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On a class of fully nonlinear elliptic equations containing gradient terms on compact almost Hermitian manifolds

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Abstract. We investigate a class of second order fully nonlinear elliptic equations containing gradient terms on compact almost Hermitian manifolds and show some a priori estimates for solutions of these equations under some assumptions.

1. Introduction

Fully nonlinear elliptic equations have been considered important in complex geometry, which includes complex Monge-Ampère equations. Recent years, as a more general case, fully nonlinear elliptic equations containing gradient terms have been attracted attention, and which have played a central role in differential geometry and mathematical physics. For instance, the Fu-Yau equation which was introduced by J. Fu and S.-T. Yau as a natural generalization of Strominger system in string theory is included in these kind of generalized fully nonlinear equations. The equation related to Gauduchon conjecture, which is a generalization of Calabi conjecture, is also included, and this conjecture has been solved by showing the existence of the solution of the equation by G. Székelyhidi, V. Tosatti and B. Weinkove. In terms of these equations, it can be said that these fully nonlinear elliptic equation containing gradient terms are widely considered as an interesting subject from the view of not only differential geometry but also mathematical physics. However, the existence of gradient terms results in having the terms include $\nabla\nabla u$ for a smooth solution u in some computations, and then controlling these terms becomes a crucial part for obtaining a priori estimates. Székelyhidi has obtained a priori estimates for solutions of a class of fully nonlinear equations on compact Hermitian manifolds (cf. [5]). By applying some of his methods and dealing well with the terms including $\nabla\nabla u$, R. Yuan has derived

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a priori estimates for a class of second order fully nonlinear elliptic equations containing gradient terms on compact Hermitian manifolds (cf. [8]). Additionally, we have been able to find generalization of some results such as for the Monge-Ampère equation in the complex geometry to the almost complex geometry (cf. [3]). In the present paper, we extend Yuan's results to the almost Hermitian geometry.

Let (M^{2n}, J, g) be a compact almost Hermitian manifold of real dimension $2n$ with smooth boundary ∂M and $\overline{M} = M \cup \partial M$. Let ω be the associated real $(1,1)$-form with respect to the metric g on M. On (M^{2n}, J, g), we consider the following equation containing gradient terms of the form

$$\begin{cases} b_1(\mathfrak{g}[u])^{n-1} \wedge \omega + b_2(\mathfrak{g}[u])^{n-2} \wedge \omega^2 = \psi(\mathfrak{g}[u])^n \\ \mathfrak{g}[u] := \chi(\cdot, du) + \sqrt{-1}\partial\bar{\partial}u > 0, \end{cases} \tag{1.1}$$

where b_1 and b_2 are two nonnegative constants with $b_1 + b_2 > 0$, ψ is a smooth positive function on M, and $\chi(p, \zeta)$, $(p, \zeta) \in T^*_{\mathbb{C}}M$, is a smooth real $(1,1)$-form.

We crucially need the following subsolution in order to derive estimates.

Definition 1.1 (cf. [8, Definition 1.1]). A function $\underline{u} \in C^2(\overline{M})$ is called a $\mathscr{C}$-subsolution of the equation (1.1) if for any nonzero $(1,0)$-form γ, one obtains

$$\lim_{t \to +\infty} [\psi(\mathfrak{g}[\underline{u}, t, \gamma])^n - b_1(\mathfrak{g}[\underline{u}, t, \gamma])^{n-1} \wedge \omega - b_2(\mathfrak{g}[\underline{u}, t, \gamma])^{n-2} \wedge \omega^2] > 0, \tag{1.2}$$

where $\mathfrak{g}[\underline{u}, t, \gamma] := \mathfrak{g}[\underline{u}] + t\sqrt{-1}\gamma \wedge \bar{\gamma}$, $\mathfrak{g}[\underline{u}] = \chi(p, d\underline{u}) + \sqrt{-1}\partial\bar{\partial}\underline{u}$.

