A generalization of P. Roquette's theorems

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction

Throughout this paper, we assume that every ring has an identity 1, every module over a ring is unitary and a ring extension A/B has the same identity 1. For a commutative ring R, we consider only R-algebras which are finitely generated as R-modules. By [5], an R-algebra Λ is called left semisimple if any finitely generated left Λ-module is (Λ, R)-projective. Similarly we can define right semisimple R-algebras, and an R-algebra Λ is called semisimple if Λ is left and right semisimple. When R is indecomposable, an R-algebra Λ is called simple if (1) Λ is semisimple, (2) there exists left Λ-module ${}_{\Lambda}E$ which is finitely generated projective completely faithful and (Λ, R)-irreducible ([12]). We call an R-algebra Δ a division R-algebra if Δ is semisimple and (Δ, R)-irreducible. Obviously division algebras are simple algebras.

The followings are well known. Let K be a field (a field means commutative field) and let A be a finite dimensional central simple K-algebra. Then there exists a central division K-algebra D such that $A \cong (D)_n$ ($n \times n$ full matrix ring over D), and the free rank of D over $K([D:K])$ equals s^2 where $s(\geqq 1)$ is an integer. This s is called the Schur index of A and D is called a division algebra to which A belongs.

Let Δ be a division R-algebra and Λ be a simple R-algebra. If there exists a Morita module ${}_{\Lambda}M_{\Delta}$ ([9]), Δ is called a division R-algebra to which Λ belongs. By [12], any simple R-algebra belongs to some division R-algebra. Now, let R be a Hensel ring ([2], [10]) and Λ be a simple R-algebra. Then $\Lambda \cong (\Delta)_n$ where Δ is a division R-algebra to which Λ belongs. Moreover, Δ is uniquely determined up to isomorphisms and n is uniquely determiend ([12]).

The purpose of this paper is to extend some properties with respect to the Schur index concerning fields to the case of that R is a Noetherian Hensel ring.

We prove the followings.

THEOREM 2.2. *Let R be a semilocal ring (not necessarily Noetherian* *and has maximal ideals of finite numbers) which has no proper idempotents (i, e, R has no idempotents except* 0 *and* 1), *S be a commutative ring, a ring extension S/R be a finite Galois extension with Galois group G, and Λ be a central separable R-algebra. We put $\Gamma=\Lambda\otimes_R S$. Tnen, $H^1(G, I(\Gamma))\overset{\delta}{\rightarrow}H^2(G, U(S))$ is injective. Here, $U(S)$ denotes the unit group of S, $I(\Gamma)=U(\Gamma)/U(S)$, and $U(\Gamma)$ denotes the unit group of Γ.*

THEOREM 2.7. *Let R be a Noetherian Hensel ring, S be a commutative ring and a ring extension S/R be a finite Galois extension with Galois group G such that S has no proper idempotents. Let $[c_{\sigma,\tau}]\in H^2(G, U(S))$, $\Lambda=(R)_l$ and $T=\Delta(c_{\sigma,\tau}, S, G)$ (crossed product). Then $[c_{\sigma,\tau}]$ is contained in the image of δ if and only if the Schur index of T (see definition* 1.3.) *divides l.*

THEOREM 2.2 was proved in [11], when R is a field and $\Lambda=(R)_t$ for an integer $t\geqq 1$. THEOREM 2.7 was proved in [11], when R is a field.

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§ 1. The Schur indexes of central separable algebras

In this section, so far as we don't especially state, let R be a Noetherian Hensel ring with unqiue maximal ideal $\mathfrak{m}$.

LEMMA 1.1. ([6]. THEOREM 4.) *If Λ is a central separable R-algebra, then it is a central simple R-algebra.*

PROPOSITION 1.2. *Let Λ be a central separable R-algebra, and Δ be a division R-algebra to which Λ belongs. Then Δ is free R-module and $[\Delta; R]=s^2$ (s is an integer $\geqq 1$).*

PROOF. Δ is a central separable R-algebra ([12]. PROPOSITION 3.) and R is a local ring (not necessarily Noetherian). Hence Δ is a free R-module. $\Delta/\mathfrak{m}\Delta$ is a central division $R/\mathfrak{m}$-algebra ([12]. THEOREM 8, [1]. COROLLARY 1.6.). $[\Delta: R]=[\Delta/\mathfrak{m}\Delta: R/\mathfrak{m}]=s^2$. Q.E.D.

