On direct modules

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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Y. Utumi obtained that if a ring R is left self-injective then so is the residue class ring R/J modulo the Jacobsn radical J of R. And B. L. Osofsky [5] extended this result to the case of endomorphism rings of quasi-injective modules. In this note we study endomorphism rings of those modules which are weaker than quasi-injectives, conforming to the method by Utumi [8].

1. Preliminaries. We will assume throughout that R is a nonzero ring with identity and that $M={}_RM$ denotes a nonzero unital left R-module. Let ${}_RA$ be an (R-)submodule of ${}_RM$. A complement ${}_RA^c$ of ${}_RA$ in ${}_RM$ is a maximal submodule of ${}_RM$ such that $A\cap A^c=0$. And, a double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$ is a complement of a complement of ${}_RA$ in ${}_RM$ such that $A\subset A^{cc}$. Zorn's lemma ensures the existence of ${}_RA^c$ and ${}_RA^{cc}$ for every submodule ${}_RA$ of ${}_RM$. ${}_RA$ is called complemented in ${}_RM$ if ${}_RA$ is a complement of some submodule of ${}_RM$ in ${}_RM$. To be easily seen, every direct summand of ${}_RM$ is complemented in ${}_RM$. Moreover, ${}_RA$ is essential in ${}_RA^{cc}$ and ${}_RA^{cc}$ is (essentially) closed in ${}_RM$, i.e., ${}_RA^{cc}$ has no proper essential extension in ${}_RM$.

The above leads the following smoothly:

LEMMA 1. *Let ${}_RA$ be a submodule of ${}_RM$. Then the following conditions are equivalent*:

(i) *${}_RA$ is closed in ${}_RM$.*

(ii) *${}_RA$ is complemented in ${}_RM$.*

(iii) *$A=A^{cc}$ for some double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$.*

(iv) *$A=A^{cc}$ for every double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$.*

(v) *Let ${}_RB$ be any submodule of ${}_RM$ contained in A. If ${}_RB$ is essential in ${}_RA$, then there exists such a double complement ${}_RB^{cc}$ of ${}_RB$ in ${}_RM$ that $B^{cc}=A$.*

The following notations will be adopted henceforth. Let ${}_RM$ be a left R-module and let S be the (R-)endomorphism ring of ${}_RM$, acting on the right side. Therefore $M={}_RM_S$ is a left R- and right S-bimodule. For ${}_RM$ we set

$$Z({}_RM)=\{a\in M\mid {}_R{}^Ra \text{ is essential in } {}_RR\},$$
$$Z(M_S)=\{a\in M\mid a^S{}_S \text{ is essential in } S_S\}$$

and $Y(S)=\{x\in S \mid {}_R^M x \text{ is essential in } {}_RM\}$,
where ${}^Ra=\{r\in R \mid ra=0\}$, $a^S=\{x\in S \mid ax=0\}$ and ${}^Mx$ means the kernel of x. Thus, to be easily proved, both $Z({}_RM)$ and $Z(M_S)$ are (R-S-)submodules of ${}_RM_S$, $Y(S)$ is a two-sided ideal of S, having no nonzero idempotent of S and $MY(S)\subset Z({}_RM)$.

2. Quasi-injective modules and pseudo-injective modules. ${}_RM$ is called quasi-injective (or pseudo-injective *)) if every (R-)homomorphism (or every (R-)monomorphism) of any submodule ${}_RA\subset {}_RM$ into ${}_RM$ can be extended to an (R-)endomorphism of ${}_RM$. Let ${}_R\hat{M}$ be an injective hull of ${}_RM$ and T its endomorphism ring, acting on the right: $\hat{M}={}_R\hat{M}_T$. Then we recall the following characterization of quasi-injective modules:

[Johnson-Wong] *${}_RM$ is quasi-injective if and only if $M=MT$.*

Let T' be the subset of T composed of all monomorphisms of ${}_R\hat{M}$ into ${}_R\hat{M}$. ${}_RM$ is called to be finite-dimensional if every independent set of submodules of ${}_RM$ is finite. Then we have:

Proposition 1. *Let ${}_RM$ be finite-dimensional. Then ${}_RM$ is pseudo-injective if and only if $M=MT'$.* (Cf. [6, Theorem 3.7].)

Proof. It is proved similarly to the quasi-injective case that if ${}_RM$ is pseudo-injective then $Mx\subset M$ for all $x\in T'$ (without the assumption of ${}_RM$ finite-dimensional).