Choosing a local $(1,0)$-frame $\{Z_r\}$, we write $\chi_{i\bar{j}} = \chi(p, du)(Z_i, Z_{\bar{j}})$,

$$\chi_{i\bar{j}k} = \nabla_{Z_k}\chi_{i\bar{j}} = \chi_{i\bar{j},k} + \chi_{i\bar{j},\zeta_\alpha}u_{\alpha k} + \chi_{i\bar{j},\bar{\zeta}_\alpha}u_{\bar{\alpha}k},$$

where ∇ is the Chern connection.

We assume that the real $(1,1)$-form χ satisfies the following structural condition

$$\chi_{i\bar{j},\zeta_\alpha,\zeta_\beta} = 0, \qquad \chi_{i\bar{j},\zeta_\alpha,\bar{\zeta}_\beta} = 0, \qquad \chi_{i\bar{j},\bar{\zeta}_\alpha,\bar{\zeta}_\beta} = 0. \tag{1.3}$$

By assuming the condition (1.3), we show the gradient estimates for the equation (1.1).

Theorem 1.1. *Suppose that* (1.3) *holds and there exists a* $\mathscr{C}$*-subsolution* $\underline{u} \in C^2(\overline{M})$ *of the equation* (1.1) *in the sense of Definition* 1.1. *Then for any solution* $u \in C^3(M) \cap C^1(\overline{M})$ *of the equation* (1.1) *with* $\mathfrak{g}[u] > 0$, *there is a uniform constant* C *depending on* $|u|_{C^0(\overline{M})}$ *and some geometric quantities such as*

Christoffel symbols, coefficients of Lie bracket, Chern curvature, torsion and their derivatives, such that

$$\max_{\overline{M}} |\nabla u| \leq C\left(1 + \max_{\partial M} |\nabla u|\right),$$

where $|\cdot| := |\cdot|_g$, $|\nabla u|_g^2 = g^{i\bar{j}} \nabla_i u \nabla_{\bar{j}} u$.

We also prove a priori second order estimates.

Theorem 1.2. *Let* $u \in C^4(M) \cap C^2(\overline{M})$ *be a solution of the equation* (1.1) *such that* $\mathfrak{g}[u]$ *is positive. Suppose that the condition* (1.3) *holds and that at any fixed point* $p \in M$, *where* $g_{i\bar{j}} = \delta_{ij}$ *and* $\mathfrak{g}_{i\bar{j}} = \lambda_i \delta_{ij}$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

$$\sum_{\alpha > 1} |\mathrm{Re}(f_\alpha \chi_{\alpha\bar{1}, \zeta_\beta})| \leq \rho(\lambda) \lambda_1 f_\beta, \qquad \text{for } \forall \beta, \tag{1.4}$$

where $\rho(\lambda) : \Gamma_n \to \mathbb{R}^+$ *is a positive continuous function with* $\rho(\lambda) \to 0$ *as* $|\lambda| \to +\infty$, $|\lambda| = \sqrt{\sum \lambda_i^2}$, $\Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$ *and*

$$f(\lambda) = -\sum_i \frac{d_1}{\lambda_i} - \sum_{i<j} \frac{d_2}{\lambda_i \lambda_j}, \qquad d_1 = \frac{b_1}{n}, \qquad d_2 = \frac{2b_2}{n(n-1)}. \tag{1.5}$$

Then, there exists a uniform bounded positive constant C depending on $|u|_{C^1(\overline{M})}$, $|\psi|_{C^{1,1}(\overline{M})}$, $|\underline{u}|_{C^{1,1}(\overline{M})}$, *and some geometric quantities such as Christoffel symbols, coefficients of Lie bracket, Chern curvature, torsion and their derivatives, such that*

$$\sup_{\overline{M}} |\Delta u| \leq C\left(1 + \sup_{\partial M} |\Delta u|\right),$$

provided that there exists a $\mathscr{C}$-*subsolution* $\underline{u} \in C^2(\overline{M})$ *of the equation* (1.1) *in the sense of Definition* 1.1.