DEFINITION 1.3. *The s which is obtaihed in Proposition* 1.2 is called the Schur index of Λ.

PROPOSITION 1.4. *Let Λ be a central separable R-algebra, and Δ be a division R-algebra to which Λ belongs. Then a division $R/\mathfrak{m}$-algebra to which $\Lambda/\mathfrak{m}\Lambda$ belongs is $\Delta/\mathfrak{m}\Delta$, and the Schur index of Λ equals the Schur index of $\Lambda/\mathfrak{m}\Lambda$.*

PROOF. By our assumptios, $\Lambda\cong(\Delta)_n$ and $\Lambda/\mathfrak{m}\Lambda\cong(\Delta/\mathfrak{m}\Delta)_n$. Q.E.D.

When R is a field, the following PROPOSITION 1.5, 1.6 and 1.7 are well known. By $Br(R)$, we denote the Brauer group of R. When R is a Hensel ring (not necessarily Noetherian) with unique maximal ideal $\mathfrak{m}$, if we use the fact that $Br(R)\cong Br(R/\mathfrak{m})$ ([2]), these PROPOSITIONS are easily proved. By $[\Lambda]$, we denote the element of $Br(R)$ represented by the central separable R-algebra Λ.

PROPOSITION 1.5. *For any $[\Lambda]\in Br(R)$, $[\Lambda]^s=[R]$ where s is the Schur index of Λ.*

PROPOSITION 1.6. *Let e be the exponent of $[\Lambda]\in Br(R)$ (that is, e is the minimal integer $n\geqq 1$ such that $[\Lambda]^n=[R]$), and p be a prime number such that p divides s. Then p divides e.*

PROPOSITION 1.7. *Let Δ be a central separable division R-algebra, and the Schur index of $\Delta=\prod_{i=1}^{n} p_i^{\alpha_i}$ (unique factorization to prime numbers). Then there exist central separable division R-algebras $\Delta_1,\cdots,\Delta_n$ such that $\Delta\cong\Delta_1\otimes_R\cdots\otimes_R\Delta_n$, and the Schur index of Δ_i equals a power of p_i $(i=1,\cdots,n)$.*

PROOF. $\Delta/\mathfrak{m}\Delta$ is a central division $R/\mathfrak{m}$-algebra, and the Schur index of $\Delta/\mathfrak{m}\Delta=\prod_{i=1}^{n} p_i^{\alpha_i}$. Hence $\Delta/\mathfrak{m}\Delta=U_1\otimes_{R/\mathfrak{m}}\cdots\otimes_{R/\mathfrak{m}}U_n$ where each U_i is a central division $R/\mathfrak{m}$-algebra, and the Schur index of U_i equals a power of p_i. As R is a Hensel ring, there exists a central separable division R-algebra Δ_i such that $\Delta_i/\mathfrak{m}\Delta_i\cong U_i$ $(i=1,\cdots,n)$ ([12]. PROPOSITION 14, THEOREM 8.). Hence $\Delta/\mathfrak{m}\Delta\cong\Delta_1/\mathfrak{m}\Delta_1\otimes_{R/\mathfrak{m}}\cdots\otimes_{R/\mathfrak{m}}\Delta_n/\mathfrak{m}\Delta_n=(\Delta_1\otimes_R\cdots\otimes_R\Delta_n)\otimes_R R/\mathfrak{m}$, and $\Delta\cong\Delta_1\otimes_R\cdots\otimes_R\Delta_n$ ([12]. PROPOSITION 14.). The Schur index of Δ_i equals that of U_i. Q.E.D.

