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The best constant of the L^p Sobolev-type inequality corresponding to elliptic operator in $\mathbf{R}^N$

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ABSTRACT. The L^p Sobolev-type inequality shows that the supremum of $|u(y)|$ defined on $\mathbf{R}^N$ is estimated from above by constant C multiples of the L^p norm of $(-\Delta + a^2)u(x)$. Among such constant C, the smallest constant is the best constant C_0. If we replace C by C_0 in the L^p Sobolev-type inequality, then the equality holds for the best function $U(x)$. The aim of this paper is to find C_0 and $U(x)$ of the L^p Sobolev-type inequality. The Green function $G(x-y)$ of partial differential equation of elliptic type $(-\Delta + a^2)u(x) = f(x)$ defined on $\mathbf{R}^N$ is an important factor in this paper because C_0 and $U(x)$ consist of the Green function.

1. Introduction

Let $N = 1, 2, 3, \ldots$, $a > 0$ and $p, q > 1$ satisfying $1/p + 1/q = 1$. Let $x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$ be an independent variable. We put the Laplacian:

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_N}^2, \qquad \partial_{x_i} = \frac{\partial}{\partial x_i},$$

the surface area of N dimensional unit sphere [8, p. 517]:

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \qquad \omega_1 = 2, \qquad \omega_2 = 2\pi, \qquad \omega_3 = 4\pi, \ldots$$

and the heat kernel $H(x,t)$:

$$H(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/(4t)} \qquad (x \in \mathbf{R}^N,\ 0 < t < \infty). \tag{1.1}$$

ASSUMPTION 1.1. *We assume that*

$$\max\left\{1, \frac{N}{2}\right\} < p < \infty.$$

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LEMMA 1.2. *For any bounded continuous function $f(x)$, the partial differential equation of elliptic type*

$$(-\Delta + a^2)u = f(x) \qquad (x \in \mathbf{R}^N) \tag{1.2}$$

has a unique solution

$$u(x) = \int_{\mathbf{R}^N} G(x-y)f(y)dy, \tag{1.3}$$

where $G(x-y)$ is the Green function. $G(x)$ is given by

$$G(x) = \int_0^\infty e^{-a^2 t} H(x,t)dt \tag{1.4}$$

$$= \frac{a^{(N-2)/2}}{(2\pi)^{N/2}|x|^{(N-2)/2}} K_{(N-2)/2}(a|x|) \tag{1.5}$$

$$= \begin{cases} \dfrac{a^{n-1/2}}{(2\pi)^{n+1/2}|x|^{n-1/2}} K_{n-1/2}(a|x|) & (N = 2n+1) \\ \dfrac{a^n}{(2\pi)^{n+1}|x|^n} K_n(a|x|) & (N = 2n+2) \end{cases} \qquad (n = 0,1,2,\dots), \tag{1.6}$$

where $K_\nu(z)$ is the modified Bessel function [10, p. 170] *as*

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \qquad (\nu \notin \mathbf{Z}),$$

$$K_n(z) = K_{-n}(z) = \frac{(-1)^n}{2}\left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=n} \qquad (n \in \mathbf{Z}).$$

Here, $I_\nu(z)$ is the modified Bessel function [10, p. 170] *as*

$$I_\nu(z) = \sum_{n=0}^\infty \frac{1}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}.$$

COROLLARY 1.3. *We enumerate $G(x)$ in the case of $N = 1,2,3$ as*

$$G(x) = \begin{cases} \dfrac{1}{2a} e^{-a|x|} & (N = 1), \\ \dfrac{1}{2\pi} K_0(a|x|) & (N = 2), \\ \dfrac{1}{4\pi|x|} e^{-a|x|} & (N = 3). \end{cases}$$

We introduce the function space W^p and the L^p norm $\|\cdot\|_p$ as

$$W^p = \{u,\ \Delta u \in L^p(\mathbf{R}^N)\}, \qquad \|u\|_p = \left(\int_{\mathbf{R}^N} |u(x)|^p dx\right)^{1/p},$$

where $dx = dx_1 dx_2 \cdots dx_N$.