The following structural condition of the equation (1.1) plays a crucial role for proving Theorem 1.2:

$$\liminf_{|\lambda| \to \infty} \frac{f_1 \lambda_1^2}{\sum f_i} > \rho_0 \text{ in } \left\{ \lambda \in \Gamma_n : \inf_{\overline{M}} \psi \leq f(\lambda) \leq \sup_{\overline{M}} \psi \right\}, \tag{1.6}$$

where f denotes the function appears in (4.1), $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and ρ_0 is a positive constant. We can verify the equation (1.1) satisfies the condition (1.6) in the same way as in [8]. For readers convenience, we introduce the argument: If $d_2 = 0$ and $d_1 > 0$, $f(\lambda) = -\sum \frac{d_1}{\lambda_i}$, then $f_i \lambda_i^2 = f_1 \lambda_1^2 = d_1$, $\sum f_i = \sum \frac{d_1}{\lambda_i^2}$. Note that we have $\sum \frac{d_1}{\lambda_i} = \psi$. Since we have $\frac{1}{nd_1} \psi^2 \leq \sum f_i \leq \frac{1}{d_1} \psi^2$ from the Cauchy-Schwarz inequality, the condition (1.6) holds for $d_2 = 0$.

Assuming $d_2 > 0$, then we have that using $f_i \lambda_i \le \psi$ for any $i = 1, 2, \ldots, n$,

$$f_1 \lambda_1^2 = d_1 + \sum_{j>1} \frac{d_2}{\lambda_j}, \qquad \frac{f_1 \lambda_1^2}{\sum f_i} \ge \frac{f_1 \lambda_1^2}{n f_n} \ge \frac{f_1 \lambda_1^2 \lambda_n}{n\psi} \ge \frac{d_2}{n\psi},$$

which completes the proof for verifying that the equation (1.1) satisfies (1.6).

Since combining Theorem 1.1 and 1.2, the equation (1.1) becomes a uniform elliptic equation, by assuming a priori C^0 estimate for a smooth solution, then one can derive the $C^{2,\alpha}$ estimate for some $0 < \alpha < 1$ by applying the Evans-Krylov theorem (cf. [1]) and one can apply the Schauder theory to derive uniform C^k estimates for all $k \ge 0$, and we have the following corollary for closed manifolds.

Corollary 1.1. *Let (M, J, ω) be a closed Hermitian manifold and let u be a smooth solution to equation* (1.1). *Suppose conditions* (1.3), (1.4) *hold and that there is a function $\underline{u} \in C^2(M)$ such that $\mathfrak{g}[\underline{u}] > 0$ and*

$$n\psi(\mathfrak{g}[\underline{u}])^{n-1} - (n-1) b_1 (\mathfrak{g}[\underline{u}])^{n-2} \wedge \omega - (n-2) b_2 (\mathfrak{g}[\underline{u}])^{n-3} \wedge \omega^2 > 0,$$

i.e., $\underline{u}$ is a $\mathscr{C}$-subsolution of the equation (1.1). *In addition, we assume that we have a uniform bound for $|u|_{C^0(M)}$. Then, there are uniform C^∞ a priori estimates for u.*