PROPOSITION 1.8. *Let Λ be a central separable R-algebra, and Δ and Δ' be division R-algebras such that $\Lambda=(\Delta)_n=(\Delta')_n$. Then an R-algebra isomorphism $\beta:\Delta\to\Delta'$ (see introduction) is a restriction of an inner automorphism of Λ.*

PROOF. As R is a Hensel ring, β can be extended to an inner automorphism of Λ ([3]. THEOREM 1.2). Hence there exists a $\lambda\in U(\Lambda)$ (the unit group of Λ such) that $\Delta'=\lambda\Delta\lambda^{-1}$. Q.E.D.

§2. A generalization of P. Roquette's theorems

In this section, we state about a generalization of [11] §3.

LEMMA 2.1. *Let R be a commutative ring and A be an R-algebra which is flat and faithful as an R-module (not necessarily finitely generated). Let B be an R-module which is finitely generated, projective and faithful.*

Then the followings are true.

(1) *If S is a subset of A, then $V_{A\otimes_R B}(S)=V_A(S)\otimes_R B$ where we can consider $A\otimes_R B$ (A, A)-bimodule under $(\sum_i a_i\otimes b_i)a=\sum_i a_ia\otimes b_i$ and $a(\sum_i a_i\otimes b_i)=\sum_i aa_i\otimes b_i$.*

(2) *Moreover, let B be an R-algebra. Let S and T be subrings of A and B respectively. If $V_A(S)$ is a finitely generated and projective R-module, then $V_{A\otimes_R B}(S\otimes T)=V_A(S)\otimes_R V_B(T)$ where $S\otimes T=\{\sum_i s_i\otimes t_i\in A\otimes_R B|s_i\in S,\ t_i\in T\}$.*

Here, $V_A(S)=\{a\in A|as=sa$ for all $s\in S\}$ and $V_{A\otimes_R B}(S\otimes T)=\{\sum_i a_i\otimes b_i\in A\otimes_R B|$ $(\sum_i a_i\otimes b_i)x=x(\sum_i a_i\otimes b_i)$ for all $x\in S\otimes T\}$.

PROOF. (1) First, we prove in the case that B is a free R-module. $V_{A\otimes_R B}(S)\supset V_A(S)\otimes_R B$ is trivial. Let $\{b_i|i=1,\cdots,l\}$ be a free base of B. For any $\sum_{i=1}^{l} a_i\otimes b_i\in V_{A\otimes_R B}(S)$ $(a_i\in A)$, $(\sum a_i\otimes b_i)s=\sum sa_i\otimes b_i=\sum sa_i\otimes b_i=s(\sum a_i\otimes b_i)$. As $1\otimes b_1,\cdots,1\otimes b_l$ are linearly independent over A in $A\otimes_R B$, $a_is=sa_i$ *for all* $i=1,\cdots,l$. Hence $a_i\in V_A(S)$. In the case that B is a finitely generated, projective and faithful, there exists a finitely generated and free R-module F such that $F=B\oplus B'$ (direct sum as an R-module).

$$\begin{aligned} V_{A\otimes_R B}(S) &= A\otimes_R B\cap V_{A\otimes_R F}(S) \\ &= A\otimes_R B\cap(V_A(S)\otimes_R F) \\ &= A\otimes_R B\cap\{(V_A(S)\otimes_R B)\oplus(V_A(S)\otimes_R B')\} \\ &= V_A(S)\otimes_R B. \end{aligned}$$

(2)
$$\begin{aligned} V_{A\otimes_R B}(S\otimes T) &= V_{A\otimes_R B}(S)\cap V_{A\otimes_R B}(T) \\ &= V_{V_A(S)\otimes_R B}(T) \\ &= V_A(S)\otimes_R V_B(T) \qquad \text{(by (1).)} \end{aligned}$$

Q.E.D.

