilon\to 0} W(V'_{1-\varepsilon'}, z, G)$$
$$=W(q, z, G)>0,$$

where $E[z\in G\colon\ \omega(q, z, G)\geqq 1-\varepsilon]=V'_{1-\varepsilon}\subset V_{1-\varepsilon}=E[z\in R-R_0\colon\ \omega(p, z, R-R_0)\geqq 1-\varepsilon]$. Hence $q\in_{S,2}\Delta(G, N)$ implies $p\in_{S,2}\Delta(R, L)\cap G(L, \Delta)$ and the mapping f is one to one from ${}_{S,2}\Delta(R, L)\cap G(L, \Delta)$ onto ${}_{S,2}\Delta(G, N)$. By ${}_{S,1}\Delta={}_S\Delta-{}_{S,2}\Delta$ and by Theorem 2 we have the theorem.

LEMMA 2. *Let G be a domain in $R-R_0$ such that $G\overset{L}{\ni}p\colon p\in_S\Delta(R, L)$. Then we can find another domain G' such that $\bar{G}'\subset G$ and $G'\overset{L}{\ni}p$.*

PROOF. Let $q=f(p)$ and let $G'=E[z\in G\colon\ N(z, q)>\varepsilon]\colon 0<\varepsilon<\sup N(z, q)$. Since every point of ∂G is regular for the Dirichlet problem in G, $\partial G\cap\partial G'=0$. We shall show $G'\overset{L}{\ni}p$. By the Dirichlet principle $D(\omega(G', z, R-R_0))$

$\leqq \frac{1}{\varepsilon^2} D(\min(\varepsilon, N(z,q))) \leqq \frac{2\pi}{\varepsilon}$. Put $\omega'(z) = \omega(G', z, R-R_0)$ in $R-R_0-G'$, $=1$ in G'. Then ${}_{CG}L(z,p)+\varepsilon\omega'(z) \geqq L(z,p)$ on $R-R_0-G'$ and $L(z,p)+\varepsilon\omega'(z)$ is full-superharmonic in $R-R_0$, whence ${}_{CG}L(z,p) \leqq {}_{CG}L(z,p)+\varepsilon$, $L(z,p) \neq {}_{CG'}L(z,p)$ and $G' \overset{L}{\ni} p$.

Let G_0 be a disk in G. When we regard G as a Riemann surface, we can define L'-topology over $G-G_0+\varDelta(G, L')$ introduced by N-Green's function $L'(z,p)$, $p \in (G-G_0)+\varDelta(G, L')$ of $G-G_0$ vanishing on ∂G_0. As L-topology over $R-R_0+\varDelta(R, L')$, the sets, $\varDelta(G, L')$, and $G'(L', \varDelta)$ are defined, where G' is a subdomain of $G-\bar{G}_0$. Then we have the following

THEOREM 4. *Let G_0 be a disk in G. Then there exists a one to one mapping $\hat{f}$ from ${}_{S,i}\varDelta(G, N)$ into ${}_{S,i}\varDelta(G, L')$ $(i=1, 2)$.*

PROOF. Let $q \in {}_{S,i}\varDelta(G, N)$. Since G is a domain in $R-R_0$, there exists a uniquely determined point $p \in {}_{S,i}\varDelta(R, L) \cap G(L, \varDelta)$ such that

$$L(z,p) - {}_{CG}L(z,p) = N(z,q): \quad q = f(p).$$

We can find a domain $G_1 \subset G$ such that $\bar{G}_1 \subset G$ and $G_1 \overset{L}{\ni} p$. Since the fact $G_1 \overset{L}{\ni} p$ depends only the behaviour of G_1 in a neighbourhood of the boundary $\varDelta(R, L)$, we can suppose without loss of generality that the above mentioned G_1 satisfies the condition $\bar{G}_1 \cap \bar{G}_0 = 0$. Since $G_1 \overset{L}{\ni} p$, there exists one to one mapping f_1 from ${}_{S,i}\varDelta(R, L) \cap G(L, \varDelta)$ onto ${}_{S,i}\varDelta(G_1, N_1)$ such that

$$L(z,p) - {}_{CG_1}L(z,p) = N_1(z, q_1), \quad q_1 \in {}_{S,i}\varDelta(G_1, N_1), \quad q_1 = f_1(p), \tag{8}$$

where $N_1(z,p)$ is an N-Green's function of G_1 vanishing on ∂G_1. Since G_1 is a subdomain of G, there exists a mapping g_1 from ${}_{S,i}\varDelta(G, L') \cap G_1(L', \varDelta)$ onto ${}_{S,i}\varDelta(G_1, N_1)$ and

$$\begin{aligned} &L'(z,p^1) - {}_{G-G_1}L'(z,p^1) = N_1(z, q_1), \quad q_1 \in {}_{S,i}\varDelta(G_1, N_1), \\ &p^1 \in {}_{S,i}\varDelta(G, L') \cap G_1(R', \varDelta), \quad q_1 = g_1(p^1). \end{aligned} \tag{8'}$$