Theorem 1.4. *Under Assumption* 1.1, *for any* $u \in W^p$, *there exists a positive constant* C *which is independent of* u, *such that the* L^p *Sobolev-type inequality*

$$\sup_{y \in \mathbf{R}^N} |u(y)| \le C\|(-\Delta + a^2)u\|_p \tag{1.7}$$

holds. Among such C, *the best constant* $C_0 = \|G\|_q$ *is*

$$\|G\|_q^q = \frac{a^{(N-2)q-N}\omega_N}{(2\pi)^{(N/2)q}} \int_0^\infty r^{-((N-2)/2)q+N-1}(K_{(N-2)/2}(r))^q dr. \tag{1.8}$$

If we replace C *by* C_0, *then the equality holds for* $u(x) = cU(x)$, *where* c *is an arbitrary constant and* $U(x)$ *is given by*

$$U(x) = \int_{\mathbf{R}^N} G(x-y)(G(y))^{q/p} dy. \tag{1.9}$$

We refer to $U(x)$ *as the best function.*

The L^p Sobolev-type inequality (1.7) shows that the supremum of $|u(y)|$ is estimated in constant multiples of the L^p norm of $(-\Delta + a^2)u(x)$. The Green function $G(x-y)$ of $(-\Delta + a^2)u(x)$ is an important factor in this paper. The best constant of (1.7) is given by the L^q norm of the Green function. The best function of (1.7) consists of the Green function.

Applying $N = 1, 2, 3$ and the formula in [10, p. 172] to (1.8), we have Corollary 1.5.

Corollary 1.5. *We enumerate* $\|G\|_q^q$ *in the case of* $N = 1, 2, 3$ *as*

$$\|G\|_q^q = \begin{cases} \dfrac{1}{2^{q-1}a^{q+1}q} & (N = 1,\ 1 < q < \infty), \\ \dfrac{1}{(2\pi)^{q-1}a^2} \displaystyle\int_0^\infty r(K_0(r))^q dr & (N = 2,\ 1 < q < \infty), \\ \dfrac{\Gamma(3-q)}{(4\pi)^{q-1}(aq)^{3-q}} & (N = 3,\ 1 < q < 3), \end{cases}$$

where $\Gamma(z)$ *is the Gamma function.*

Furthermore, applying $p = q = 2$ to Theorem 1.4, we obtain Theorem 1.6. We note that N satisfies $N = 1, 2, 3$ by Assumption 1.1.

THEOREM 1.6. *For any function $u \in W^2$, there exists a positive constant C which is independent of u such that the L^2 Sobolev-type inequality*

$$\sup_{y \in \mathbf{R}^N} |u(y)| \le C \|(-\Delta + a^2)u\|_2 \qquad (N = 1, 2, 3) \tag{1.10}$$

holds. Among such C, the best constant $C_0 = \|G\|_2$ is

$$\|G\|_2^2 = \begin{cases} \dfrac{1}{4a^3} & (N = 1), \\ \dfrac{1}{4\pi a^2} & (N = 2), \\ \dfrac{1}{8\pi a} & (N = 3). \end{cases} \tag{1.11}$$

If we replace C by C_0, then the equality holds for $u(x) = cU(x)$, where c is an arbitrary constant and $U(x)$ is given by

$$U(x) = \begin{cases} \dfrac{1}{4a^3}(a|x| + 1)e^{-a|x|} & (N = 1,\ x \in \mathbf{R}), \\ \dfrac{|x|}{4\pi a} K_{-1}(a|x|) & (N = 2,\ x \in \mathbf{R}^2), \\ \dfrac{1}{8\pi a} e^{-a|x|} & (N = 3,\ x \in \mathbf{R}^3). \end{cases} \tag{1.12}$$

