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On meromorphic functions sharing three one-point or two-point sets CM

Manabu SHIROSAKI

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ABSTRACT. We show that if three nonconstant meromorphic functions on the complex plane share three one-point or two-point sets CM, then there exist two of the meromorphic functions such that one of them is a Möbius transform of the other. The cases that all three sets are one-point and that all three sets are two-point are obtained by H. Cartan ([C1]) and by the author ([S4]), respectively.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\bar{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where we consider $1/f$ and $1/g$ for $f - f(z_0)$ and $g - g(z_0)$ if $f(z_0) = \infty$ and $g(z_0) = \infty$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [C1], H. Cartan showed the following theorem:

THEOREM A. *Let f, g and h be nonconstant meromorphic functions on $\boldsymbol{C}$ and let a_1, a_2 and a_3 be three distinct points in $\bar{\boldsymbol{C}}$. If f, g and h share a_j CM for $j = 1, 2, 3$, then at least two of f, g and h are identical.*

On the other hand the author proved

THEOREM B ([S3], see also [S2] and [ST]). *Let S_1, S_2, S_3, S_4 be four one-point or two-point sets in $\bar{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 and S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_j CM for $j = 1, \dots, 4$, then f is a Möbius transform of g, i.e., $f = (ag + b)/(cg + d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$.*

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Theorem B contains partially the result of Nevanlinna ([N1] and [N2]):

Theorem C. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and let $a_1, \dots, a_4$ be four distinct points in $\bar{\boldsymbol{C}}$. If f and g share each of $a_1, \dots, a_4$ CM, then f is a Möbius transform of g. Moreover, there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}$ and $a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

After Theorems A and B, the author proved in [S4]

Theorem D. *Let S_1, S_2 and S_3 be pairwise disjoint two-point sets in $\bar{\boldsymbol{C}}$. If three nonconstant meromorphic functions f, g and h on $\boldsymbol{C}$ share each of S_1, S_2, S_3 CM, then one of f, g and h is a Möbius transform of one of the others.*

However, this theorem does not contain the cases where one-point sets and two-point sets are mixed, differing from Theorem B, and so, in this paper we consider two cases: one is the case where three meromorphic functions on $\boldsymbol{C}$ share two one-point sets and one two-point set in $\bar{\boldsymbol{C}}$ CM, and the other is the case where three meromorphic functions on $\boldsymbol{C}$ share one one-point set and two two-point sets in $\bar{\boldsymbol{C}}$ CM. The results are as follows.

Theorem 1.1. *Let S_1 be a two-point set in $\bar{\boldsymbol{C}}$, and let S_2 and S_3 be two one-point sets in $\bar{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 are pairwise disjoint. If three nonconstant meromorphic functions f, g and h on $\boldsymbol{C}$ share each of S_1, S_2, S_3 CM, then one of f, g and h is a Möbius transform of one of the others.*

Theorem 1.2. *Let S_1 and S_2 be two two-point sets in $\bar{\boldsymbol{C}}$, and let S_3 be a one-point set in $\bar{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 are pairwise disjoint. If three nonconstant meromorphic functions f, g and h on $\boldsymbol{C}$ share each of S_1, S_2, S_3 CM, then one of f, g and h is a Möbius transform of one of the others.*

For the proofs of Theorem 1.1 and Theorem 1.2, by considering compositions $T \circ f$, $T \circ g$, $T \circ h$ of f, g, h and a suitable Möbius transformation T, it is enough to prove the following theorems in the case where all S_j are in $\boldsymbol{C}$.

Theorem 1.3. *Let S_1 be a two-point set in $\boldsymbol{C}$, and let S_2 and S_3 be two one-point sets in $\boldsymbol{C}$. Suppose that S_1, S_2, S_3 are pairwise disjoint. If three nonconstant meromorphic functions f, g and h on $\boldsymbol{C}$ share each of S_1, S_2, S_3 CM, then one of f, g and h is a Möbius transform of one of the others.*

Theorem 1.4. *Let S_1 and S_2 be two two-point sets in $\boldsymbol{C}$, and let S_3 be a one-point set in $\boldsymbol{C}$. Suppose that S_1, S_2, S_3 are pairwise disjoint. If three* *nonconstant meromorphic functions f, g and h on $\boldsymbol{C}$ share each of S_1, S_2, S_3 CM, then one of f, g and h is a Möbius transform of one of the others.*

2. Preliminaries of proofs

In [S4], the following theorem is the key for the proof of Theorem D.