Let G_2 be another domain in G such that $\bar{G}_2 \subset G-G_0$ and $G_2 \overset{L}{\ni} p$. Then as above

$$\begin{aligned} &L(z,p) - {}_{CG_2}L(z,p) = N_2(z, q_2), \quad q_2 \in {}_{S,i}\varDelta(G_2, N_2), \quad q_2 = f_2(p), \\ &L'(z,p^2) - {}_{G-G_2}L'(z,p^2) = N_2(z, q_2), \quad q_2 = g_2(p^2), \\ &p^2 \in {}_{S,i}\varDelta(G, L') \cap G_2(L', \varDelta), \end{aligned}$$

where $N_2(z,p)$ is an N-green's function of G_2 vanishing on ∂G_2. We shall show $p^1 = p^2$, in other words p^i $(i=1,2,)$ does not depend on the choice of the auxiliary domain G_i under the condition that $\bar{G}_i \cap \bar{G}_0 = 0$. To prove it we use a supplementary domain G_3, where $G_3=G_1\cap G_2$. Then $G_3\overset{L}{\ni}p$ (because two thin sets is also thin). We have

$$L(z,p)-{}_{CG_3}L(z,p)=N_3(z,q_3),\quad q_3\in{}_{S,i}\Delta(G_3,N_3),\quad q_3=f_3(p),\tag{9}$$

$$L'(z,p^3)-{}_{G-G_3}L'(z,p^3)=N_3(z,q_3)\text{ and}$$
$$p^3\in{}_{S,i}\Delta(G,L')\cap G_3(L',\Delta),\quad q_3=g_3(p^3),\tag{10}$$

where $N_3(z,q)$ is an N-green's function of G_3 vanishing on ∂G_3. Consider $L'(z,p^1)$ in G_3. Since $L'(z,p^1)={}_{G-G_1}L'(z,p^1)+N_1(z,q_1)$ on $\bar{G}_3$ and ${}_{G-G_3}({}_{G-G_1}L'(z,p^1))={}_{G-G_1}L'(z,p^1)$, we have ${}_{G-G_3}L'(z,p^1)={}_{G-G_1}L'(z,p^1)+{}_{G-G_3}N_1(z,q_1)$ and

$$L'(z,p^1)-{}_{G-G_3}L'(z,p^1)=N_1(z,q_1)-{}_{G-G_3}N_1(z,q_1).\tag{11}$$

Similarly we have $L(z,p)-{}_{CG_3}L(\ z,p)=N_1(z,q_1)-{}_{G-G_3}N(z,q_1)$. Hence by (11), (9) and (10)

$$L'(z,p^1)-{}_{G-G_3}L'(z,p^1)=N_3(z,q_3)=L'(z,p^3)-{}_{G-G_3}L'(z,p^3).\tag{12}$$

However by Theorem 3 there exists a one to one mapping φ form ${}_{S,i}\Delta(G,L')\cap G_3(L',\Delta)$ onto ${}_{S,i}\Delta(G_3,N_3)$. Hence by (12) $\varphi(p^1)=q=\varphi(p^3)$ and $p^1=p^3$. Similary $p^2=p^3$. Hence $g_2^{-1}f_2f^{-1}(q)=g_1^{-1}f_1f^{-1}(q)$. We define the mapping $\hat{f}=g_j^{-1}\cdot f_j\cdot f^{-1}$ for any domain G_j in G such that $\partial G_j\subset G$, where f_j is one to one mapping from ${}_{S,i}\Delta(R,L)\cap G_j(L,\Delta)$ onto ${}_{S,i}\Delta(G_j,N_j)$ and g_j is from ${}_{S,i}\Delta(G,L')\cap G_j(L',\Delta)$ onto ${}_{S,i}\Delta(G_j,N_j)$. Then $\hat{f}$ does not depend on the domain G_j. We shall show $\hat{f}(q_1)\neq\hat{f}(q_2)$ for $q_1,q_2\in{}_{S,i}\Delta(G,N)$ and $q_1\neq q_2$. Let $G_1\overset{L}{\ni}f^{-1}(q_1)$ and $G_2\overset{L}{\ni}f^{-1}(q_2)$, $\bar{G}_1\subset G-G_0$ and $\bar{G}_2\subset G-G_0$. We can find a relatively compact domain G' in $G-\bar{G}_0$ such that $G'\cap G_1\neq 0$, $G\cap G_2\neq 0$. Put $G_3=G_1+G_2+G'$. Then $G_3\overset{L}{\ni}f^{-1}(q_1)$, $G_3\overset{L}{\ni}f^{-1}(q_2)$ and there exists one to one mappings f_3 and g_3 from ${}_{S,i}\Delta(R,L)\cap G_3(L,\Delta)$ onto ${}_{S,i}\Delta(G_3,N_3)$ and from ${}_{S,i}\Delta(G,L')\cap G_3(L',\Delta)$ onto ${}_{S,i}\Delta(G_3,N_3)$ respectively. Then we see at once $\hat{f}(q_1)\neq\hat{f}(q_2)$. Hence the mapping $\hat{f}$ is one to one and from ${}_{S,i}\Delta(G,N)$ into ${}_{S,i}\Delta(G,L')$ and the theorem is proved.