Assume the finite-dimensional ${}_RM=MT'$. Let ${}_RA$ be a submodule of ${}_RM$, and φ any monomorphism of ${}_RA$ into ${}_RM$. Since ${}_RA$ is a finite-dimensional submodule of ${}_R\hat{M}$, Miyashita [4, Corollary 2, p. 175] implies that φ can be extended to an automorphism $x\in T'$. Hence, as $Mx\subset M$, the contraction of x to M is an endomorphism of ${}_RM$, which is an extension of φ.

3. Direct modules. Now, although quasi-injectivity implies pseudo-injectivity evidently, we want to extract another type of property from quasi-injective modules. Let ${}_RA$, ${}_RA'$ be submodules of ${}_RM$. Then ${}_RA'$ will be called a *direct hull* of ${}_RA$ in ${}_RM$, if ${}_RA'$ is an essential extension of ${}_RA$ and ${}_RA'$ is a direct summand of M_R. And, ${}_RM$ will be called *direct* if every submodule of ${}_RM$ has a direct hull in ${}_RM$. Moreover, a direct ${}_RM$ is called *uniquely direct* if for any submodules ${}_RA$, ${}_RB\subset {}_RM$ every isomorphism between ${}_RA$ and ${}_RB$ can be extended to an isomorphism between any direct hulls ${}_RA'$ and ${}_RB'$ of ${}_RA$ and ${}_RB$ in ${}_RM$ respectively. If ${}_RM$ is injective, then each submodule of ${}_RM$ has an injective hull in M which is, of course,

*) See Singh and Jain [6].

a direct summand of ${}_RM$. Therefore a direct hull in an injective module is nothing but an injective hull contained in it.

${}_RM(\neq 0)$ is called uniform if every nonzero submodule of ${}_RM$ is essential in ${}_RM$, or equivalently if every pair of nonzero submodules of ${}_RM$ has a nonzero intersection. Hence, ${}_RM$ is uniform if and only if ${}_RM$ is direct and indecomposable.

LEMMA 2. *${}_RM$ is direct if and only if every submodule of ${}_RM$ which is closed in ${}_RM$ is a direct summand of ${}_RM$.*

PROOF. Let ${}_RM$ be direct, and ${}_RA$ any closed submodule of ${}_RM$. Then ${}_RA$ has a direct hull ${}_RA'$ in ${}_RM$. Since ${}_RA$ is essential in ${}_RA'$, the closed ${}_RA$ coincides with ${}_RA'$, which is a direct summand of ${}_RM$.

Conversely, assume that each closed submodule of ${}_RM$ is a direct summand of ${}_RM$. For any submodule ${}_RA\subset{}_RM$, there exists a double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$. By assumption ${}_RA^{cc}$ is a direct summand of ${}_RM$. Since ${}_RA$ is essential in ${}_RA^{cc}$, ${}_RA^{cc}$ is a direct hull of ${}_RA$ in ${}_RM$.

If a submodule ${}_RA\subset{}_RM$ is contained in a direct summand ${}_RM'$ of ${}_RM$, then within M' we can find a certain double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$, just as mentioned in [4, Theorem 2.3]. Therefore, if ${}_RM$ is direct ${}_RA^{cc}$ is a direct summand of ${}_RM$ and accordingly of ${}_RM'$. Namely, every direct summand of a direct module is direct.

If $Z({}_RM)=0$, then any submodule of ${}_RM$ has a unique closed essential extension in ${}_RM$. Actually, let ${}_RA'$ and ${}_RA''$ be two essential extensions of ${}_RA$ in ${}_RM$ which are both closed in ${}_RM$. Then $Z({}_RM)=0$ implies that ${}_RA$ is essential in ${}_R A'+A''$, and hence $A'=A''$. Thus we obtain the following:

PROPOSITION 2. *If ${}_RM$ is direct with $Z({}_RM)=0$, then every submodule of ${}_RM$ has a unique direct hull in ${}_RM$.*

It is to be noted here that each submodule ${}_RA\subset{}_RM$ is a direct summand of ${}_RM$ if and only if $A=Me$ for some idempotent $e\in S$.