In our previous studies, we have the best constant of the L^2 Sobolev inequality corresponding to some differential equations with boundary conditions. We enumerate its differential equations. Here, we introduce the characteristic polynomial $P_M(z) = (z + a_0^2)(z + a_1^2) \cdots (z + a_{M-1}^2)$, where $0 < a_0 < a_1 < \cdots < a_{M-1}$. Let $D = d/dx$. We have the best constant of the L^2 Sobolev inequality corresponding to $P_1(-D^2)u = f(x)$ $(x \in (0, L))$ [7, 22], its discrete version [26, 27], $P_2(-D^2)u = f(x)$ $(x \in (0, \infty))$ [6, 15, 18], $P_2(-D^2)u = f(x)$ $(x \in (0, L))$ [22, 25], $P_M(-D^2)u = f(x)$ $(x \in \mathbf{R})$ [1] and $P_M(-\Delta)u = f(x)$ $(x \in \mathbf{R}^N)$ [4, 5]. Here, we also consider $2M$-th order simple type $(a_0 = \cdots = a_{M-1} = 0)$ differential equation on an interval $(-1)^M D^{2M} u = f(x)$ $(x \in (0, L))$ [17, 20, 21, 23] and its discrete version [11]. Moreover, we extend the L^2 Sobolev inequality into the L^p Sobolev inequality corresponding to $(-1)^M D^{2M} u = f(x)$ $(x \in (0, L))$ [2, 12, 13, 14, 19, 28, 29]. On the other hand, we have the best constant of the L^2 Sobolev-type inequality corresponding to $P_M(D)u = f(x)$ $(x \in \mathbf{R})$ [3], $P_M(D)u = f(x)$ $(x \in (0, 1))$ with periodic boundary condition [16], $P_M(-\Delta)u = f(x)$ in an N dimensional torus [24]. In this paper, we extend the argument of the L^2 Sobolev-type inequality into the L^p Sobolev-type inequality. We have the best constant and the best function of the L^p Sobolev-type inequality corresponding to the second order partial differential equation $(-\Delta + a^2)u = f(x)$ $(x \in \mathbf{R}^N)$.

This paper is organized as follows. In section 2, we explain the Green function. The sections 3 and 4 are devoted to the proof of Theorem 1.4 and 1.6, respectively. In section 5, we state the relation between the L^2 Sobolev-type inequality and the L^2 Sobolev inequality.

2. Green function

In this section, we prove Lemma 1.2. First, we prepare Lemma 2.1 which states the initial value problem of heat equation. Second, using Lemma 2.1, we prove Lemma 1.2. Finally, we state the L^q norm of the Green function.

Lemma 2.1. *For any bounded continuous function* $f(x)$ $(x \in \mathbf{R}^N)$, *the initial value problem*

IVP

$$\begin{cases} (\partial_t - \Delta + a^2)u = f(x) & (x \in \mathbf{R}^N,\ 0 < t < \infty), \\ u(x,0) = 0 & (x \in \mathbf{R}^N) \end{cases}$$

has a unique solution

$$u(x,t) = \int_{\mathbf{R}^N} \int_0^t e^{-a^2 s} H(x-y,s)\,ds f(y)\,dy. \tag{2.1}$$

Proof of Lemma 2.1. If we consider $u(x,t) = e^{-a^2 t} v(x,t)$, then we have

$\text{IVP}^{\sim}$

$$\begin{cases} (\partial_t - \Delta)v = e^{a^2 t} f(x) & (x \in \mathbf{R}^N,\ 0 < t < \infty), \\ v(x,0) = 0 & (x \in \mathbf{R}^N). \end{cases}$$

For $x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$ and $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbf{R}^N$, using

$$\langle \xi, x \rangle = \sum_{j=1}^N \xi_j x_j, \qquad |\xi|^2 = \langle \xi, \xi \rangle = \sum_{j=1}^N |\xi_j|^2,$$

we define the Fourier transform as

$$v(x) \overset{\frown}{\longrightarrow} \hat{v}(\xi) = \int_{\mathbf{R}^N} e^{-\sqrt{-1}\langle \xi, x \rangle} v(x)\,dx.$$

Through the Fourier transform, IVP$^{\sim}$ is transformed into

IVP$^{\sim\wedge}$

$$\begin{cases} (\partial_t + |\xi|^2)\hat{v} = e^{a^2 t}\hat{f}(\xi) & (\xi \in \mathbf{R}^N,\ 0 < t < \infty), \\ \hat{v}(\xi, 0) = 0 & (\xi \in \mathbf{R}^N). \end{cases}$$

IVP$^{\sim\wedge}$ has a solution as

$$\hat{v}(\xi, t) = \int_0^t \hat{H}(\xi, t-s) e^{a^2 s}\, ds \hat{f}(\xi), \qquad \hat{H}(\xi, t) = e^{-|\xi|^2 t}.$$