Let G be a domain in $R-R_0$. We suppose L'-top. over $G-G_0$ and L-top over $R-R_0$ are defined, where G_0 and R_0 are disks in R and G respectively. The identity mapping z (relative to L'-top.)$\to z$ (relative to L-top.) can be considered as an analytic function $z=f(z)$. Now $f(z)$ is a almost finitely sheeted covering.[15] Hence by Beurling's theorem

$$\bigcap_{\tau}\overline{f(G_\tau)}=\text{one point in }R+\Delta(R,L),\tag{13}$$

for $z\in{}_1\Delta(G,L')$ except at most a set of capacity zero, where $\overline{f(G_\tau)}$ is the closure of $f(G_\tau)$ relative to L-top. and $\{G_\tau\}$ runs over all domains G_τ such that $G_\tau \overset{L'}{\ni} p$. Let $p \in_S \Delta(G, L')$. Then $\bigcap_\tau \overline{f(G_\tau)}$ is one point by Cap $(p)>0$ $(\omega(p, z, G-G_0)>0)$. Put $f(p)=\bigcap_\tau \overline{f(G_\tau)}$. Then we have the following.

THEOREM 5. *If* $p \in_{S,i} \Delta(G, L')$, *then* $f(p) \in_{S,i} \Delta(R, L)$.

PROOF. Although this theorem can be deduced from Beurlings and Riesz's theorems, we shall prove the theorem more explicity. Let $p \in_S \Delta(G, L')$ and put $q=f(p)$. Let $G_\tau \overset{L'}{\ni} p$. Then

$$\omega\Big((G-G_\tau)\cap p, z, G-G_0\Big)=\lim_{n=\infty}\omega\Big((G-G_\tau)\cap \bar{v}_n(p), z, G-G_0\Big)=0, \quad (14)$$

where $v_n(p)$ is a neighbourhood of p pelative to L'-top. By (13) for any $v_n(q)$ there exists a domain G_τ such that $G_\tau \overset{L'}{\ni} p$ and $\bar{G}_\tau \subset v_n(q)$, where $\bar{G}_\tau$ is the closure of G_τ relative to L-top. and $v_n(q)$ is a neighbourhood of q relative to L-top. Hence by (14)

$$0<\omega(p, z, G-G_0)=\lim_{n=\infty}\omega\Big((\bar{v}_n(p)\cap G)\cap \bar{G}_\tau, z, G-G_0\Big)\leqq$$

$$\lim_n \omega\Big(\bar{v}_n(q)\cap G, z, G-G_0\Big).$$

Let $U_n(z)$ be a harmonic function in $R-R_0-G_0-v_n(q)$ such that $U_n(z)=1$ on $\partial(v_n(q)\cap R),=0$ on $\partial G_0+\partial R_0$ and $U_n(z)$ has M.D.I. (because $\partial G_0+\partial R_0$ is compact and $\bar{v}_n(q)\cap(\bar{R}_0+G_0)=0$). Since $U_n(z)=0$ on ∂G_0 and $=1$ on $\partial v_n(q) \cap R$, $D(\omega(\bar{v}_n(q)\cap G, z, G-G_0))\leqq$Dirichlet integral of $U_n(z)$ over $G-G_0-v_n(q)$ $\leqq D(U_n(z))$. Clearly $U_n(z)=\omega(\overline{v_n(q)}\cap R, z, R-R_0-G_0)$. Let $n\to\infty$. Then $U_n(z)$ converges locally uniformly and in Dirichlet integral to $\omega(q, z, R-R_0-G_0)$ and $0<D(\omega(q, z, G-G_0))\leqq D(\lim_n \omega(v_n(q), z, R-R_0-G_0))$ and $\omega(q, z, R-R_0-G_0)>0$. On the other hand, since G_0 is compact in R, $\omega(q, z, R-R_0)>0$ if and only if $\omega(q, z, R-R_0-G_0)>0$. Hence $\omega(q, z, R-R_0)>0$. This shows that $q=f(p)\in_S \Delta(R, L)$ from $p \in_S \Delta(G, L')$.

Let $p \in_{S,2}\Delta(G, L')$. Then by $W((G-G_\tau)\cap p, z, G-G_0)\leqq \omega((G-G_\tau)\cap p, z, G-G_0)=0$ we have $0<W(p, z, G-G_0)=W(p\cap G_\tau, z, G-G_0)\leqq W(G_\tau, z, R-R_0)\leqq W(\overline{v_n(q)}\cap R, z, R-R_0)$. Let $n\to\infty$. Then

$$W(q, z, R-R_0)>0 \quad \text{and} \quad q\in_{S,2}\Delta(R, L).$$

Analytic functions defined in neighbourhoods of singular points. In the following we suppose the value of analytic functions $w=f(z)$ falls on the w-sphere. In the previous paper[16] we proved the following two theorems:

THEOREM 6. *Let* $p \in_S \Delta(R, L)$ *and* G *be a domain in* R *such that* $G \overset{L}{\ni} p$.