Theorem 2.1 (Theorem 6 in [S4]). *Let $f = f_1/f_0$, $g = g_1/g_0$ and $h = h_1/h_0$ be nonconstant meromorphic functions on $\boldsymbol{C}$, where f_0 and f_1 are entire functions without common zero and so are g_0 and g_1, and h_0 and h_1. Let $P_j(z) = z^2 + a_j z + b_j$ $(j = 1, 2, 3)$ be polynomials such that $P_j(z)$ and $P_k(z)$ have no common zero for distinct j, k. Assume that there exist entire functions α_j, β_j without zeros such that*

$$\alpha_j({f_1}^2 + a_j f_1 f_0 + b_j {f_0}^2) = {g_1}^2 + a_j g_1 g_0 + b_j {g_0}^2 \tag{2.1}$$

and

$$\beta_j({f_1}^2 + a_j f_1 f_0 + b_j {f_0}^2) = {h_1}^2 + a_j h_1 h_0 + b_j {h_0}^2 \tag{2.2}$$

for $j = 1, 2, 3$. Then at least one of the following occurs: (A) *α_1/α_2 and α_1/α_3 are constant;* (B) *β_1/β_2 and β_1/β_3 are constant;* (C) *$(\alpha_1/\beta_1)/(\alpha_2/\beta_2)$ and $(\alpha_1/\beta_1)/(\alpha_3/\beta_3)$ are constant;* (D) *α_j/α_k and β_j/β_k are constant for some $1 \leq j < k \leq 3$.*

Remark. Note that we do not assume that P_j has no double zeros in Theorem 2.1, and hence, it is possible to use it for the proof of our theorems.

Also, we use the following results.

Theorem 2.2 ([C2] and pp. 45–46 in [H]). *Let f be a nonconstant meromorphic function on $\boldsymbol{C}$ and $a_1, \ldots, a_q$ distinct complex numbers. If all the zeros of $f - a_j$ have multiplicity at least m_j $(j = 1, \ldots, q)$ and all the poles of f have order at least m_0, where $m_0, m_1, \ldots, m_q$ are fixed positive integers, then*

$$\sum_{j=0}^{q}\left(1 - \frac{1}{m_j}\right) \leq 2.$$

For a nonconstant meromorphic function f on $\boldsymbol{C}$, we call $c \in \boldsymbol{C}$ a completely multiple value of f if all the zeros of $f - c$ have multiplicity at least 2, and also, ∞ is defined to be a completely multiple value if f has no simple poles. Note that for an exceptional value c of f we can consider that all the zeros of $f - c$ have multiplicity greater than an arbitrarily large positive integer. Therefore, we get

COROLLARY 2.3. (i) *A nonconstant meromorphic function on* $\boldsymbol{C}$ *has at most four completely multiple values in* $\bar{\boldsymbol{C}}$. (ii) *A nonconstant meromorphic function on* $\boldsymbol{C}$ *with one exceptional value has at most two completely multiple values in* $\bar{\boldsymbol{C}}$ *different from the exceptional value.* (iii) *A nonconstant meromorphic function on* $\boldsymbol{C}$ *with two exceptional values has no completely multiple values in* $\bar{\boldsymbol{C}}$ *different from the exceptional values.*

The following lemma is necessary for the proof of Theorem 1.4.

LEMMA 2.4. *Let* $P_j(z) = z^2 + a_j z + b_j$ $(j = 1, 2)$ *be two quadratic polynomials. Assume that each of them has two distinct zeros and that they have no common zeros. Let* ξ_j, η_j *be their zeros. Then for any* $\tau \in \boldsymbol{C} \backslash \{\xi_1, \eta_1, \xi_2, \eta_2\}$, *except finite specific values, the polynomials* $\tilde{P}_j(z) = z^2 + \tilde{a}_j z + \tilde{b}_j$ *with zeros* $\frac{1}{\xi_j - \tau}$ *and* $\frac{1}{\eta_j - \tau}$ *have different determinants, that is,* $\tilde{\Delta}_1 \neq \tilde{\Delta}_2$, *where* $\tilde{\Delta}_j := \tilde{a}_j{}^2 - 4\tilde{b}_j$.