Let R be a semi local ring (not necessarily Noetherian and has maximal ideals of finite numbers) which has no proper idempotents (i.e. has no idempotents except 0 and 1), S be a commutative ring, a ring extension S/R be a finite Galois extension with Galois group G, and Λ be a central separable R-algebra. If we put $\Gamma=\Lambda\otimes_R S$, Γ/Λ is a Galois extension with Galois group G ([8]). For a ring A, we denote the unit gryup of A by $U(A)$. Then we have a G-exact sequence

$$1 \longrightarrow U(S) \longrightarrow U(\Gamma) \overset{h}{\longrightarrow} I(\Gamma) \longrightarrow 1$$

where $I(\Gamma)=U(\Gamma)/U(S)$ and h is the canonical map. From this exact sequence, we obtain an exact sequence

$$(*)\quad H^1\big(G, U(S)\big) \longrightarrow H^1\big(G, U(\Gamma)\big) \longrightarrow H^1\big(G, I(\Gamma)\big) \overset{\delta}{\longrightarrow} H^2\big(G, U(S)\big)$$

([11]. § 2.).

THEOREM 2. 2. (*cf.* [11]. § 3. COROLLARY *of* PROPOSITION 3.) *Under the above assumptions, δ is injective.*

PROOF. Let $\Delta(\Gamma, G)=\sum_{\sigma\in G}\oplus\sigma\Gamma$ and $\Delta(S, G)=\sum_{\sigma\in G}\oplus\sigma S$ be trivial crossed products. Then $\Delta(\Gamma, G)=\Lambda\otimes_R \Delta(S, G)$. Hence $\Delta(\Gamma, G)$ is a central separable R-algebra ([1]. PROPOSITION 1.5.). When we put $\mathfrak{G}=\bigcup_{\sigma\in G}\sigma U(\Gamma)\subset U(\Delta(\Gamma, G))$, $\mathfrak{G}$ is a splitting extension of $U(\Gamma)$ by G as a G-group. That is, $GU(\Gamma)=\mathfrak{G}$, $G\cap U(\Gamma)=1$ and $U(\Gamma)\lhd\mathfrak{G}$ (normal subgroup). We put $\mathscr{H}=\{\mathfrak{H}\subset\mathfrak{G}\,|\,\mathfrak{H}$ is a G-subgroup of $\mathfrak{G}$, $\mathfrak{H}\cap U(\Gamma)=U(S)$ and $\mathfrak{H}U(\Gamma)=\mathfrak{G}\}$. That is, each element of $\mathscr{H}$ is an extension of $U(S)$ by G as a G-group. For $\mathfrak{H}$ and $\mathfrak{H}'\in\mathscr{H}$, we define $\mathfrak{H}\sim\mathfrak{H}'$ by existence of $a\in U(\Gamma)$ such that $\mathfrak{H}'=a^{-1}\mathfrak{H}a$. It is well known that $\mathfrak{H}\sim\mathfrak{H}'$ implies that $\mathfrak{H}$ and $\mathfrak{H}'$ are the same extension type. Then by [11] § 2 PROPOSITION 1, the following diagram is commutative.

$$\begin{array}{ccc} H^1(G, I(\Gamma)) & \overset{\delta}{\longrightarrow} & H^2(G, U(S)) \\ f\uparrow\downarrow f^{-1} & & \uparrow g \\ \mathscr{H}/\sim & \underset{\alpha}{\longrightarrow} & ext\,(G, U(S)) \end{array}$$