Then on G there exists no analytic function $w=f(z)$ such that the image of G by $f(z)$ is a covering surface of a finite number of sheets over the w-sphere.

THEOREM 7. *Let $p\in{}_{S,2}\Delta(R, L)$ and G be a domain in R such that $G\overset{L}{\ni}p$. Then on G there exists no analytic function $w=f(z)$ such that the image of G by $f(z)$ has a finite spherical area.*

Suppose there exists a point $p\in{}_{S,i}\Delta(R, L)\cap G(L, \Delta)$ $(i=1,2)$. Then by Theorem 3 there exists a uniquely determined point $q\in{}_{S,i}\Delta(G, L')$ relative to L'-top. over $G-G_0$, where G_0 is a disk in G and G is considered as a Riemann surface. Then the above two theorems will be proved more easily than the previous proofs.

Let w_0 be a point in the Riemann sphere S_w and $A(r)$ be the spherical area of the image of G by $f(z)$ over a spherical circle: $E[w$: spherical distance $(w, w_0)<r]$.

If
$$\varliminf_{r\to 0}\frac{A(r)}{r^2}<\infty,$$
w_0 is called an ordinary point.

We shall prove the following theorem which is an extension of Theorem 6.

THEOREM 8. *Let R be a Riemann surface and let $p\in{}_S\Delta(R, L)$. Let G be a domain in R with the following property: $G\overset{L}{\ni}p$ relative to L-topology over $R-R_0$. Then there exists no analytic function $f(z)$ in G such that every point w in S_w is an ordinary point with respect to the image of G by $f(z)$.*

PROOF. By Theorem 2 and 3 the existence of a point $p\in{}_S\Delta(R, L)\cap G(L, \Delta)$ implies the existence of a point $q\in{}_S\Delta(G, L')$. Let $f(z)$ be an analytic function in G such that every point w is an ordinary point. We can apply the Beurling's theorem in regarding G as a Riemann surface. Since every point w is an ordinary point, the spherical area of the image of G is finite and $f(z)$ has a fine limit $w_0\in S_w$ at q by $q\in{}_S\Delta(G, L')$. On the other hand, w_0 is an ordinary point. Hence by Beurling's[17] theorem $f(z)$ must be a constant$=w_0$. This is a contradiction and we have Theorem 8.

THEOREM 9. *Let G_j $(j=1,2,\cdots)$ be a domain in R such that $\{G_j\}$ clusters nowhere in R, $G_i\cap G_j=0$ for $i\neq j$ and $G_i\cap \bar{R}_0=0$ and*
$$\omega\Big(\big\{(R-G)\cap(R-R_n)\big\}, z, R-R_0\Big)=0,\quad G=\sum G_j.$$
Then ${}_{S,i}\Delta(R, L)\neq 0$ if and only if ${}_{S,i}\Delta(G_j, L_j)\neq 0$ for some j $(i=1,2)$, where

L_j is L top. over $G_j-G_{j,0}$ and $G_{j,0}$ si disk in G_j.

PROOF. The "if" part of the theorem is deduced from Theorem 5 directly. It is sufficient to prove the "only if" part. Let $p\in_{S,i}\Delta(R, L)$ and z_0 be a point in $R-R_0$. Let δ be a positive number such that

$$\omega(p, z_0, R-R_0)>\delta>0\,.$$

By the assumption for any given positive number ε $(<\delta)$, there exists a number $n_0=n(\varepsilon, z_0)$ such that $\omega((R-G)\cap(R-R_{n_0})z_0, R-R_0)<\varepsilon$. Hence by $\omega(p, z, R-R_0)\leqq 1$

$$\omega(p, z_0, R-R_0)-\underset{(R-G\cap(R-R_{n_0})}{\omega(p, z_0, R-R_0)}>\delta-\varepsilon>0\,.$$

This implies $p\overset{L}{\in}(G+R_{n_0})$ and $G\overset{L}{\ni}p$, since R_{n_0} is compact. There exists the only one component G_j of G such that $G_j\overset{L}{\ni}p$. Hence by Theorem 2 and 3 there exists a point q(corresponding to $p)\in{}_{S,i}\Delta(G_j, L_j)$. Thus we have the theorem.

Let G be a domain in $R-R_0$ and G^∞ be its universal covering surface. Map G^∞ conformally onto $|\xi|<1$ by $\xi=\xi(z)$. Let $N(z, p_0)$: $p_0\in G$ be an N-Green's function such that $N(z, p_0)=0$ on ∂G. Then $N(z, p_0)$ has angular limits almost everywhere on Γ: $|\xi|=1$. ∂G is mapped ont a set A on Γ. Let E_N be the set in $(\Gamma-A)$ where $N(z, p_0)$ has angular limits$=0$. Clearly E_N dose not depend on the point p_0. Then we proved[18] that every Dirichlet finite harmonic function vanishing on ∂G has angular limits$=0$ in E_N. Let $W(z)=W(\{G\cap(R-R_n)\}, z, G)$. Let E_k be the set of $\Gamma-A$ where $W(z)$ has angular limits$=0$. Then there exists a set e on Γ such that mes $e=0$ and $E_k-e\subset E_N$.