Through the inverse Fourier transform, the solution of IVP$^{\sim}$ follows from

$$v(x, t) = \int_{\mathbf{R}^N} \int_0^t H(x-y, t-s) e^{a^2 s}\, ds f(y) dy,$$

where $H(x, t)$ is the heat kernel given by (1.1). Thus, the solution of IVP is

$$\begin{aligned} u(x, t) &= e^{-a^2 t} v(x, t) \\ &= \int_{\mathbf{R}^N} \int_0^t e^{-a^2(t-s)} H(x-y, t-s) ds f(y) dy \\ &= \int_{\mathbf{R}^N} \int_0^t e^{-a^2\sigma} H(x-y, \sigma) d\sigma f(y) dy. \end{aligned}$$

Hence, we have (2.1). This completes the proof of Lemma 2.1. ■

PROOF OF LEMMA 1.2. Taking the limit as $t \to \infty$ for Lemma 2.1, we have IVP $\to$ (1.2) and $u(x, t)$ in (2.1) $\to u(x)$ in (1.3) and (1.4). So this $u(x)$ is the stationary function of $u(x, t)$. In addition, (1.4) is equal to

$$G(x) = \frac{1}{(4\pi)^{N/2}} \int_0^\infty e^{-a^2 t} t^{-N/2} e^{-|x|^2/(4t)}\, dt.$$

Applying the above equality to the formula in [9, p. 291], we have (1.5). Substituting $N = 2n+1$ and $N = 2n+2$ into (1.5), we have (1.6). This completes the proof of Lemma 1.2. ■

LEMMA 2.2. *Under Assumption* 1.1*, the* L^q *norm of* $G(x)$ *to the power of* q (1.8) *holds.*

PROOF OF LEMMA 2.2. x is denoted by the polar coordinates

$$\begin{aligned}
x_1 &= r\cos(\theta_1),\\
x_2 &= r\sin(\theta_1)\cos(\theta_2),\\
x_3 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\\
&\ \vdots\\
x_{N-2} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-3})\cos(\theta_{N-2}),\\
x_{N-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\qquad\qquad\sin(\theta_{N-2})\cos(\varphi),\\
x_N &= r\sin(\theta_1)\sin(\theta_2)\cdots\qquad\qquad\sin(\theta_{N-2})\sin(\varphi),\\
&(0 < r = |x| < \infty,\ 0 < \theta_1, \theta_2, \dots, \theta_{N-2} < \pi,\ 0 < \varphi < 2\pi).
\end{aligned}$$

Then, its Jacobian is

$$\frac{\partial(x_1, \dots, x_N)}{\partial(r, \theta_1, \dots, \theta_{N-2}, \varphi)} = r^{N-1}(\sin(\theta_1))^{N-2}(\sin(\theta_2))^{N-3}\cdots\sin(\theta_{N-2}).$$

Let the surface area of N dimensional unit sphere ω_N be

$$\omega_N = \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(\sin(\theta_1))^{N-2}(\sin(\theta_2))^{N-3}\cdots\sin(\theta_{N-2})d\varphi d\theta_{N-2}\cdots d\theta_1.$$

For (1.5), using the polar coordinates of x, we have

$$\begin{aligned}
\|G\|_q^q &= \frac{a^{((N-2)/2)q}}{(2\pi)^{(N/2)q}}\int_{\mathbf{R}^N}|x|^{-((N-2)/2)q}(K_{(N-2)/2}(a|x|))^q dx\\
&= \frac{a^{((N-2)/2)q}\omega_N}{(2\pi)^{(N/2)q}}\int_0^{\infty} r^{-((N-2)/2)q}(K_{(N-2)/2}(ar))^q r^{N-1}\,dr\\
&= \frac{a^{(N-2)q-N}\omega_N}{(2\pi)^{(N/2)q}}\int_0^{\infty}\xi^{-((N-2)/2)q+N-1}(K_{(N-2)/2}(\xi))^q d\xi.
\end{aligned}$$

If we rewrite ξ as r, then we have (1.8).