where f is a bijection and defined by the following way. We denote an element of $\mathscr{H}/\sim$ containing $\mathfrak{H}$ by $[\mathfrak{H}]$. When a $[\mathfrak{H}]$ is given, for any $\sigma\in G$, we can write $\sigma=u_\sigma a_\sigma^{-1}$ where $u_\sigma\in\mathfrak{H}$ and $a_\sigma\in U(\Gamma)$. Put $h(a_\sigma)=b_\sigma$. Then we can find that the $\{b_\sigma|\sigma\in G\}$ is a crossed homomorphism, and when we write $[b_\sigma]\in H^1(G, I(\Gamma))$, $f([\mathfrak{H}])=[b_\sigma]$. f^{-1} is defined by the following way. That is, when $[b_\sigma]\in H^1(G, I(\Gamma))$, pick up any $a_\sigma\in h^{-1}(b_\sigma)=\{x\in U(\Gamma)|h(x)=b_\sigma\}$ $\subset U(\Gamma)$, and put $\mathfrak{H}=\bigcup_{\sigma\in G}\sigma a_\sigma U(S)\subset\mathfrak{G}$, then $\mathfrak{H}\in\mathscr{H}$. Let $\mathfrak{H}\in\mathscr{H}$ and $u_\sigma=\sigma a_\sigma$ ($\sigma\in G$ and $a_\sigma\in U(\Gamma)$), then $u_\sigma u_\tau\equiv u_{\sigma\tau}\ mod\,(U(S))$. Hence if we put $u_\sigma u_\tau=u_{\sigma\tau}c_{\sigma,\tau}(c_{\sigma,\tau}\in U(S))$, the set $\{c_{\sigma,\tau}|\sigma, \tau\in G\}$ is a factor set, and $(\delta\circ f)([\mathfrak{H}])=[c_{\sigma,\tau}]\in H^2(G, U(S))$. $\alpha([\mathfrak{H}])$ is the class of the same extension type as $\mathfrak{H}$. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be the same extension type, and by the above methods, let factor sets $\{c_{\sigma,\tau}\}$ and $\{c'_{\sigma,\tau}\}$ correspond to $\mathfrak{H}$ and $\mathfrak{H}'$ respectively. Then $[c_{\sigma,\tau}]=[c'_{\sigma,\tau}]\in H^2(G, U(S))$. That is, there exists the set $\{c_\sigma|\sigma\in G\}\subset U(S)$ such that $c'_{\sigma,\tau}=c_{\sigma,\tau}c_\sigma^\tau c_\tau c_{\sigma\tau}^{-1}$. Moreover $\Delta(c_{\sigma,\tau}, S, G)\overset{\Phi}{\rightarrow}\Delta(c'_{\sigma,\tau}, S, G)$ $(\sum_{\sigma\in G} v_\sigma s_\sigma \underset{\Phi}{\mapsto} \sum_{\sigma\in G} v'_\sigma c_\sigma^{-1} s_\sigma)$ is an isomorphism where $\Delta(c_{\sigma,\tau}, S, G)$ and $\Delta(c'_{\sigma,\tau}, S, G)$ are crossed products, and $\{v_\sigma|\sigma\in G\}$ and $\{v'_\sigma|\sigma\in G\}$ are free S-basis of $\Delta(c_{\sigma,\tau}, S, G)$ and $\Delta(c'_{\sigma,\tau}, S, G)$ respectively. Then

$$\begin{array}{ccc}
\Delta(c_{\sigma,\tau}, S, G) & \xrightarrow{\Phi} & \Delta(c'_{\sigma,\tau}, S, G) \\
\varphi \downarrow & & \downarrow \psi \\
\Delta(\Gamma, G)\supset \sum_{\sigma\in G} u_\sigma S & \xrightarrow[\Phi']{} & \sum_{\sigma\in G} u'_\sigma S\subset \Delta(\Gamma, G)
\end{array}$$

is a commutative diagram, and Φ, Φ', φ and ψ are R-algebra isomorphisms where $\varphi(\sum v_\sigma s_\sigma)=\sum u_\sigma s_\sigma$ and $\psi(\sum v'_\sigma s_\sigma)=\sum u'_\sigma s_\sigma$. The facts that φ and ψ are isomorphisms due to the followings. $\sum_{\sigma\in G} u_\sigma S=\sum_{\sigma\in G}\oplus\sigma a_\sigma S\subset\sum_{\sigma\in G}\oplus\sigma\Gamma=\Delta(\Gamma, G)$. If $\sum u_\sigma s_\sigma=0$, $a_\sigma s_\sigma=0$ for all $\sigma\in G$. As $a_\sigma\in U(\Gamma)$, $s_\sigma=0$ for all $\sigma\in G$. By φ and ψ, we can identity $\Delta(c_{\sigma,\tau}, S, G)$ with $\sum u_\sigma S$ and $\Delta(c'_{\sigma,\tau}, S, G)$ with $\sum u'_\sigma S$. Then Φ' is the restriction map of Φ on $\sum_{\sigma\in G} u_\sigma S$. As R is a semilocal ring and has no proper idempotents, by [3] THEOREM 1.2, Φ can be extended to an inner automorphism Φ^* of $\Delta(\Gamma, G)$. That is, there exists a unit element $a\in U(\Delta(\Gamma, G))$ such that $\Phi^*(x)=a^{-1}xa$ for all $x\in\Delta(\Gamma, G)$.