THEOREM 10. *Let G_j $(j=1, 2, \cdots)$ be a domain in $R-R_0$ such that $\{G_j\}$ clusters nowhere in R, $G_i\cap G_j=0$ for $j\neq i$ and $\bar{R}_0\cap\bar{G}_j=0$. Put $G=\sum G_j$. Suppose*

$$W\Big(\big\{(R-G)\cap(R-R_n)\big\}, z, R-R_0\Big)=0 \quad \textit{and}$$

$$\textit{mes}\, E_N=\textit{mes}\ E_k \quad \textit{for any } G_j\,.$$

Then ${}_{S,2}\Delta(R, L)\neq 0$ if and only if ${}_{S,2}\Delta(G_j, L_j)\neq 0$ for some j, where L_j is L-top, over G_j.

At first we shall prove the following.

LEMMA 3. *Let G be a damin in $R-R_0$. Let $\{f_n(z)\}$: $f_n(z)\geqq 0$ be a sequence of continuous functions on ∂G such that $f_n(z)\downarrow f(z)\geqq 0$ and $f(z)$ is continuous. Let ${}_{f_n}U(z)$ $({}_fU(z))$ be the least positive harmonic function in G such that ${}_{f_n}U(z)=f_n(z)\,({}_fU(z)=f(z))$ on ∂G. Then*

$$\lim_{n} {}_{f_n}U(z) = {}_{f}U(z).$$

In fact, let $U_{n,m}(z)$ be harmonic function in $G\cap R_m$ such that $U_{n,m}(z)=f_n(z)$ on $\partial G\cap R_m$ and $=0$ on $\partial R_m\cap G$. Then $U_{n,m}(z)=\frac{1}{2\pi}\int_{\partial G\cap R_m} f_n(\zeta)\frac{\partial}{\partial n}G_m(\zeta, z)\,ds$, where $G_m(\zeta, z)$ is a Green's function of $G\cap R_m$. Let $G(\zeta, z)$ be a Green's fuction of G. Then $\frac{\partial}{\partial n}G_m(\zeta, z)\uparrow\frac{\partial}{\partial n}G(\zeta, z)$ on ∂G. Hence $\lim_{m=\infty} U_{n,m}(z)={}_{f_n}U(z)=\frac{1}{2\pi}\int_{\partial G} f_n(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds$. Since $f_n(z)\downarrow f(z)$, we have by Lebesgue's theorem

$${}_{f}U(z)=\frac{1}{2\pi}\int_{\partial G} f(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\mathrm{ds}=\lim_{n} {}_{f_n}U(z).$$

PROOF. of the theorem. As Theorem 9, in is sufficient to prove the "only if" part. Let D be a domain in $R-R_0$ and let $U(z)$ be a positive harmonic function in $R-R_0$. Let ${}_{\widehat{CD}}U(z)$ be the lower envelope of non negative superharmonic functions in $R-R_0$ larger than $U(z)$ on CD. Let $q\in_{s,2}\Delta(R, L)$, then $W(q, z, R-R_0)=\lim_{n} W(v_n(q)\cap R, z, R-R_0)>0$. By the assumption, for any point z_0 in $R-R_0$ and any positive number ε, there exists a number $n_0=n(z_0, \varepsilon)$ such that $W((R-G)\cap(R-R_{n_0}), z_0, R-R_0)<\varepsilon< W(q, z_0, R-R_0)$. Since $W(q, z, R-R_0)\leqq 1$, we have ${}_{\widehat{C\tilde{G}}}W((q, z_0, R-R_0)< W(R-G)\cap(R-R_{n_0}), z_0, R-R_0)$ and

$${}_{\widehat{C\tilde{G}}}W(q, z, R-R_0)\not\equiv W(q, z, R-R_0), \tag{15}$$

where $\tilde{G}=G+(R_{n_0}-R_0)$.

Since ${}_{\widehat{C\tilde{G}}}W(R\bar{v}_n(q), z, R-R_0)$ is the least positive harmonic function in $\tilde{G}$ larger than $W(R\cap\bar{v}_n(q), z, R-R_0)$ on $\partial\tilde{G}$,

$$W(\overline{v_n(q)}\cap\tilde{G}, z, \tilde{G})\geqq W(R\cap\overline{v_n(q)}, z, R-R_0)-{}_{\widehat{C\tilde{G}}}W(R\cap\overline{v_n(q)}, z, R-R_0).$$

Consider $W(\overline{v_n(q)}, z, R-R_0)$ as $f_n(z)$ on $\partial\tilde{G}$ in Lemma 3, then we have by (15) and by letting $n\to\infty$

$$W(q, z, \tilde{G})\geqq W(q, z, R-R_0)-{}_{\widehat{C\tilde{G}}}W(q, z, R-R_0)>0. \tag{16}$$