In almost Hermitian geometry, we need to handle some geometric quantities which do not appear in Hermitian geometry, especially one of components of torsion with respect to the Chern connection which is denoted by T'' in the present paper appears as a quite troublesome quantity in some computations. For example, the computation in transferring $\nabla_k \nabla_{\bar{k}} \nabla_1 \nabla_{\bar{1}} u$ to $\nabla_1 \nabla_{\bar{1}} \nabla_k \nabla_{\bar{k}} u$, which appears in the proof of Theorem 1.2 (see Section 5), plays a crucial role, and which includes, say $Z_k(B^r_{\bar{k}\bar{1}} Z_r Z_1(u))$ for a local $(1,0)$-frame $\{Z_r\}$ and a smooth real function u satisfies the equation (1.1), that is, we need to control some terms including third order derivatives of u such as $T^r_{\bar{1}\bar{k}} \nabla_k \nabla_r \nabla_1 u$. This causes a problem for generalizing some estimates in the complex case to the almost complex case. In the complex case, since we have $T^r_{\bar{k}\bar{1}} = 0$, indeed we do not need to worry about the terms including third order derivatives of u and what we have to deal with is up to second order derivatives of u such as $\nabla\overline{\nabla} u$, $\nabla\nabla u$ and so on. In this sense, we need a new method for overcoming this difficulty in order to establish a priori estimates on almost Hermitian manifolds. In Lemma 3.1, we show that, for instance, $B^r_{\bar{k}\bar{1}} Z_1 Z_r(u)$ can be in fact expressed by $T^r_{\bar{k}\bar{1}}(T^s_{1r} Z_s(u) + T^{\bar{s}}_{1r} Z_{\bar{s}}(u))$. As a result, it turns out that the terms including third order derivatives of u do not appear also in the almost Hermitian case, and which works as a breakthrough for obtaining a priori estimates as in the complex case.

This paper is organized as follows: in section 2, we recall some basic definitions and computations on an almost Hermitian manifold (M, J, g). In section 3, for an arbitrary chosen smooth function φ and a local $(1,0)$-frame $\{Z\}$ on M, we show the result that on a particular occasion, $ZZ(\varphi)$ and $\bar{Z}\bar{Z}(\varphi)$ depend only on $Z(\varphi)$, $\bar{Z}(\varphi)$ and some geometric quantities of (M, J, g). In section 4, 5, we prove Theorem 1.1 and 1.2. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes and we employ the standard raising and lower convention for indices in all this paper.

2. Preliminaries

2.1. The Nijenhuis tensor of the almost complex structure. Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$, where TM is the real tangent vector bundle of M. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

which is the Nijenhuis tensor of J.

Let $\{Z_r\}$ be a local $(1,0)$-frame on (M, J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e., $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \ldots, n$. Since g is almost Hermitian, its components satisfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$. Using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N^k_{\bar{i}\bar{j}} Z_k, \qquad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}},$$

and

$$N = \frac{1}{2}\overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N^k_{\bar{i}\bar{j}} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}). \tag{2.2}$$

The complexified tangent vector bundle is given by $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ for the real tangent vector bundle TM. By extending J $\mathbb{C}$-linearly and g $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent vector bundle $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$\begin{aligned} T^{1,0}M &= \{X - \sqrt{-1}JX \mid X \in TM\}, \\ T^{0,1}M &= \{X + \sqrt{-1}JX \mid X \in TM\}. \end{aligned} \tag{2.3}$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q} M = \Lambda^p(\Lambda^{1,0} M) \otimes \Lambda^q(\Lambda^{0,1} M)$,

$$\Lambda^{1,0} M = \{\eta + \sqrt{-1} J\eta \mid \eta \in \Lambda^1 M\}, \qquad \Lambda^{0,1} M = \{\eta - \sqrt{-1} J\eta \mid \eta \in \Lambda^1 M\} \tag{2.4}$$

and $\Lambda^1 M$ denotes the dual of $T^{\mathbf{C}} M$.

Let (M, J, g) be an almost Hermitian manifold with $\dim_{\mathbf{R}} M = 2n$. An affine connection D on $T^{\mathbf{C}} M$ is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [6]).