PROPOSITION 3. *If ${}_RM$ is direct with $Z({}_RM)=0$, then $Z(M_S)=0$.*

PROOF. Let $a\in Z(M_S)$. Then there exists an idempotent $e\in S$ such that ${}_RRa$ is essential in ${}_RMe$. Take any elements $x\in a^S\cap eS$ and $b\in M$. Since

$$^R(be+Ra)=\{r\in R\mid rbe\in Ra\}$$

is an essential left ideal of R, $^R(be+Ra)bx=0$ implies that $bx\in Z({}_RM)$, i.e., $bx=0$. Therefore $a^S\cap eS=0$. As $a^S{}_S$ is essential in S_S, $eS=0$ or $e=0$. Thus $a=0$, as required.

Now we state some conditions concerning ${}_RM$.

CONDITION (I): Every submodule of ${}_RM$ isomorphic to a direct summand of ${}_RM$ is also a direct summand of ${}_RM$.

CONDITION (I′): Every submodule of ${}_RM$ isomorphic to a closed submodule of ${}_RM$ in ${}_RM$ is also closed in ${}_RM$.

By Lemma 2, Conditions (I) and (I′) are equivalent if ${}_RM$ is direct. And, if R is a (von Neumann) regular ring, then (${}_RM=$) ${}_RR$ sitisfies Condition (I).

CONDITION (II): If ${}_RA$ and ${}_RB$ are direct summands of ${}_RM$ such that $A\cap B=0$, then ${}_R A\oplus B$ is also a direct summand of ${}_RM$.

It will be proved readily that this condition is equivalent to the next:

CONDITION (II′): If $Me\cap Mf=0$ for idempotents e, $f\in S$, then there exists an idempotent $g\in S$ such that $Me=Mg$ and $Mf\subset M(1-g)$.

For ${}_RM$ Condition (I) yields Condition (II), proved in this way. Suppose $Me\cap Mf=0$ for $e=e^2$, $f=f^2\in S$. Since ${}_RMf(1-e)$ is isomorphic to ${}_RMf$, by Condition (I) $Mf(1-e)=Mg$ for some $g=g^2\in S$. Hence ${}_R Me\oplus Mf$ is isomorphic to ${}_R Me\oplus Mg=M(e+g-eg)$, where $e+g-eg\in S$ is an idempotent. Therefore, Condition (I) implies again that ${}_R Me\oplus Mf$ is a direct summand of ${}_RM$.

We already know another characterization of quasi-injective modules:

[FAITH-UTUMI] *${}_RM$ is quasi-injective if and only if ${}_RM$ satisfies the following: let ${}_RA$ and ${}_RC$ be submodules of ${}_RM$ and let ${}_RC$ be closed in ${}_RM$. Then every homomorphism of ${}_RA$ into ${}_RC$ can be extended to a homomorphism of ${}_RM$ into ${}_RC$.*

As an immediate consequence of this theorem we obtain that any closed submodule of ${}_RM$ is a direct summand of ${}_RM$ if ${}_RM$ is quasi-injective. Thus, we can set up the following:

PROPOSITION 4. *Every quasi-injective module is pseudo-injective and direct.*

In case ${}_RM$ is pseudo-injective, the following holds by a similar manner: let ${}_RA$, ${}_RB$ and ${}_RC$ be submodules of ${}_RM$ such that ${}_RB$ is an essential extension of ${}_RA$ and ${}_RC$ is closed in ${}_RM$. Then every monomorphism φ of ${}_RA$ into ${}_RC$ can be extended to a monomorphism φ' of ${}_RB$ into ${}_RC$.

In this condition, if φ is particularly an isomorphism of ${}_RA$ onto ${}_RC$, then A must coincide with B. Hence, a pseudo-injective ${}_RM$ satisfies Condition (I′).

THEOREM 1. *The following are equivalent*:

(i) *${}_RM$ is uniquely direct.*

(ii) *${}_RM$ is pseudo-injective and direct.*

(iii) *Let ${}_RA$ and ${}_RC$ be submodules of ${}_RM$ and let ${}_RC$ be closed in ${}_RM$.*

Then every monomorphism of ${}_RA$ *into* ${}_RC$ *can be extended to a homomorphism of* ${}_RM$ *into* ${}_RC$.