Here, we investigate the condition of integrability of (1.8). For $N = 1, 2, 3, \dots$, using the formula in [10, p. 173], we have asymptotic expansion of $r \to \infty$ as

$$K_{(N-2)/2}(r) \sim \sqrt{\frac{\pi}{2r}}e^{-r}\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{N-1}{2}+j\right)}{j!\,\Gamma\left(\frac{N-1}{2}-j\right)}\frac{1}{2^j}\frac{1}{r^j} \to 0 \qquad (r \to \infty).$$

In the case of $N = 2n+1$, using the formula in [10, p. 172], we have

$$r^{-((2n-1)/2)q+2n}(K_{(2n-1)/2}(r))^q$$
$$= \begin{cases} \left(\sqrt{\dfrac{\pi}{2}}e^{-r}\right)^q & (n=0), \\ r^{-(n-1/2)q+2n}\left(\sqrt{\dfrac{\pi}{2r}}e^{-r}\displaystyle\sum_{j=0}^{n-1}\frac{(n-1+j)!}{j!(n-1-j)!}\frac{1}{2^j}\frac{1}{r^j}\right)^q & (n=1,2,3,\ldots). \end{cases}$$

Thus, we have asymptotic expansion of $r \to 0$ as

$$r^{-((2n-1)/2)q+2n}(K_{(2n-1)/2}(r))^q$$
$$\sim \begin{cases} \sqrt{\dfrac{\pi}{2}} & (n=0), \\ \text{const. } r^{-(2n-1)q+2n} & (n=1,2,3,\ldots), \end{cases} \qquad (r \to 0).$$

Hence, the condition of integrability of (1.8) are $1 < q < \infty$ in $n = 0$ and $1 < q < \frac{2n+1}{2n-1} = \frac{N}{N-2}$ in $n = 1, 2, 3, \ldots$. In the case of $N = 2n+2$, using the formula in [10, p. 170], we have

$$r^{-nq+2n+1}(K_n(r))^q$$
$$= \begin{cases} r\left[-I_0(r)\left(\gamma+\log\dfrac{r}{2}\right)+\displaystyle\sum_{j=0}^{\infty}\frac{1}{(j!)^2}\left(\frac{r}{2}\right)^{2j}\sum_{k=1}^{j}\frac{1}{k}\right]^q & (n=0), \\ r^{-nq+2n+1}\left[(-1)^{n+1}I_n(r)\left(\gamma+\log\dfrac{r}{2}\right)\right. \\ \quad +\dfrac{(-1)^n}{2}\displaystyle\sum_{j=0}^{\infty}\frac{1}{j!(n+j)!}\left(\frac{r}{2}\right)^{n+2j}\left(\sum_{k=1}^{j}\frac{1}{k}+\sum_{k=1}^{j+n}\frac{1}{k}\right) \\ \quad \left.+\dfrac{1}{2}\displaystyle\sum_{j=0}^{n-1}(-1)^j\frac{(n-j-1)!}{j!}\left(\frac{r}{2}\right)^{2j-n}\right]^q & (n=1,2,3,\ldots), \end{cases}$$

where $I_\nu(z)$ is the modified Bessel function [10, p. 170] and $\gamma = 0.57721\cdots$ is Euler's constant. Thus, we have asymptotic expansion of $r \to 0$ as

$$r^{-nq+2n+1}(K_n(r))^q \sim \begin{cases} 0 & (n=0), \\ \text{const. } r^{-2nq+2n+1} & (n=1,2,3,\ldots), \end{cases} \qquad (r \to 0),$$

where we use $I_0(0) = 1$ and $r \log r \to 0$ $(r \to 0)$. Hence, the condition of integrability of (1.8) are $1 < q < \infty$ in $n = 0$ and $1 < q < \frac{n+1}{n} = \frac{N}{N-2}$ in $n = 1, 2, 3, \ldots$. As a result, we have the condition of integrability of (1.8) as

$$\begin{cases} N = 1,2 & \Rightarrow \quad 1 < q < \infty, \\ N = 3,4,5,\cdots & \Rightarrow \quad 1 < q < \dfrac{N}{N-2} \end{cases} \quad \Leftrightarrow \quad \max\left\{1, \frac{N}{2}\right\} < p < \infty.$$

So, under Assumption 1.1, we have $\|G\|_q^q < \infty$ for $N = 1,2,3,\dots$.

This completes the proof of Lemma 2.2. ■

3. L^p Sobolev-type inequality

In this section, we prove Theorem 1.4.