$$\begin{array}{ccccc}
\Delta(\Gamma, G) & \xrightarrow{\Phi^*} & \Delta(\Gamma, G) & \qquad & x\mapsto a^{-1}xa \\
\Big| & & \Big| & & \\
\Delta(c_{\sigma,\tau}, S, G) & \xrightarrow[\Phi]{} & \Delta(c'_{\sigma,\tau}, S, G) & &
\end{array}$$

By the definition of Φ, Φ fixes all elements of S. Hence $a\in V_{\Delta(\Gamma,G)}(S)$. On the other hand, $\Gamma=V_{\Delta(\Gamma,G)}(V_{\Delta(\Gamma,G)}(\Gamma))=V_{\Delta(\Gamma,G)}(S)\ni a$. Because, by [7] THEOREM 2,

$$\begin{aligned}
\Gamma &= V_{\Delta(\Gamma,G)}(V_{\Delta(\Gamma,G)}(\Gamma)), \quad \text{and} \\
V_{\Delta(\Gamma,G)}(\Gamma) &= V_{\Lambda\otimes_R\Delta(S,G)}(\Lambda\otimes_R S) \\
&= R\otimes_R V_{\Delta(S,G)}(S) \\
&= S \qquad\qquad \text{(by LEMMA 2.1.).}
\end{aligned}$$

As $\mathfrak{H}=\bigcup_{\sigma\in G} u_\sigma U(S)$ and $\mathfrak{H}'=\bigcup_{\sigma\in G} u'_\sigma U(S)$, $\mathfrak{H}'=a^{-1}\mathfrak{H}a$. That is, $\mathfrak{H}$ and $\mathfrak{H}'$ are conjugate under an element of $U(\Gamma)$. Hence our THEOREM follows from [11] §2 COROLLARY of PROPOSITION 1. Q.E.D.

COROLLARY 2.3. *Under the same assumptions as in* THEOREM 2.2, *we obtain* $H^1(G, U(\Gamma))=1$.

PROOF. The fact that $H^1(G, U(S))=1$ (*Hilbert's* THEOREM 90, [1]. THEOREM *A*. 9.) and the exact sequence (*) lead us to the conclusion. Q.E.D.

COROLLARY 2.4. *Under the same assumptions as in* THEOREM 2.2, *and if S has no proper idempotents, we obtain a one to one onto correspondence between the image of* δ *and* $\mathfrak{T}=\{$*isomorphism class of* $T|R\subset S\subset T\subset \Delta(\Gamma, G)$, *T is a central separable R-algebra such that T contains S as a maximal commutative subalgebra*$\}$.