If one of these conditions holds, then:

(iv) ${}_RM$ *is direct with Condition (I).*

And this implies that

(v) ${}_RM$ *is direct with Condition (II).*

Proof. (i) $\Rightarrow$ (ii): Let φ be any monomorphism of a submodule ${}_RA \subset {}_RM$ into ${}_RM$. Then ${}_RA$ and the image ${}_RA\varphi$ have direct hulls ${}_RMe$ and ${}_RMf$, $e=e^2$, $f=f^2 \in S$, respectively. And, there exists, by the uniqueness of direct hulls, an isomorphism φ' of ${}_RMe$ onto ${}_RMf$ which induces φ on ${}_RA$. Therefore $e\varphi'$ gives an endomorphism of M_R which is an extension of φ. This shows that ${}_RM$ is pseudo-injective.

(ii) $\Rightarrow$ (iii): By Lemma 2 we deduce that a closed ${}_RC$ is a direct summand of ${}_RM$, say $C=Me$, $e=e^2 \in S$. Let φ be a monomorphism of ${}_RA$ into ${}_RC$. Then the monomorphism $\varphi\nu$, where ν is the natural injection of ${}_RC$ into ${}_RM$, can be extended to an endomorphism $x \in S$, since ${}_RM$ is pseudo-injective. Hence, the homomorphism xe of ${}_RM$ into ${}_RC$ is an extension of φ.

Immediately (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): In order to prove the uniqueness of direct hulls, settle an isomorphism φ of ${}_RA$ onto ${}_RB$ for two submodules ${}_RA$, ${}_RB \subset {}_RM$. Then since ${}_RM$ is pseudo-injective, φ is induced by an endomorphism $x \in S$. Take any direct hulls ${}_RA' \subset {}_RM$ of ${}_RA$ and ${}_RMe$ of ${}_RB$, $e=e^2 \in S$. The contraction of x to ${}_RA'$ and e compose a homomorphism φ' of ${}_RA'$ into ${}_RMe$, which is clearly an extension of φ. However, as ${}_RA$ is essential in ${}_RA'$, φ' is monomorphic. Since pseudo-injectivity implies Condition (I$'$), as noticed before, ${}_RA'\varphi'$ is closed in ${}_RM$. On the other hand ${}_RB$ is essential in ${}_RMe$, $B \subset A'\varphi' \subset Me$, and therefore $A'\varphi' = Me$. Thus φ' is an isomorphism of ${}_RA'$ onto ${}_RMe$.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v): These implications have already been shown previously, completing the proof.

In our theorem if ${}_RM = {}_RR$ is suited to the statement of (iv), then R is what is called Utumi's left continuous ring. And, [7, Example 3] is to be seen yet.

If a submodule of ${}_RM$ has two direct hulls ${}_RMe$, ${}_RMf$ ($e=e^2$, $f=f^2 \in S$) in ${}_RM$, then $Me \cap M(1-f)=0$ and ${}_RMe$ is isomorphic to ${}_RMef$. Furthermore, if ${}_RM$ satisfies Condition (II), then $Me \oplus M(1-f) = Mg$ for some $g=g^2 \in S$. Therefore $Mef=Mgf$, where gf is an idempotent of S, must coincide with Mf since ${}_RMgf$ is essential in ${}_RMf$. Thus we have: if ${}_RM$ is direct with Condition (II), then any direct hulls of a submodule ${}_RA\subset{}_RM$ in ${}_RM$ are isomorphic leaving A elementwise fixed.

An application of Miyashita's uniform dimension theorem to a uniquely direct module yields the following, the proof of which will be omitted because of its similarity to that of [4, Theorem 4.5].

PROPOSITION 5. *Let ${}_RM$ be uniquely direct.*

(i) *Let $\{{}_RA_\lambda|\lambda\in\Lambda\}$ and $\{{}_RB_\gamma|\gamma\in\Gamma\}$ be maximal independent sets of uniform submodules of ${}_RM$, and let ${}_RA'_\lambda$ and ${}_RB'_\gamma$ be any direct hulls of ${}_RA_\lambda$ and ${}_RB_\gamma$ $(\lambda\in\Lambda,\ \gamma\in\Gamma)$ respectively. Then there exist a one-to-one correspondence χ of Λ onto Γ and an automorphism $x\in S$ such that $A'_\lambda x=B'_{(\lambda)\chi}$ for all $\lambda\in\Lambda$.*

(ii) *Moreover, let ${}_RM$ be finite-dimensional. Then M is a direct sum of a finite number of pseudo-injective uniform submodules and such a representation of M is unique up to isomorphism.*

(iii) *Let ${}_RA$ and ${}_RB$ be finite-dimensional submodules of ${}_RM$. Then every isomorphism between ${}_RA$ and ${}_RB$ can be extended to an automorphism of ${}_RM$.*