Let $V_{n_0}(z)$ be the least positive harmonic function in $\tilde{G}$ such that $V_{n_0}(z)=1$ on $\partial G\cap R_{n_0}$. Let $\tilde{V}_{n_0}(z)$ be the least positive harmonic function in $R-R_{n_0}$ such that $\tilde{V}_{n_0}(z)=1$ on ∂R_{n_0}. Then $V_{n_0}(z)\leqq\tilde{V}_{n_0}(z)$ on $G-R_{n_0}$. Since R is a Riemann surface with positive boundary, $\sup \tilde{V}_{n_0}(z)<K$: $K<1$ in $R-R_{n_0+1}$ by $\partial R_{n_0+1}\cap\bar{R}_{n_0}=0$. Now $W(q, z, \tilde{G})>0$ implies $\sup W(q, z, \tilde{G})=1$. Hence there exists at least a point z_0 in $G-R_{n_0+1}$ such that $W(q, z_0, \tilde{G})-\tilde{V}_{n_0}(z_0)>0$. Since $G-R_{n_0+1}$ consists of $G_i-R_{n_0+1}$, there exists at least a component $G_j-R_{n_0+1}$ containing z_0. Then $W(q, z_0, \tilde{G})>\tilde{V}_{n_0}(z_0)$ and $W(q, z, G_j)\geqq W(q, z, \tilde{G})-V_{n_0}(z)$ and

$$W(q, z, G_j)>0\,. \tag{17}$$

Let $N(z, p_0)$ and $G(z, p_0)$: $p_0\in G_j$ be a N-Green's function and Green's function of G_j respectively. Put $\Omega_\delta=E[z\in G_j:\ N(z, \mathrm{p}_0)>\delta]$ and $\tilde{\Omega}_\alpha=E[z\in G_j:\ G(z, p_0)>\alpha]$. Under the condition that mes $E_N=$ mes E_k we shall show

$$\lim_{\delta\to 0}\lim_{n=\infty} W\Big(C\Omega_\delta\cap(R-R_n), z, G_j\Big)=0\,.$$

Put $\overset{*}{W}(z)=\lim\limits_{\delta\to 0}\lim\limits_{n=\infty} W(C\Omega_\delta\cap R-R_n), z, G_j)$ and assume $\overset{*}{W}(z)>0$. Map G_j^∞, the universal covering surface of G_j conformally onto $|\xi|<1$ by $\xi=\xi(z)$. Let A be the image of ∂G_j on Γ: $|\xi|=1$. Then since $0<\overset{*}{W}(z)\leqq 1$, $\overset{*}{W}(z)=0$ on A and has angular limits$=0$ a.e. (almost everywhere) on E_k. By $E_N\supset E_k-e$ (e is a set with mes $e=0$) and mes $(E_N-E_k)=0$, we can find a set E in $\Gamma-A-E_N$ such that mes $E>0$, both $\overset{*}{W}(z)$ and $N(z, p_0)$ have angular limits >0 a.e. on E. Hence for any given positive number ε: $3\varepsilon<$mes E, we can find a set E_1 in E such that mes $(E-E_1)<\varepsilon$, $\overset{*}{W}(z)$ and $N(z, p_0)$ converge uniformly and $G(z, p_0)$ converges uniformly to zero as $\xi\to e^{i\theta}\in E$ inside of an angular domain $A(\theta)$: $A(\theta)=E\Big[\xi:\ \Big|\arg\dfrac{\xi-e^{i\theta}}{e^{i\theta}}\Big|<\dfrac{\pi}{4}\Big]$. Also we can find a closed set E_2 in E_1 and const. $\delta>0$ such that mes $(E-E_2)<2\varepsilon$ and

$$N(z, p_0)>\delta \ \text{ and } \ \overset{*}{W}(z)>\delta \ \text{ in } C_m\cap(\bigcup_{e^{i\theta}\in E_2} A(\theta))\,, \tag{18}$$

where C_m: $1-\dfrac{1}{m}<|\xi|<1$.

$C_m\cap(\bigcup\limits_{e^{i\theta}\in E_2} A(\theta))$ consists of a finite number of components. Let D be a component such that mes $(\partial D\cap\Gamma)>0$. We consider $\overset{*}{W}(z)$ in D. Let γ_α be the image of $\partial\tilde{\Omega}_\alpha$ and let Γ_m: $|\xi|=1-\dfrac{1}{n}$. Then $\gamma_\alpha\cap\partial D\cap\Gamma_m=0$ for $\alpha<\min\limits_{z\in\partial D\cap\Gamma_m} G(z, p_0)$ and since $G(z, p_0)\to 0$ as $\xi\to\Gamma$ in D, γ_α must separate $\bar{D}\cap\Gamma$ from $\partial D\cap\Gamma_m$. By (18) the image of $\partial\Omega_\delta$ does not fall in D. Hence

$$W(C\Omega_\delta, z, \tilde{\Omega}_\alpha)\leqq W(\xi) \ \text{ on } D\cap\tilde{\Omega}_\alpha,\ \alpha<\min_{z\in\partial D\cap\Gamma_m} G(z, p_0) \tag{19}$$