PROOF. The correspondence from an element $[c_{\sigma,\tau}]$ of the image of δ to an element an isomorphism class of $T=\Delta(c_{\sigma,\tau}, S, G)$ of $\mathfrak{T}$ gives its correspondence. For, let $[T]\in\mathfrak{T}$ be given. As R is a semilocal ring and S has no proper idempotents, each element of G can be extended to an inner automorphism of T ([3]. THEOREM 1.2.). Hence by [1] PROPOSITION *A*. 13, $T=\Delta(c_{\sigma,\tau}, S, G)=\sum_{\sigma\in G}\oplus w_\sigma S$ where $\{w_\sigma|\sigma\in G\}$ is a free S-base of T. If we put $\mathfrak{H}=\bigcup_{\sigma\in G} w_\sigma U(S)\subset T$, then $\mathfrak{H}\in\mathscr{K}$. For, if we put $\sigma^{-1}w_\sigma=a_\sigma$, for any $\alpha\in S$, $\alpha a_\sigma=\alpha\sigma^{-1}w_\sigma=\sigma^{-1}\alpha^{\sigma^{-1}}w_\sigma=\sigma^{-1}w_\sigma(\alpha^{\sigma^{-1}})^\sigma=\sigma^{-1}w_\sigma\alpha=a_\sigma\alpha$. Hence $a_\sigma\in V_{\Delta(\Gamma,G)}(S)=\Gamma$ (see PROOF of THEOREM 2.2) and $a_\sigma=\sigma^{-1}w_\sigma\in\Gamma\cap U(\Delta(\Gamma, G))=U(\Gamma)$. Hence $w_\sigma=\sigma a_\sigma(a_\sigma\in U(\Gamma))$. $\mathfrak{H}U(\Gamma)=(\bigcup_{\sigma\in G} w_\sigma U(S))U(\Gamma)=\bigcup_{\sigma\in G} w_\sigma U(\Gamma)=\bigcup_{\sigma\in G}\sigma a_\sigma U(\Gamma)=\bigcup_{\sigma\in G}\sigma U(\Gamma)=\mathfrak{G}$. For any $\beta\in\mathfrak{H}\cap U(\Gamma)$ we can write $\beta=w_\sigma s(s\in U(S))$. Then σ must be 1. That is, $\beta=w_1 s=c_{1,1}s\in U(S)$. Hence $\mathfrak{H}\in\mathscr{K}$. So, [11] §2 COROLLARY of PROPOSITION 1 leads us to the conclusion. Q.E.D.

LEMMA 2.5. (*cf.* [11]. §3. LEMMA 2.). *Let R be a Noetherian Hensel ring, S be a commutative ring which has no proper idempotents and S/R be a finite Galois extension with Galois group G.* (*In this case, by* [10] (43, 15) *and* (43, 16), *S is also a Hensel ring.*) *We put* $T=\Delta(c_{\sigma,\tau}, S, G)$. *Then there exists a right T-module* N_T *such that* N_T *is finitely generated projective and* (T, R)-*irreducible uniquely up to an isomorphism and* $[N:S]$ *equals the Schur index of T.*

PROOF. There exists a division R-algebra Δ such that $T=(\Delta)_n$. We put e_{ij} the matrix in $(\Delta)_n$ with 1 *in the* (i, j)-position and zeros elsewhere. We put $N=\sum_{j=1}^{n} e_{1j}\Delta$. Then this LEMMA is similarly proved as [11] §3 LEMMA 2. Q.E.D.

PROPOSITION 2.6. *Let R be a Noetherian Hensel ring, S be a com-*

mutative ring which has no proper idempotents, S/R be a finite Galois extension with Galois group G, Λ be a central separable R-algebra, $\Gamma=\Lambda\underset{R}{\otimes}S$, $[c_{\sigma,\tau}]\in H^2(G, U(S))$, $T=\Delta(c_{\sigma,\tau}, S, G)$ and M_Λ be a finitely generated projective and (Λ, R)-irreducible right Λ-module. Then if $[c_{\sigma,\tau}]$ is contained in the image of δ (i.e. $T\subset\Delta(\Gamma, G)$), s divides $[M:R]$ where s is the Schur index of T.