PROOF. Put $\Delta_j := a_j{}^2 - 4b_j$. We have

$$\tilde{a}_j = -\left(\frac{1}{\xi_j - \tau} + \frac{1}{\eta_j - \tau}\right) = \frac{a_j + 2\tau}{P_j(\tau)}, \qquad \tilde{b}_j = \frac{1}{\xi_j - \tau} \cdot \frac{1}{\eta_j - \tau} = \frac{1}{P_j(\tau)},$$

and, hence,

$$\tilde{\Delta}_j = \frac{(a_j + 2\tau)^2 - 4(\tau^2 + a_j\tau + b_j)}{P_j(\tau)^2} = \frac{\Delta_j}{P_j(\tau)^2}.$$

So, $\tilde{\Delta}_1 = \tilde{\Delta}_2$ implies that

$$\begin{aligned} \Delta_1 P_2(\tau)^2 - \Delta_2 P_1(\tau)^2 = {} & (\Delta_1 - \Delta_2)\tau^4 + 2(a_2\Delta_1 - a_1\Delta_2)\tau^3 \\ & + (a_2{}^2\Delta_1 - a_1{}^2\Delta_2 + 2b_2\Delta_1 - 2b_1\Delta_2)\tau^2 \\ & + 2(a_2b_2\Delta_1 - a_1b_1\Delta_2)\tau + (b_2{}^2\Delta_1 - b_1{}^2\Delta_2) \end{aligned}$$

is zero. If the conclusion of the lemma does not hold, then the above is a zero polynomial about τ. Then we have $\Delta_1 = \Delta_2$, $a_2\Delta_1 = a_2\Delta_2$ and $(a_2{}^2 + 2b_2)\Delta_1 = (a_1{}^2 + 2b_1)\Delta_2$. As $\Delta_1 \neq 0$, we get $a_1 = a_2$, $b_1 = b_2$, which is a contradiction. This completes the proof. □

3. Proof of Theorem 1.3

Now, we start the proof of Theorem 1.3.

Let

$$S_1 = \{\xi_1, \eta_1\} = \{z; z^2 + a_1 z + b_1 = 0\}$$

and

$$S_j = \{\xi_j\} = \{z; z^2 + a_j z + b_j = (z - \xi_j)^2 = 0\} \qquad (j = 2, 3)$$

be pairwise disjoint sets in $\boldsymbol{C}$, where $a_1 = -(\xi_1 + \eta_1)$, $b_1 = \xi_1\eta_1$ and $a_j = -2\xi_j$, $b_j = \xi_j{}^2$ $(j = 2, 3)$, and let f, g, h be nonconstant meromorphic functions on $\boldsymbol{C}$ sharing each S_j CM. Then we can take $P_j(z) = z^2 + a_j z + b_j$ in Theorem 2.1 and there exist some entire functions α_j without zeros satisfying (2.1) and (2.2) for $j = 1, 2, 3$, where f_0, f_1, g_0, g_1, h_0, h_1 are as in Theorem 2.1. By Theorem 2.1, one of (A), (B), (C) and (D) holds.

First, we consider the case where (A) holds. Then α_2/α_3 is a nonzero constant, and from

$$\frac{\alpha_2}{\alpha_3} \cdot \frac{(f - \xi_2)^2}{(f - \xi_3)^2} = \frac{(g - \xi_2)^2}{(g - \xi_3)^2},$$

we have

$$c\frac{f - \xi_2}{f - \xi_3} = \frac{g - \xi_2}{g - \xi_3},$$

which is the conclusion. Here, c is a nonzero constant such that $c^2 = \alpha_2/\alpha_3$.

Similarly, we get the conclusion in each case (B) and (C).