PROPOSITION 6. *Let ${}_RM$ be direct with Condition (I). If φ is a homomorphism of any submodule ${}_RA\subset{}_RM$ into ${}_RM$ such that $A\cap A\varphi=0$, then φ can be extended to an endomorphism of ${}_RM$.*

PROOF. Take direct hulls ${}_RMe$ and ${}_RMf$ of ${}_RA$ and ${}_RA\varphi$ respectively, where $e=e^2$, $f=f^2\in S$. Then since $Me\cap Mf=0$, we may assume $ef=fe=0$ by Condition (II) for ${}_RM$. Set ${}_RB=\{a+a\varphi|a\in A\}$, which is a submodule of ${}_RM$ contained in $M(e+f)$. Since ${}_RM$ is direct, there exists $g=g^2=g(e+f)\in S$ such that ${}_RB$ is essential in ${}_RMg$. Because $B\cap M(1-e)=0$, $Mg\cap M(1-e)=0$ and so ${}_RMg$ is isomorphic to ${}_RMge$. Hence by Condition (I) for ${}_RM$, ${}_RMge$ is a direct summand of ${}_RM$. However, since ${}_RA=Be$ is essential in ${}_RMe$, $Mge=Me$. Therefore given any element $a\in M$, there exists a unique element $bg\in M$ $(b\in M)$ such that $ae=bge$, i.e., there exists an endomorphism $x\in S$ such that $ax=bg$. Hence $e=xe$ and $x=xg$. And an easy verification implies that $a\varphi-axf\in Mf\cap Mg=0$ for all $a\in A$. Thus xf is an extension endomorphism of φ.

Proposition 6 will be used as a lemma to obtain the following:

Let ${}_RM$ be direct with Condition (I). And let ${}_RM$ be a direct sum of n submodules, $n>1$, say $M=A_1\oplus A_2\oplus\cdots\oplus A_n$, ${}_RA_i\subset{}_RM$ $(i=1,2,\cdots,n)$, such that each $\sum\oplus_{j\neq i}A_j$ contains an isomorphic image of ${}_RA_i$. Then ${}_RM$ is quasi-injective.

To establish the proof see Utumi [8, Theorem 7.1].

4. Endomorphism rings of uniquely direct modules. For the endomorphism ring S of ${}_RM$, we let $\bar{S}=S/Y(S)$ denote the residue class ring of S modulo $Y(S)$. And for $x\in S$, $\bar{x}$ will denote the residue class of x modulo $Y(S)$.

On lifting idempotents modulo $Y(S)$ we have the following:

LEMMA 3. *Let ${}_RM$ be direct with Condition (II). And let x, $e=e^2\in S$. If $\bar{x}=\bar{x}\bar{e}=\bar{x}^2$, then there exists an idemotent $f=fe=f^2\in S$ such that $\bar{x}=\bar{f}$.*

PROOF. By our assumption $x-xe$, $x-x^2\in Y(S)$, there exists an essential submodule ${}_RA$ of ${}_RM$ such that $A(x-xe)=A(x-x^2)=0$. As ${}_RM$ is direct, we can take direct hulls ${}_RMg$ of ${}_RAx=Axe$ in ${}_RMe$ and ${}_RMh$ of ${}_RA(1-x)$ in ${}_RM$, where $g=ge=g^2$, $h=h^2\in S$. And since $Ax\cap A(1-x)=0$, $Mg\cap Mh=0$. It follows by Condition (II′) that there exists $f=f^2\in S$ such that $Mg=Mf$ and $Mh\subset M(1-f)$. Thus $Ax(1-f)=A(1-x)f=0$ and so $x(1-f)$, $(1-x)f\in Y(S)$. Hence $x-f\in Y(S)$. And $f=fe$ since $Mf\subset Me$, completing the proof.

PROPOSITION 7. *Let ${}_RM$ be direct with Condition (I). Then $Y(S)$ coincides with the Jacobson radical $J(S)$ of S, and $\bar{S}$ is a regular ring.*

PROOF. Let first $x\in Y(S)$. Then since ${}_R^Mx$ is essential in ${}_RM$, ${}^Mx\cap {}^M(1+x)=0$ implies ${}^M(1+x)=0$. Hence ${}_RM$ is isomorphic to ${}_RM(1+x)$, which is a direct summand of ${}_RM$ by Condition (I). On the other hand, ${}_RM(1+x)$ is essential in ${}_RM$ as ${}^Mx\subset M(1+x)$. Hence $M(1+x)=M$. Thus $1+x$ is an automorphism of ${}_RM$, meaning that x is a quasi-regular element of S; $x\in J(S)$. This shows the inclusion $Y(S)\subset J(S)$.