Proof of Theorem 1.4. Exchanging x and y in (1.3) and using $G(y-x) = G(x-y)$, we have

$$u(y) = \int_{\mathbf{R}^N} G(x-y)f(x)dx.$$

For any fixed $y \in \mathbf{R}^N$, applying the Hölder inequality to the above relation and using $\|G(\cdot - y)\|_q = \|G\|_q$ and $f(x) = (-\Delta + a^2)u(x)$, we have

$$|u(y)| \le \|G(\cdot - y)\|_q \|f\|_p = \|G\|_q \|(-\Delta + a^2)u\|_p.$$

Taking the supremum of the both sides with respect to y, we have the L^p Sobolev-type inequality

$$\sup_{y \in \mathbf{R}^N} |u(y)| \le \|G\|_q \|(-\Delta + a^2)u\|_p. \tag{3.1}$$

We suppose $U(x)$ satisfying

$$(-\Delta + a^2)U = (G(x))^{q/p}. \tag{3.2}$$

From Lemma 1.2, $U(x)$ is given by (1.9). By putting $x = 0$ and using (3.2), we have

$$\begin{aligned} U(0) &= \int_{\mathbf{R}^N} G(-y)(G(y))^{q/p} dy = \int_{\mathbf{R}^N} (G(y))^q dy \\ &= \left(\int_{\mathbf{R}^N} (G(y))^q dy\right)^{1/q} \left(\int_{\mathbf{R}^N} ((G(y))^{q/p})^p dy\right)^{1/p} \\ &= \|G\|_q \|(-\Delta + a^2)U\|_p, \end{aligned}$$

where we note $1/p + 1/q = 1$. Combining this with (3.1), we have

$$\|G\|_q \|(-\Delta + a^2)U\|_p = U(0) \le \sup_{y \in \mathbf{R}^N} |U(y)| \le \|G\|_q \|(-\Delta + a^2)U\|_p.$$

Hence we have

$$\sup_{y \in \mathbf{R}^N} |U(y)| = \|G\|_q \|(-\Delta + a^2) U\|_p.$$

This shows that $\|G\|_q$ in (1.8) is the best constant of the L^p Sobolev-type inequality (1.7) and the equality holds for $U(x)$ defined by (1.9). The concrete form of $\|G\|_q^q$ in (1.8) is shown by Lemma 2.2. This completes the proof of Theorem 1.4. ■

4. L^2 Sobolev-type inequality

In this section, we prove Theorem 1.6.

PROOF OF THEOREM 1.6. Applying $p = q = 2$ to Theorem 1.4, we have the L^2 Sobolev-type inequality (1.10), its best constant

$$\|G\|_2^2 = \int_{\mathbf{R}^N} |G(x)|^2 dx \tag{4.1}$$

and its best function

$$U(x) = \int_{\mathbf{R}^N} G(x-y)G(y)dy. \tag{4.2}$$

First, we show that the best function (4.2) equals to (1.12). Using $G(x)$ in (1.4) and the semi-group property of the heat kernel

$$\int_{\mathbf{R}^N} H(x-y,t)H(y,s)dy = H(x,t+s),$$

we have

$$\begin{aligned} U(x) &= \int_{\mathbf{R}^N} G(x-y)G(y)dy \\ &= \int_{\mathbf{R}^N} \int_0^\infty e^{-a^2 t} H(x-y,t)dt \int_0^\infty e^{-a^2 s} H(y,s)dsdy \\ &= \int_0^\infty \int_0^\infty \int_{\mathbf{R}^N} H(x-y,t)H(y,s)dy e^{-a^2(t+s)}\, dtds \\ &= \int_0^\infty \int_0^\infty H(x,t+s) e^{-a^2(t+s)}\, dtds. \end{aligned}$$

By the change of variables

$$\begin{cases} \tau = t+s \\ \sigma = t-s \end{cases} \Leftrightarrow \begin{cases} t = \dfrac{1}{2}(\tau+\sigma) \\ s = \dfrac{1}{2}(\tau-\sigma) \end{cases}, \qquad dtds = \frac{1}{2}\, d\sigma d\tau,$$