Now, we consider the case (D). If $j = 2$, $k = 3$, then the conclusion is obtained in the same way as the above three cases. So, without loss of generality, we may assume that $j = 1$, $k = 2$. Then

$$c\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = \frac{g^2 + a_1 g + b_1}{g^2 + a_2 g + b_2}$$

and

$$c'\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = \frac{h^2 + a_1 h + b_1}{h^2 + a_2 h + b_2}$$

hold, where $c := \alpha_1/\alpha_2$, $c' := \beta_1/\beta_2$ are nonzero constants. If $c = 1$ or $c' = 1$ or $c = c'$, then we get the conclusion by a simple calculation. Now assume that $c \neq 1$, $c' \neq 1$ and $c \neq c'$. Then there is no $z \in \boldsymbol{C}$ such that $f(z) = g(z) = h(z) = \xi_3$.

Now, we consider quadratic homogeneous polynomials $Q_j(w_0, w_1) = w_1{}^2 + a_j w_1 w_0 + b_j w_0{}^2$ $(j = 1, 2)$. Then

$$Q_1(w_0, w_1) - \lambda Q_2(w_0, w_1) = (1 - \lambda)w_1{}^2 + (a_1 - \lambda a_2)w_1 w_0 + (b_1 - \lambda b_2)w_0{}^2 \quad (3.1)$$

has a double zero in the 1-dimensional complex projective space $\boldsymbol{P}^1(\boldsymbol{C})$ with the homogeneous coordinate system $(w_0 : w_1)$ if and only if

$$D := (a_1 - \lambda a_2)^2 - 4(1-\lambda)(b_1 - \lambda b_2) = \Delta_2\lambda^2 - 2(a_1a_2 - 2b_1 - 2b_2)\lambda + \Delta_1$$
$$= -2(a_1a_2 - 2b_1 - 2b_2)\lambda + \Delta_1 = 0,$$

where $\Delta_j = a_j{}^2 - 4b_j$ $(j = 1, 2)$ are the discriminants of $P_j(z) = 0$. By the assumption $\xi_2 \notin S_1$, we see $a_1a_2 - 2b_1 - 2b_2 = -2(\xi_2{}^2 + a_1\xi_2 + b_1) \neq 0$. Hence, the quadratic polynomial (3.1) has a double zero in $\boldsymbol{P}^1(\boldsymbol{C})$ if and only if $\lambda = \lambda_0 := \frac{\Delta_1}{2(a_1a_2 - 2b_1 - 2b_2)} (\neq 0)$. By using $\alpha_2 = \alpha_1/c$, we have

$$\alpha_1\left\{Q_1(f_0, f_1) - \frac{\lambda_0}{c}Q_2(f_0, f_1)\right\} = Q_1(g_0, g_1) - \lambda_0 Q_2(g_0, g_1).$$

The right-hand side is a square of a linear homogeneous polynomial of g_0 and g_1. If $Q_1(w_0, w_1) - \frac{\lambda_0}{c}Q_2(w_0, w_1) = 0$ expresses two distinct points in $\boldsymbol{P}^1(\boldsymbol{C})$, then f has two completely multiple values. The same thing holds for $Q_1(f_0, f_1) - \frac{\lambda_0}{c'}Q_2(f_0, f_1)$. Note that $Q_1(w_0, w_1) - \lambda Q_2(w_0, w_1)$ and $Q_1(w_0, w_1) - \mu Q_2(w_0, w_1)$ have no common zero if $\lambda \neq \mu$. By Corollary 2.3, the number of simple zeros, different from ξ_3, of one of two quadratic homogeneous polynomials

$$Q_1(w_0, w_1) - \frac{\lambda_0}{c}Q_2(w_0, w_1)$$

and

$$Q_1(w_0, w_1) - \frac{\lambda_0}{c'}Q_2(w_0, w_1)$$

is at most two. However, it is impossible since $c \neq 1$, $c' \neq 1$ and $c \neq c'$. So, we complete the proof.

4. Proof of Theorem 1.4

Now, we give the proof of Theorem 1.4.