LEMMA 2.1. *Let (M, J, g) be an almost Hermitian manifold with $\dim_{\mathbf{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1 \le i \le n}$, there exists a unique almost Hermitian connection ∇ on (M, J, g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connection is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇. Now let ∇ be the Chern connection on M. We denote the structure coefficients of Lie bracket by

$$[Z_i, Z_j] = B_{ij}^r Z_r + B_{ij}^{\bar{r}} Z_{\bar{r}}, \qquad [Z_i, Z_{\bar{j}}] = B_{i\bar{j}}^r Z_r + B_{i\bar{j}}^{\bar{r}} Z_{\bar{r}},$$
$$[Z_{\bar{i}}, Z_{\bar{j}}] = B_{\bar{i}\bar{j}}^r Z_r + B_{\bar{i}\bar{j}}^{\bar{r}} Z_{\bar{r}}.$$

We have $B_{ij}^k = -B_{ji}^k$ since $[Z_i, Z_j] = -[Z_j, Z_i]$. Notice that J is integrable if and only if the $B_{ij}^{\bar{r}}$'s vanish.

For any p-form ψ, there holds that

$$\begin{aligned} d\psi(X_1, \dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\psi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{p+1}) \end{aligned} \tag{2.5}$$

for any vector fields $X_1, \dots, X_{p+1}$ on M (cf. [9]). We directly compute that

$$d\zeta^s = -\frac{1}{2} B_{kl}^s \zeta^k \wedge \zeta^l - B_{k\bar{l}}^s \zeta^k \wedge \zeta^{\bar{l}} - \frac{1}{2} B_{\bar{k}\bar{l}}^s \zeta^{\bar{k}} \wedge \zeta^{\bar{l}}. \tag{2.6}$$

For any real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$, we have

$$\partial\eta = \frac{\sqrt{-1}}{2}(Z_i(\eta_{j\bar{k}}) - Z_j(\eta_{i\bar{k}}) - B^s_{ij}\eta_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\eta_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\eta_{i\bar{s}})\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}. \quad (2.7)$$

We can split the exterior differential operator $d : \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p+1} M \otimes_{\mathbb{R}} \mathbb{C}$, into four components

$$d = A + \partial + \bar{\partial} + \bar{A}$$

with

$$\partial : \Lambda^{p,q} M \to \Lambda^{p+1,q} M, \qquad \bar{\partial} : \Lambda^{p,q} M \to \Lambda^{p,q+1} M,$$

$$A : \Lambda^{p,q} M \to \Lambda^{p+2,q-1} M, \qquad \bar{A} : \Lambda^{p,q} M \to \Lambda^{p-1,q+2} M.$$

In terms of these components, the condition $d^2 = 0$ can be written as

$$A^2 = 0, \qquad \partial A + A\partial = 0, \qquad \bar{\partial}\bar{A} + \bar{A}\bar{\partial} = 0, \qquad \bar{A}^2 = 0,$$

$$A\bar{\partial} + \partial^2 + \bar{\partial}A = 0, \qquad A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0, \qquad \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0. \quad (2.8)$$

A direct computation yields for any $\varphi \in C^\infty(M, \mathbb{R})$,

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\varphi)\zeta^i \wedge \zeta^{\bar{j}}, \quad (2.9)$$

so we write locally

$$\partial_i \partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi. \quad (2.10)$$

We shall use the following notations: for a function φ,

$$\varphi_{i\bar{j}} := \partial_i \partial_{\bar{j}}\varphi = \nabla_{Z_i}\nabla_{Z_{\bar{j}}}\varphi,$$

where $\partial_i \partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi$, $\nabla_i \nabla_{\bar{j}}\varphi = Z_i Z_{\bar{j}}(\varphi) - B^{\bar{s}}_{i\bar{j}} Z_{\bar{s}}(\varphi)$. Since we have $[Z_i, Z_{\bar{j}}]^{(0,1)}\varphi = B^{\bar{s}}_{i\bar{j}} Z_{\bar{s}}(\varphi)$, we obtain that $\partial_i \partial_{\bar{j}}\varphi = \nabla_{Z_i}\nabla_{Z_{\bar{j}}}\varphi$.