$U(x)$ is given as

$$\begin{aligned} U(x) &= \int_0^\infty \int_{-\tau}^{\tau} e^{-a^2\tau} H(x,\tau)\frac{1}{2}\, d\sigma d\tau \\ &= \int_0^\infty \tau e^{-a^2\tau} H(x,\tau)d\tau = \int_0^\infty \left(-\frac{1}{2a}\partial_a(e^{-a^2\tau})\right) H(x,\tau)d\tau \\ &= -\frac{1}{2a}\partial_a\left(\int_0^\infty e^{-a^2\tau} H(x,\tau)d\tau\right) = -\frac{1}{2a}\partial_a G(x). \end{aligned}$$

Here, setting $v = \dfrac{N-2}{2}$ for $G(x)$ in (1.5), we have

$$G(x) = \frac{a^v}{(2\pi)^{v+1}|x|^v} K_v(a|x|).$$

Hence, we have

$$\begin{aligned} U(x) &= -\frac{1}{2a}\partial_a G(x) = -\frac{1}{2a}\frac{1}{(2\pi)^{v+1}|x|^v}\partial_a[a^v K_v(a|x|)] \\ &= -\frac{1}{2a}\frac{a^{v-1}}{(2\pi)^{v+1}|x|^v}[vK_v(a|x|) + a|x|K_v'(a|x|)] \\ &= -\frac{1}{2a}\frac{a^{v-1}}{(2\pi)^{v+1}|x|^v}[-a|x|K_{v-1}(a|x|)], \end{aligned}$$

where we use one of the properties of the modified Bessel function [10, p. 174]:

$$vK_v(z) + zK_v'(z) = -zK_{v-1}(z).$$

So we have

$$U(x) = \frac{a^{v-1}}{2^{v+2}\pi^{v+1}|x|^{v-1}} K_{v-1}(a|x|).$$

By rewriting $v = \dfrac{N-2}{2}$ and collecting the above argument, we have

$$U(x) = \int_{\mathbf{R}^N} G(x-y)G(y)dy = \int_0^\infty te^{-a^2t}H(x,t)dt \tag{4.3}$$

$$= \frac{a^{N/2-2}}{2^{N/2+1}\pi^{N/2}|x|^{N/2-2}} K_{N/2-2}(a|x|). \tag{4.4}$$

Applying $N = 1, 2, 3$ and the formula in [10, p. 172] to (4.4), we have the best function (1.12).

Second, we show that the best constant (4.1) equals to (1.11). From $U(x)$ in (4.3), $\|G\|_2^2$ is expressed as

$$\|G\|_2^2 = \int_{\mathbf{R}^N} |G(y)|^2 dy = \int_{\mathbf{R}^N} G(0-y)G(y)dy = U(0)$$

$$= \int_0^\infty te^{-a^2t}H(0,t)dt = \frac{1}{(4\pi)^{N/2}} \int_0^\infty t^{1-N/2}e^{-a^2t}\,dt$$

$$= \frac{1}{(4\pi)^{N/2}a^{4-N}} \Gamma\left(\frac{4-N}{2}\right). \tag{4.5}$$

Applying $N = 1, 2, 3$ to (4.5), we have the best constant (1.11).

This completes the proof of Theorem 1.6. ■

From the uniqueness of the best constant, we have the following non-trivial identity.

Proposition 4.1.

$$\int_0^\infty r(K_0(r))^2 dr = \frac{1}{2}.$$

Proof of Proposition 4.1. Applying $p = q = 2$ to (1.8), we have

$$\|G\|_2^2 = \frac{\omega_N}{(2\pi)^N a^{4-N}} \int_0^\infty r(K_{(N-2)/2}(r))^2 dr. \tag{4.6}$$

Because the best constant is unique, (4.6) equals to (4.5) as

$$\frac{\omega_N}{(2\pi)^N a^{4-N}} \int_0^\infty r(K_{(N-2)/2}(r))^2 dr = \|G\|_2^2 = \frac{1}{(4\pi)^{N/2}a^{4-N}} \Gamma\left(\frac{4-N}{2}\right),$$

that is,

$$\int_0^\infty r(K_{(N-2)/2}(r))^2 dr = \frac{1}{2}\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{4-N}{2}\right).$$

If we insert $N = 1, 3$ into the above relation, then we have well-known identity from the concrete form of the modified Bessel function [10, p. 172]. If we insert $N = 2$ into the above relation, then we have the non-trivial identity which is stated by Proposition 4.1. ■

5. Relation to Sobolev inequality

In this section, we state the relation between the Sobolev-type inequality and the Sobolev inequality. We set $p = q = 2$ and $N = 1$ for simplification. Let $D = d/dx$. In the background of two inequalities, the differential equation and the Green function exist. Applying $N = 1$ to Lemma 1.2, we have Corollary 5.1.