Let

$$S_j = \{\xi_j, \eta_j\} = \{z; z^2 + a_jz + b_j = 0\} \qquad (j = 1, 2)$$

and

$$S_3 = \{\xi_3\} = \{z; z^2 + a_3z + b_3 = (z - \xi_3)^2 = 0\}$$

be pairwise disjoint sets in $\boldsymbol{C}$, where $a_j = -(\xi_j + \eta_j)$, $b_j = \xi_j\eta_j$ $(j = 1, 2)$ and $a_3 = -2\xi_3$, $b_3 = \xi_3{}^2$, and let f, g, h be nonconstant meromorphic functions on $\boldsymbol{C}$ sharing each S_j CM. By Lemma 2.4, we may assume that $\Delta_1 \neq \Delta_2$, where

$\Delta_j = a_j^2 - 4b_j$ $(j = 1, 2)$. Then we can take $P_j(z) = z^2 + a_j z + b_j$ in Theorem 2.1 and there exist some entire functions α_j without zeros satisfying (2.1) and (2.2) for $j = 1, 2, 3$, where f_0, f_1, g_0, g_1, h_0, h_1 are as in Theorem 2.1. By Theorem 2.1, one of (A), (B), (C) and (D) holds.

First, we consider the case where (A) holds. Then we have

$$\frac{\alpha_j}{\alpha_3} \cdot \frac{f^2 + a_j f + b_j}{f^2 + a_3 f + b_3} = \frac{g^2 + a_j g + b_j}{g^2 + a_3 g + b_3} \qquad (j = 1, 2).$$

Here, α_j/α_3 are nonzero constants. If $\alpha_1/\alpha_3 = 1$ or $\alpha_2/\alpha_3 = 1$, then we have the conclusion by a simple calculation. Assume that $\alpha_j/\alpha_3 \neq 1$ for $j = 1, 2$. Then there exists no $z \in \boldsymbol{C}$ such that $f(z) = g(z) \in \overline{\boldsymbol{C}} \backslash S_3$. Hence, by assumption, $f^{-1}(\xi_j) = g^{-1}(\eta_j)$, $f^{-1}(\eta_j) = g^{-1}(\xi_j)$ $(j = 1, 2)$. Consider the Möbius transformation T such that $T(\xi_j) = \eta_j$, $T(\eta_j) = \xi_j$ $(j = 1, 2)$. Then f and $T \circ g$ share four values ξ_1, η_1, ξ_2 and η_2 CM, and we get the conclusion by Theorem C. Similarly, we get the conclusion in each case (B) and (C).

Now, we consider the case (D). First consider the case where $j = 1$, $k = 3$. Then we have

$$c \frac{f^2 + a_1 f + b_1}{f^2 + a_3 f + b_3} = \frac{g^2 + a_1 g + b_1}{g^2 + a_3 g + b_3}$$

and

$$c' \frac{f^2 + a_1 f + b_1}{f^2 + a_3 f + b_3} = \frac{h^2 + a_1 h + b_1}{h^2 + a_3 h + b_3},$$

where $c := \alpha_1/\alpha_3$, $c' := \beta_1/\beta_3$. If $c = 1$ or $c' = 1$ or $c = c'$, we can get the conclusion. Otherwise, f, g and h take different values on $f^{-1}(S_2)$, or $f^{-1}(S_2) = \varnothing$. However, the former is impossible since $\sharp S_2 = 2$, and the latter is the case where f, g and h share three one-point sets $\{\xi_2\}$, $\{\eta_2\}$, $\{\xi_3\}$ and one two-point set S_1 CM, which derives the conclusion by Theorem B.

The case where $j = 2$, $k = 3$ is the same as this one.

Finally, we consider the case where $j = 1$, $k = 2$. Then

$$c \frac{P_1(f)}{P_2(f)} = \frac{P_1(g)}{P_2(g)}$$

and

$$c' \frac{P_1(f)}{P_2(f)} = \frac{P_1(h)}{P_2(h)}$$

hold, where $c := \alpha_1/\alpha_2$, $c' := \beta_1/\beta_2$ are nonzero constants. If $c = 1$ or $c' = 1$ or $c = c'$, then we get the conclusion. Now assume that $c \neq 1$, $c' \neq 1$ and $c \neq c'$. Then there is no $z \in \boldsymbol{C}$ such that $f(z) = g(z) = h(z) = \xi_3$.