Let next $y\in S$. Setting ${}_RA={}^My$ and ${}_RA^c=Me$, $e=e^2\in S$, we have an isomorphism of ${}_RA^c$ onto ${}_RA^cy$. Hence by Condition (I) there exists $f=f^2\in S$ such that $A^cy=Mf$. Therefore, for any element $a\in M$ we can find a unique $b\in A^c$ such that $af=by$; there exists $z\in S$ such that $f=zy$. Since ${}_R A\oplus A^c$ is essential in ${}_RM$, it follows from $(A\oplus A^c)(y-yzy)=0$ that $y-yzy\in Y(S)$. Thus $\bar{S}$ is regular.

If in particular $y\in J(S)$, then $y\in Y(S)$ since $1-yz$ is a unit of S. This completes the proof.

LEMMA 4. *Let ${}_RM$ be direct with Condition (I) and let $e_\lambda=e_\lambda^2\in S$ $(\lambda\in\Lambda)$. If $\{{}_{\bar{S}}\bar{S}\bar{e}_\lambda|\lambda\in\Lambda\}$ is an independent set, then so is $\{{}_RMe_\lambda|\lambda\in\Lambda\}$.*

PROOF. We have only to prove the lemma under $\#\Lambda<\infty$; we deduce that if $\{{}_{\bar{S}}\bar{S}\bar{e}_1, {}_{\bar{S}}\bar{S}\bar{e}_2, \cdots, {}_{\bar{S}}\bar{S}\bar{e}_n\}$ is independent, then so is $\{{}_RMe_1, {}_RMe_2, \cdots, {}_RMe_n\}$ for idempotents $e_1, e_2, \cdots, e_n\in S$. First we treat the case of $n=2$. Since $\bar{S}$ is regular, there exists $f=f^2\in S$ by Lemma 3 such that $\bar{S}\bar{e}_1=\bar{S}\bar{f}$ and $\bar{S}\bar{e}_2\subset \bar{S}(\bar{1}-\bar{f})$. Evidently, $(Me_1\cap Me_2)\cap {}^M(e_2f)\subset {}^M(1-e_1+e_1f)$. Since $\bar{e}_1=\bar{e}_1\bar{f}$, ${}_R^M(e_1-e_1f)$ is essential in ${}_RM$ and so ${}^M(1-e_1+e_1f)=0$. Therefore $(Me_1\cap Me_2)\cap {}^M(e_2f)=0$. However, ${}_R^M(e_2f)$ is essential in ${}_RM$ since $\bar{e}_2\bar{f}=0$. This yields

$Me_1 \cap Me_2 = 0$. Next, assume $n \geqq 3$ and that our assertion holds for $n-1$ idempotents of S. By assumption $\{_RMe_1, {}_RMe_2, \cdots, {}_RMe_{n-1}\}$ is independent. Hence by Condition (II) for $_RM$, there exists $e=e^2 \in S$ such that $Me_1 \oplus Me_2 \oplus \cdots \oplus Me_{n-1} = Me$. Therefore, we can find idempotents $e_1', e_2', \cdots, e_{n-1}' \in S$ such that $Me_i' = Me_i$ $(i=1, 2, \cdots, n-1)$ and $e_1' + e_2' + \cdots + e_{n-1}' = e$. Since $Se_i' = Se_i \subset Se$ $(i=1, 2, \cdots, n-1)$, $\bar{S}\bar{e}_1 \oplus \bar{S}\bar{e}_2 \oplus \cdots \oplus \bar{S}\bar{e}_{n-1} = \bar{S}\bar{e}_1' \oplus \bar{S}\bar{e}_2' \oplus \cdots \oplus \bar{S}\bar{e}_{n-1}' = \bar{S}\bar{e}$. Accordingly, $\bar{S}\bar{e} \cap \bar{S}\bar{e}_n = 0$ implies $Me \cap Me_n = 0$, whence it follows that $\{_RMe_1, {}_RMe_2, \cdots, {}_RMe_n\}$ is independent. This completes the proof by induction.