At a point x_0 where φ attains its local maximum (resp. local minimum), there holds (cf. [4])

$$\{\varphi_{i\bar{j}}\}(x_0) = \{(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi\}(x_0) \le 0 \qquad (\text{resp. } \ge 0).$$

2.2. The torsion and the curvature on almost complex manifolds. Since the Chern connection ∇ preserves J, we have

$$\nabla_i Z_j := \nabla_{Z_i} Z_j = \Gamma^r_{ij} Z_r, \qquad \nabla_i Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}} Z_{\bar{r}},$$

where $\Gamma^r_{ij} = g^{r\bar{s}} Z_i(g_{j\bar{s}}) - g^{r\bar{s}} g_{j\bar{l}} B^{\bar{l}}_{i\bar{s}}$. Notice that $B^{\bar{q}}_{j\bar{b}}$, $B^q_{\bar{j}b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar{b}}$, $\nabla_{\bar{j}} Z_b$ do not depend on g. Since we have $B^q_{b\bar{j}} = -B^q_{\bar{j}b}$, we have that $B^q_{b\bar{j}}$, $B^{\bar{q}}_{\bar{b}j}$'s do not depend on g (cf. [6]). Also note that $B_{ri}^{\bar{s}}$, $B_{\bar{r}\bar{i}}^{s}$ do not depend on g, depend only on J.

Let $\{\gamma_j^i\}$ be the connection form, which is defined by $\gamma_j^i = \Gamma_{sj}^i \zeta^s + \Gamma_{\bar{s}j}^i \zeta^{\bar{s}}$. The torsion T of the Chern connection ∇ is given by

$$T^i = d\zeta^i - \zeta^p \wedge \gamma_p^i, \qquad T^{\bar{i}} = d\zeta^{\bar{i}} - \zeta^{\bar{p}} \wedge \gamma_{\bar{p}}^{\bar{i}},$$

which has no $(1,1)$-part and the only non-vanishing components are as follows:

$$\begin{aligned} T_{ij}^s &= T^s(Z_i, Z_j) \\ &= -\zeta^s([Z_i, Z_j]) - (\Gamma_{sp}^i \zeta^p \wedge \zeta^s + \Gamma_{\bar{s}p}^i \zeta^p \wedge \zeta^{\bar{s}})(Z_i, Z_j) \\ &= -B_{ij}^s - \Gamma_{ji}^s + \Gamma_{ij}^s, \\ T_{ij}^{\bar{s}} &= T^{\bar{s}}(Z_i, Z_j) = d\zeta^{\bar{s}}(Z_i, Z_j) = -\zeta^{\bar{s}}([Z_i, Z_j]) = -B_{ij}^{\bar{s}}. \end{aligned}$$

These tell us that $T = (T^i)$ splits into $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, $T'' \in \Gamma(\Lambda^{0,2}M \otimes T^{1,0}M)$. Since the torsion T of the Chern connection ∇ has no $(1,1)$-part;

$$\begin{aligned} 0 = T_{i\bar{j}}^{\bar{s}} &= T^{\bar{s}}(Z_i, Z_{\bar{j}}) \\ &= -\zeta^{\bar{s}}([Z_i, Z_{\bar{j}}]) - (\Gamma_{r\bar{p}}^{\bar{s}} \zeta^{\bar{p}} \wedge \zeta^r + \Gamma_{\bar{r}\bar{p}}^{\bar{s}} \zeta^{\bar{p}} \wedge \zeta^{\bar{r}})(Z_i, Z_{\bar{j}}) \\ &= -B_{i\bar{j}}^{\bar{s}} + \Gamma_{i\bar{j}}^{\bar{s}}, \end{aligned}$$

we obtain that

$$\Gamma_{i\bar{j}}^{\bar{s}} = B_{i\bar{j}}^{\bar{s}}.$$

By taking the complex conjugate, we have that

$$B_{\bar{j}i}^{s} = \overline{B_{j\bar{i}}^{\bar{s}}} = \overline{\Gamma_{j\bar{i}}^{\bar{s}}} = \Gamma_{\bar{j}i}^{s}.$$