Now, we consider quadratic homogeneous polynomials $Q_j(w_0, w_1) = {w_1}^2 + a_j w_1 w_0 + b_j {w_0}^2$ $(j = 1, 2)$. Then

$$Q_1(w_0, w_1) - \lambda Q_2(w_0, w_1) = (1 - \lambda){w_1}^2 + (a_1 - \lambda a_2) w_1 w_0 + (b_1 - \lambda b_2){w_0}^2$$

has a double zero in $\boldsymbol{P}^1(\boldsymbol{C})$ with the homogeneous coordinate system $(w_0 : w_1)$ if and only if $D := (a_1 - \lambda a_2)^2 - 4(1 - \lambda)(b_1 - \lambda b_2) = \Delta_2 \lambda^2 - 2(a_1 a_2 - 2b_1 - 2b_2)\lambda + \Delta_1 = 0$. Since $(a_1 a_2 - 2b_1 - 2b_2)^2 - \Delta_1 \Delta_2 = 4R(P_1, P_2) \neq 0$, where $R(P_1, P_2)$ is the resultant of P_1 and P_2, there exist two distinct λ, say λ_1 and λ_2, such that $D = 0$ for $\lambda = \lambda_1, \lambda_2$. Trivially, $\lambda_1, \lambda_2 \neq 0$. By using $\alpha_2 = \alpha_1/c$, we have

$$\alpha_1 \left\{ Q_1(f_0, f_1) - \frac{\lambda_j}{c} Q_2(f_0, f_1) \right\} = Q_1(g_0, g_1) - \lambda_j Q_2(g_0, g_1) \qquad (j = 1, 2).$$

The right-hand side is a square of a linear homogeneous polynomial of g_0 and g_1. If $Q_1(w_0, w_1) - \frac{\lambda_j}{c} Q_2(w_0, w_1) = 0$ expresses two distinct points in $\boldsymbol{P}^1(\boldsymbol{C})$, then f has two completely multiple values. The same thing holds for $Q_1(f_0, f_1) - \frac{\lambda_j}{c'} Q_2(f_0, f_1)$. Note that $Q_1(w_0, w_1) - \lambda Q_2(w_0, w_1)$ and $Q_1(w_0, w_1) - \mu Q_2(w_0, w_1)$ have no common zero if $\lambda \neq \mu$. By Corollary 2.3, the number of simple zeros different from ξ_3 of one of four quadratic homogeneous polynomials

$$Q_1(w_0, w_1) - \frac{\lambda_1}{c} Q_2(w_0, w_1), \tag{4.1}$$

$$Q_1(w_0, w_1) - \frac{\lambda_1}{c'} Q_2(w_0, w_1), \tag{4.2}$$

$$Q_1(w_0, w_1) - \frac{\lambda_2}{c} Q_2(w_0, w_1), \tag{4.3}$$

$$Q_1(w_0, w_1) - \frac{\lambda_2}{c'} Q_2(w_0, w_1) \tag{4.4}$$

is at most two. By (4.1) and (4.3), we see that $\lambda_1/c = \lambda_2$ or $\lambda_2/c = \lambda_1$ must hold. Without loss of generality, we may assume that $\lambda_2/c = \lambda_1$. In this case, since $\lambda_2/c' \neq \lambda_1$, (4.4) has only simple zeros. Furthermore, if $\lambda_1/c = \lambda_2$, then $\lambda_1/c' \neq \lambda_2$, and (4.2) has only simple zeros. If $\lambda_1/c \neq \lambda_2$, then (4.1) has only simple zeros.

In each case, the number of simple zeros different from ξ_3 of one of (4.1), (4.2), (4.3) and (4.4) is greater than two, which contradicts Corollary 2.3. Thus, the proof of Theorem 1.4 is completed.

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Manabu Shirosaki
Department of Mathematics
Osaka Metropolitan University
Sakai 599-8531 *Japan*
E-mail: shirosaki@omu.ac.jp