where $W(C\Omega_\delta, z, \tilde{\Omega}_\alpha)$ is the H.M. of $C\Omega_\delta$ relative to $\tilde{\Omega}_\alpha$ and $W(\xi)$ is a harmonic function in D such that $W(\xi)=1$ on $\partial D-\Gamma$ and $=0$ a.e. on $\partial D\cap\Gamma$. Since ∂D is rectifiable, $W(\xi)=0$ a.e. on $\partial D\cap\Gamma$. Since $\tilde{\Omega}_\alpha\nearrow G_j$ as $\alpha\to 0$ and $W(C\Omega_\delta, z, \tilde{\Omega}_\alpha)\to W(C\Omega_\delta, z, G_j)$ as $\alpha\to 0$,

$$\overset{*}{W}(z)\leqq W\left(C\Omega_\delta\cap(R-R_n), z, G_j\right)\leqq W(C\Omega_\delta, z, G_j)\leqq W(\xi).$$

Hence $\overset{*}{W}(z)$ has angular limits $=0$ a.e. on $\partial D\cap\Gamma$. This contradicts (18) and we have

$$\overset{*}{W}(z)=0. \tag{20}$$

Now $\lim_{m=\infty} W(\overline{v_n(q)}\cap C\Omega_\delta\cap(R-R_m), G_j)+\lim_{m=\infty} W(\overline{v_n}(q)\cap\tilde{\Omega}_\delta\cap(R-R_m), z, G_j)$

$$\lim_{m=\infty} W\left(\overline{v_n(q)}\cap(R-R_m), z, G_j\right)\geqq W(q, z, G_j)>0.$$

By (20) there exists a number $\delta>0$ such that

$$\lim_{n=\infty} W\left(\overline{v_n(q)}\cap\bar{\Omega}_\delta, z, G_j\right)=W(q\cap\bar{\Omega}_\delta, z, G_j)>0. \tag{21}$$

$D(\min(N(z,p_0),\delta))=2\pi\delta$ and $D(\omega(\bar{\Omega}_\delta, z, G_j))\leqq\dfrac{2\pi}{\delta}$ and $D(\omega(v_n(q)\cap\Omega_\delta, z, G_j))$ $\leqq\dfrac{2\pi}{\delta}$. By (21) $\omega(q\cap\bar{\Omega}_\delta, z, G_j)=\lim_{n=\infty}\omega(\overline{v_n(q)}\cap\bar{\Omega}_\delta, z, G_j)\geqq W(q\cap\bar{\Omega}_\delta, z, G_j)>0.$ Clearly

$$\omega(q, z, R-R_0)\geqq\omega(q\cap\bar{\Omega}_\delta, z, G_j). \tag{22}$$

${}_{R-G_j}\omega(q, z, R-R_0)$ has M.D.I. over G_j amnog all harmonic functions with the same value as $\omega(q, z, R-R_0)$ on ∂G_j. Hence by Lemma 1 there exists at least a point z_0 such that ${}_{R-G_j}\omega(q, z_0, R-R_0)<\omega(q\cap\Omega_\delta, z_0, G_j)$. By (22) we have

$$\omega(q, z, R-R_0)\not\equiv{}_{R-G_j}\omega(q, z, R-R_0) \text{ and } G_j\overset{L}{\ni}q.$$

Hence by Theorem 3 and 4 there exists a uniquely determined point $p\in{}_{S,2}\Delta(G_j, L_j)$.

REMARK. Under the condition of Theorem 9, if every G_j satisfies the condition of Theorem 8, then ${}_S\Delta(R, L)=0$. Also under the condition of Theorem 10, if every G_j satisfies the condition of Theorem 7, then ${}_{S,2}\Delta(R, L)=0$.

On Matsumoto's conditions.[19] We shall consider the relation between Theorem 1 and Theorem 10. Put $G=\sum G_j$. Clearly $\underset{G_j}{I}[1]=\lim_{n=\infty} W((R-R_n)\cap G_j, z, G_j)$ for any G_j. Let $U(z)=\underset{G_j}{I}[1]$ in G_j and $U(z)=0$ on $R-G$. We shall show that $E\,I[1]=1$ implies $\lim_{n=\infty} W((R-G)\cap(R-R_n)z, R-R_0)=0$. Let $\tilde{U}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(R-G)\cap(R_{n+i}-R_n)$ such that $\tilde{U}_{n,n+i}(z)=0$ on $(\partial R_n-G)+\partial G\cap(R_{n+i}-R_n)$, $\tilde{U}_{n,n+i}(z)=U(z)$ on $\partial R_{n+i}\cap G$. Then $\lim_{n=\infty}\lim_{i=\infty}\tilde{U}_{n,n+i}(z)=E[U(z)]=1$. Let $\hat{U}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(R-G)\cap(R_{n+i}-R_n)$ such that $\hat{U}_{n,n+i}(z)=1$ on $(\partial R_n-G)+\partial G\cap(R_{n+i}-R_n)$, $\hat{U}_{n,n+i}(z)=0$ on $\partial R_{n+i}\cap G$. Then $\lim_n\lim_i \hat{U}_{n+i}(z)=\lim_{n=\infty} W((R-G)\cap(R-R_n), z, R)$. Now $1-\tilde{U}_{n,n+i}(z)=1=\hat{U}_{n,n+i}(z)$ on $(\partial R_n-G)+\partial G\cap(R_{n+i}-R_n)$,

$$1-\tilde{U}_{n,n+i}(z)>0=\hat{U}_{n,n+i}(z) \text{ on } \partial R_{n+i}\cap G.$$