THEOREM 2. *If $_RM$ is direct with Condition (I), then so is $_S\bar{S}$.*

PROOF. Since $\bar{S}$ is regular by Proposition 7, $_S\bar{S}$ satisfies Condition (I). Hence it is enough to show $_S\bar{S}$ direct. Let $\mathfrak{A}$ be any left ideal of $\bar{S}$. Then, in virtue of using Zorn's lemma, there exist $\bar{e}_\lambda \in \mathfrak{A}$ $(\lambda \in \Lambda)$ such that the direct sum $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda$ is essential in $_S\mathfrak{A}$. Since $\bar{S}$ is regular, we can assume, by Lemma 3, $e_\lambda = e_\lambda^2 \in S$ for all $\lambda \in \Lambda$. Hence, $\{_RMe_\lambda \mid \lambda \in \Lambda\}$ is independent by Lemma 4. Set $_RMe$ $(e=e^2 \in S)$ be a direct hull of $_R\sum \oplus_{\lambda \in \Lambda} Me_\lambda$ in $_RM$. Then, it follows from this that $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda$ is essential in $_S\bar{S}\bar{e}$. Because, let $\mathfrak{B} \subset \bar{S}\bar{e}$ be a left ideal of $\bar{S}$ such that $\mathfrak{B} \cap \sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda = 0$. If $\bar{x} \in \mathfrak{B}$, $\bar{S}\bar{x} \cap \sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda = 0$; we may say $x = xe = x^2 \in S$ and hence $Mx \cap \sum \oplus_{\lambda \in \Lambda} Me_\lambda = 0$ by Lemma 4. Since $_R\sum \oplus_{\lambda \in \Lambda} Me_\lambda$ is essential in $_RMe$, we have $Mx = 0$, namely, $x = 0$. This asserts $\mathfrak{B} = 0$, consequently.

On the other hand, for every $\bar{y} \in \mathfrak{A}$, $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda \cap \bar{S}\bar{y}$ is essential in $_S\bar{S}\bar{y}$. Hence, $_S\bar{S}\bar{e} \cap \bar{S}\bar{y}$ is essential in $_S\bar{S}\bar{y}$. However, since $\bar{S}$ is regular, $_S\bar{S}\bar{e} \cap \bar{S}\bar{y}$ is a direct summand of $_S\bar{S}$. Therefore $\bar{S}\bar{e} \cap \bar{S}\bar{y} = \bar{S}\bar{y}$ and so $\bar{y} \in \bar{S}\bar{e}$. Thus $\mathfrak{A} \subset \bar{S}\bar{e}$, whence it follows that $_S\mathfrak{A}$ is essential in $_S\bar{S}\bar{e}$. This shows that $_S\bar{S}\bar{e}$ is a direct hull of $_S\mathfrak{A}$ in $_S\bar{S}$, completing the proof.

THEOREM 3. *If $_RM$ is uniquely direct, then so is $_S\bar{S}$.*

PROOF. By Theorems 1 and 2, we have only to prove that $_S\bar{S}$ is pseudo-injective. Let $\mathfrak{A}$ be a left ideal of $\bar{S}$ and let Φ be any monomorphism of $_S\mathfrak{A}$ into $_S\bar{S}$. Then we shall extend Φ to an endomorphism of $_S\bar{S}$. As in the proof of Theorem 2, we can find $e_\lambda = e_\lambda^2 \in S$ $(\lambda \in \Lambda)$ such that the direct sum $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda$ is essential in $_S\mathfrak{A}$. Let $\bar{e}_\lambda \Phi = \bar{x}_\lambda \in \bar{S}$, $x_\lambda \in S$ $(\lambda \in \Lambda)$. Then $\{_S\bar{S}\bar{x}_\lambda \mid \lambda \in \Lambda\}$ is an independent set and $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{e}_\lambda$ is isomorphic to $_S\sum \oplus_{\lambda \in \Lambda} \bar{S}\bar{x}_\lambda$. Since $\bar{S}$ is regular, by Lemma 3 for each $\lambda \in \Lambda$ there exist y_λ, $f_\lambda = f_\lambda^2 \in S$ such that $\bar{x}_\lambda = \bar{x}_\lambda \bar{y}_\lambda \bar{x}_\lambda$ and $\bar{y}_\lambda \bar{x}_\lambda = \bar{f}_\lambda$. Hence $\bar{S}\bar{x}_\lambda = \bar{S}\bar{f}_\lambda$ for all $\lambda \in \Lambda$. By Lemma 4, we can set submodules of $_RM$;