PROOF. By the facts that M_Λ is a right Λ-module and $S_{\Delta(S,G)}$ is a right $\Delta(S, G)$-module, $M\underset{R}{\otimes}S$ is a right $\Delta(\Gamma, G)$-module. That is, $(m\otimes s)(\sigma(\lambda\otimes s'))=m\lambda\otimes s^\sigma s'$ or $(m\otimes s)(\sigma\gamma_\sigma)=(m\otimes s^\sigma)\gamma_\sigma$ $(m\in M,\ s,\ s'\in S,\ \sigma\in G,\ \lambda\in\Lambda,\ \gamma_\sigma\in\Gamma)$. There exists an integer $n\geqq 1$ such that $\Lambda_\Lambda\cong M_\Lambda^{(n)}$ (an isomorphism as a right Λ-module, [12]. PROPOSITION 4.) where $M^{(n)}$ denotes a direct sum of n-copies of M. $M\underset{R}{\otimes}S$ is a finitely generated and projective right $\Delta(\Gamma, G)$-module. $\Delta(\Gamma, G)$ is a finitely generated and free right T-module. For, $\Delta(\Gamma, G)\cong V_{\Delta(\Gamma,G)}(T)\underset{R}{\otimes}T$ $(vt\longleftrightarrow v\otimes t)$ ([1]. THEOREM 3. 3), this isomorphism is an R-algebra isomorphism and an isomorphism as a right T-module, and $V_{\Delta(\Gamma,G)}(T)$ is a central separable R-algebra ([1]. THEOREM 3. 3.). Hence, $M\underset{R}{\otimes}S$ is a finitely generated and projective right T-module. Let N_T be a finitely generated, projective and (T, R)-irreducible right T-module. Then $M\underset{R}{\otimes}S_T\cong N_T^{(t)}$ (an isomorphism as a right T-module for an integer $t\geqq 1$). Hence, $[M:R]=[M\underset{R}{\otimes}S:S]=[N^{(t)}:S]=t[N:S]=ts$. Q.E.D.

THEOREM 2. 7. (*cf.* [11] COROLLARY of PROPOSITION 5.) *Under the same assumptions as in* PROPOSITION 2. 6, *when $\Lambda=(R)_l$, we obtain that $[c_{\sigma,\tau}]$ is contained in the image of δ if and only if s divides l.*

PROOF. In this case, as R is a division R-algebra and $[M:R]=l$. Hence we only require to prove if part. $[N^{(\frac{l}{s})}:S]=\frac{l}{s}[N:S]=l$. Hence $N^{(\frac{l}{s})}\cong M\underset{R}{\otimes}S$ as a S-module. As N_T is faithful, $T\subset\mathrm{End}_R(N^{(\frac{l}{s})})\cong\mathrm{End}_R(M\underset{R}{\otimes}S)\cong\Delta(\Gamma, G)$. Hence COROLLARY 2.4 leads us to the conclusion. Q.E.D.

PROPOSITION 2. 8. *Let $L\supset K\supset k$ be extensions of fields such that L/k and K/k are Galois extensions (finite or infinite) with Galois groups $G(L/k)$ and $G(K/k)$ respectively, and let Λ be a central simple k-algebra. We put $I(\Lambda\underset{k}{\otimes}K)=U(\Lambda\underset{k}{\otimes}K)/U(K)$ and $I(\Lambda\underset{k}{\otimes}L)=U(\Lambda\underset{k}{\otimes}L)/U(L)$. Then the following inflation map is injective.*

$$H^1\big(G(K/k),\ I(\Lambda\underset{k}{\otimes}K)\big)\xrightarrow[inf]{}H^1\big(G(L/k),\ I(\Lambda\underset{k}{\otimes}L)\big).$$

PROOF. By THEOREM 2.2, this is easily seen. Q.E.D.

PROPOSITION 2.9. *Let k be a finite dimensional algebraic number field, $\bar{k}$ be an algebraic closure of k, $\{v\}$ be the set of all valuations over k, k_v be the completion of k by v, $\bar{k}_v$ be an algebraic closure of k_v and m be an integer (>0). Then we can define canonical map*

$$\Phi_v: H^1\big(G(\bar{k}/k),\ PGL_m(\bar{k})\big) \longrightarrow H^1\big(G(\bar{k}_v/k_v),\ PGL_m(\bar{k}_v)\big).$$

Furthermore, for any $x \in H^1(G(\bar{k}/k), PGL_m(\bar{k}))$, $\Phi_v(x)=1$ for almost all v and

$$(\Phi_v): H_1\big(G(\bar{k}/k),\ PGL_m(\bar{k})\big) \longrightarrow \coprod_v H^1\big(G(\bar{k}_v/k_v),\ PGL_m(\bar{k}_v)\big)$$

is injective.

PROOF. By THEOREM 2.2 and Hasse's THEOREM ([4]), this is easily proved. Q.E.D.

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