Let $i\to\infty$ and then $n\to\infty$. Then

$$0=1-E[U(z)]\geqq\lim_n W((R-G)\cap(R-R_n), z, R)$$
$$\geqq\lim_n W((R-G)\cap(R-R_n), z, R-R_0).$$

Hence $E\,I[1]=1$ implies $\lim_n W((R-G)\cap(R-R_n), z, R-R_0)=0$.

We shall show $D(I[1])<\infty$ implies that mes $E_N=$ mes E_K. Suppose $D(I[1])<\infty$. Let $W(z)=I[1]$ and let E be the set on which $W(z)$ has angular limits $=1$. Then there exists a set e_1 such that mes $e_1=0$ and $E\subset\Gamma-E_N-A+e_1$, where A is the image of ∂G and Γ is: $|\xi|=1$. By the definition of E_k and since $W(z)$ has angular limits $=0$ or 1 *a.e.* on Γ, there exists a set e_2 such that $E\supset\Gamma-E_k-A-e_2$ and mes $e_2=0$. Hence $E_N\subset E_k+e_1+e_2$. On the other hand, there exists a set e_3 such that mes $e_3=0$ and $E_N+e_3\supset E_k$, whence mes $E_N=$ mes E_k. Thus we have the following

PROPOSITION. *If* $E\,I[1]=1$, *then* $\lim_n W((R-G)\cap(R-R_n), z, R-R_0)=0$.

If $D(I[1])<\infty$, *then mes* $E_N=$ *mes* E_k.

Hence Theorem 10 implies Theorem 1.

Riemann surface of almost finite genus. M. Nakai[20] introduced the notion of almost finite genus and proved that any Riemann surface of almost finite genus has no points in ${}_{S,2}\Delta(R, \mathrm{L})$. Let C_i $(i=1,2,\cdots)$ be a closed Jordan curve in R corresponding to a handle of the Riemann surface such that $C_i\cap C_j=0$ for $i\neq j$, C_i is not a dividing curve and $R-\sum_i C_i$ is of planar character. Let A_j $(j=1,2,\cdots)$ be a relatively compact domain such that $\bar{A}_i\cap\bar{A}_j=0$ for $i\neq j$, $\bar{A}_i\cap\bar{R}_0=0$, $\{A_i\}$ clusters nowhere in R and C_i is contained in some A_j. Let $U_j(z)$ be a harmonic function in $A_j-\sum_i C_i$ such $U_j(z)=0$ on ∂A_j, $U_j(z)=1$ on $\sum_i C_i$. Put $\dfrac{1}{M_j}=D(U_j(z))$. If $\sum_j\dfrac{1}{M_j}<\infty$, then R is called of *almost finite genus.* Let $G=R-\sum_i C_i$. Then G is a domain in R. Matsumoto proved $E\, I[1]=1$ and $D(I[1])<\infty$ for a Riemann surface of almost finite genus and R has no points in ${}_{S,2}\Delta(R, L)$. We shall show that R has no points in ${}_{S}\Delta(R, L)$.

We can choose an exhaustion $\{R_n\}$ such that $\partial R_n \cap \bar{A}_i=0$. Let $\hat{U}_n(z)$ be a piecewise smooth function in $R-R_0$ such that $\hat{U}_n(z)=0$ in $R-(\sum_i A_j \cap (R-R_n))$, $\hat{U}_n(z)=U_j(z)$ in $A_j \cap (R-R_n)$, Let $\omega_n(z)$ be the C.P of $(R-G)\cap (R-R_n)=\sum_{i=1}^{\infty} C_i \cap (R-R_n)$. Then $\omega_n(z)=1$ on $\sum_{i-1}^{\infty} C_i \cap (R-R_n), =0$ on ∂R_0 and has M.D.I. Then by the Dirichlet principle

$$D\left(\omega_n(z)\right) \leqq D\left(\hat{U}_n(z)\right) < \sum^{\infty} \frac{1}{M_i},$$

where the summation is over the number i such that A_i is contained in $R-R_n$. Let $n\to\infty$. Then $\lim_n \omega((R-G)\cap(R-R_n), z, R-R_n)=0$ and the condition of Theorem 9 is satisfied. Now $G=R-\sum C_i$ is of planar and can be mapped onto a domain in the w-sphere. Hence by Theorem 8, G has no point in ${}_{S}\Delta(G, L')$ and by Theorem 9 R has no points in ${}_{S}\Delta(R, L)$. Then we have

THEOREM 11. *Riemann surface of almost finite genus has no points in* ${}_{S}\Delta(R, L)$.

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