$$_RA = \sum \oplus_{\lambda \in \Lambda} Me_\lambda, \quad {}_RB = \sum \oplus_{\lambda \in \Lambda} Mf_\lambda$$

as direct sums of direct summands of $_RM$. Let y' be a homomorphism of $_RA$ into $_RB$, and z' a homomorphism of $_RB$ into $_RA$, defined as follows:

$$ {}_R A \underset{z'}{\overset{y'}{\rightleftarrows}} {}_R B , $$

$$ ay' = \sum_{\lambda\in\Lambda} a_\lambda x_\lambda f_\lambda \quad \text{for} \quad a = \sum_{\lambda\in\Lambda} a_\lambda \in A , $$
$$ bz' = \sum_{\lambda\in\Lambda} b_\lambda y_\lambda e_\lambda \quad \text{for} \quad b = \sum_{\lambda\in\Lambda} b_\lambda \in B , $$

where $a_\lambda \in Me_\lambda$, $b_\lambda \in Mf_\lambda$ for all $\lambda\in\Lambda$ and $a_\lambda=0$, $b_\lambda=0$ for almost all $\lambda\in\Lambda$. Then, it follows that $\bar{e}_\lambda\bar{x}_\lambda\bar{f}_\lambda\bar{y}_\lambda\bar{e}_\lambda\Phi=\bar{x}_\lambda$, namely, $\bar{e}_\lambda\bar{x}_\lambda\bar{f}_\lambda\bar{y}_\lambda\bar{e}_\lambda=\bar{e}_\lambda$ and so ${}^M(1+e_\lambda x_\lambda f_\lambda y_\lambda e_\lambda - e_\lambda)=0$ for all $\lambda\in\Lambda$. If $ay'=0$ for $a=\sum_{\lambda\in\Lambda} a_\lambda \in A$, $a_\lambda \in Me_\lambda$ ($\lambda\in\Lambda$), then $ay'z'=\sum_{\lambda\in\Lambda} a_\lambda x_\lambda f_\lambda y_\lambda e_\lambda=0$. Hence $a_\lambda x_\lambda f_\lambda y_\lambda e_\lambda=0$ for all $\lambda\in\Lambda$. Therefore, since $a_\lambda\in {}^M(1+e_\lambda x_\lambda f_\lambda y_\lambda e_\lambda - e_\lambda)$ for all $\lambda\in\Lambda$, $a=0$. This yields that y' is a monomorphism. Thus, we can find an endomorphism $y\in S$ which is an extension of y', since ${}_R M$ is pseudo-injective. Now, let Ψ be an endomorphism of ${}_S\bar{S}$, by defining $\alpha\Psi=\alpha\bar{y}$ for $\alpha\in\bar{S}$. Since $e_\lambda y=e_\lambda x_\lambda f_\lambda$, $\bar{e}_\lambda\Psi=\bar{x}_\lambda=\bar{e}_\lambda\Phi$ for all $\lambda\in\Lambda$, whence we obtain $\Psi=\Phi$ on ${}_S\sum\oplus_{\lambda\in\Lambda}\bar{S}\bar{e}_\lambda$. Given $\alpha\in\mathfrak{A}$, since ${}_S\sum\oplus_{\lambda\in\Lambda}\bar{S}\bar{e}_\lambda$ is essential in ${}_S\mathfrak{A}$,

$$ \mathfrak{B} = \{\beta\in\bar{S} \mid \beta\alpha\in\sum\oplus_{\lambda\in\Lambda}\bar{S}\bar{e}_\lambda\} $$

is an essential left ideal of $\bar{S}$. And since $\mathfrak{B}\alpha(\Psi-\Phi)=0$, we have $\alpha(\Psi-\Phi)\in Z({}_S\bar{S})$. However $Z({}_S\bar{S})=0$ since $\bar{S}$ is a regular ring. Consequently, we have $\mathfrak{A}(\Psi-\Phi)=0$; Ψ is an extension of Φ, as desired.

[OSOFSKY] *If ${}_R M$ is quasi-injective, then ${}_S\bar{S}$ is injective.*

The proof of this theorem has been given as a simplified form of that of our Theorem 3. Indeed, since we have only to extend any "homomorphism" Φ of ${}_S\mathfrak{A}$ into ${}_S\bar{S}$, there is no need of referring to idempotents f_λ of S.

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