COROLLARY 5.1. *For any bounded continuous function $f(x)$, the ordinary differential equation $(-D^2 + a^2)u = f(x)$ $(x \in \mathbf{R})$ has a unique solution*

$$u(x) = \int_{\mathbf{R}} G(x-y)f(y)dy, \qquad G(x) = \frac{1}{2a}e^{-a|x|}.$$

From Theorem 1.6, we have Corollary 5.2.

COROLLARY 5.2. *For any $u \in W^2$, the L^2 Sobolev-type inequality*

$$\left(\sup_{y\in\mathbf{R}} |u(y)|\right)^2 \le \|G\|_2^2 \|(-D^2 + a^2)u\|_2^2, \qquad \|G\|_2^2 = \frac{1}{4a^3}$$

holds. The equality holds for

$$u(x) = \int_{\mathbf{R}} G(x-y)G(y)dy = \frac{1}{4a^3}(a|x|+1)e^{-a|x|}.$$

We introduce the Hilbert space and the norm

$$H = \{u \,|\, u, u' \in L^2(\mathbf{R})\}, \qquad \|u\|_H^2 = \int_{\mathbf{R}} [|u'(x)|^2 + a^2|u(x)|^2]dx.$$

In the previous paper [1], we have Corollary 5.3.

COROLLARY 5.3. *For any $u \in H$, the L^2 Sobolev inequality*

$$\left(\sup_{y\in\mathbf{R}} |u(y)|\right)^2 \le G(0)\|u\|_H^2, \qquad G(0) = \frac{1}{2a}$$

holds. The equality holds for $u(x) = G(x)$.

From Corollary 5.2 and 5.3, we have Proposition 5.4 concerning the relation between the L^2 Sobolev-type inequality and the L^2 Sobolev inequality.

PROPOSITION 5.4. *For any* $u \in \{u, u', u'' \in L^2(\mathbf{R})\}$, *the inequality*

$$\left(\sup_{y\in\mathbf{R}} |u(y)|\right)^2 \le \frac{1}{2a}\|u\|_H^2 \le \frac{1}{2a^3}\|(-D^2+a^2)u\|_2^2$$

holds.

PROOF OF PROPOSITION 5.4. It is obvious that

$$\|u\|_H^2 = \int_{\mathbf{R}} [|u'(x)|^2 + a^2|u(x)|^2]dx \ge a^2 \int_{\mathbf{R}} |u(x)|^2 dx = a^2\|u\|_2^2,$$

that is,

$$\|u\|_2 \le \frac{1}{a}\|u\|_H. \tag{5.1}$$

On the other hand, we have

$$\begin{aligned}\|u\|_H^2 &= \int_{\mathbf{R}} [|u'(x)|^2 + a^2|u(x)|^2]dx \\ &= u'(x)\bar{u}(x)|_{x=-\infty}^{x=\infty} + \int_{\mathbf{R}} [-u''(x)\bar{u}(x) + a^2 u(x)\bar{u}(x)]dx \\ &= \int_{\mathbf{R}} ((-D^2+a^2)u(x))\bar{u}(x)dx.\end{aligned}$$

Applying the Schwarz inequality to the above relation, we have

$$\|u\|_H^2 \le \|(-D^2+a^2)u\|_2\|u\|_2. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\|u\|_H \le \frac{1}{a}\|(-D^2+a^2)u\|_2.$$

Applying the above inequality to Corollary 5.3, we have Proposition 5.4. ■

Because $\|G\|_2^2 = (4a^3)^{-1} \ne (2a^3)^{-1}$ holds, $(2a^3)^{-1}$ is not the best constant of the L^2 Sobolev-type inequality in Corollary 5.2. Hence, we see that Corollary 5.2 cannot be obtained directly from Corollary 5.3.

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