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2 M \otimes \Lambda^{1,1}M$, $\Omega \in \Gamma(\Lambda^2 M \otimes \Lambda^{1,1}M)$ and Ω splits in $\Omega = \mathscr{H} + \mathscr{R} + \overline{\mathscr{H}}$, where $\mathscr{R} \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $\mathscr{H} \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$. The curvature form can be expressed by $\Omega_j^i = d\gamma_j^i + \gamma_s^i \wedge \gamma_j^s$.

In terms of Z_r's, we have

$$\begin{aligned} \mathscr{R}_{i\bar{j}k}{}^r &= \Omega_k^r(Z_i, Z_{\bar{j}}) \\ &= Z_i(\Gamma_{\bar{j}k}^r) - Z_{\bar{j}}(\Gamma_{ik}^r) + \Gamma_{is}^r \Gamma_{\bar{j}k}^s - \Gamma_{\bar{j}s}^r \Gamma_{ik}^s - B_{i\bar{j}}^s \Gamma_{sk}^r + B_{\bar{j}i}^{\bar{s}} \Gamma_{\bar{s}k}^r \\ &= -(Z_{\bar{j}}(\Gamma_{ik}^r) - Z_i(\Gamma_{\bar{j}k}^r) + \Gamma_{\bar{j}s}^r \Gamma_{ik}^s - \Gamma_{is}^r \Gamma_{\bar{j}k}^s - B_{\bar{j}i}^s \Gamma_{sk}^r + B_{i\bar{j}}^{\bar{s}} \Gamma_{\bar{s}k}^r) \\ &= -\mathscr{R}_{\bar{j}ik}{}^r, \end{aligned} \tag{2.11}$$

$$\begin{aligned}\mathscr{H}_{ijk}{}^{r} &= \Omega_k^r(Z_i, Z_j)\\ &= Z_i(\Gamma_{jk}^r) - Z_j(\Gamma_{ik}^r) + \Gamma_{is}^r\Gamma_{jk}^s - \Gamma_{js}^r\Gamma_{ik}^s - B_{ij}^s\Gamma_{sk}^r - B_{ij}^{\bar{s}}\Gamma_{\bar{s}k}^r\\ &= -(Z_j(\Gamma_{ik}^r) - Z_i(\Gamma_{jk}^r) + \Gamma_{js}^r\Gamma_{ik}^s - \Gamma_{is}^r\Gamma_{jk}^s - B_{ji}^s\Gamma_{sk}^r - B_{ji}^{\bar{s}}\Gamma_{\bar{s}k}^r)\\ &= -\mathscr{H}_{jik}{}^{r},\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathscr{H}_{\bar{i}\bar{j}k}{}^{r} &= \Omega_k^r(Z_{\bar{i}}, Z_{\bar{j}})\\ &= Z_{\bar{i}}(\Gamma_{\bar{j}k}^r) - Z_{\bar{j}}(\Gamma_{\bar{i}k}^r) + \Gamma_{\bar{i}s}^r\Gamma_{\bar{j}k}^s - \Gamma_{\bar{j}s}^r\Gamma_{\bar{i}k}^s - B_{\bar{i}\bar{j}}^s\Gamma_{sk}^r - B_{\bar{i}\bar{j}}^{\bar{s}}\Gamma_{\bar{s}k}^r\\ &= -(Z_{\bar{j}}(\Gamma_{\bar{i}k}^r) - Z_{\bar{i}}(\Gamma_{\bar{j}k}^r) + \Gamma_{\bar{j}s}^r\Gamma_{\bar{i}k}^s - \Gamma_{\bar{i}s}^r\Gamma_{\bar{j}k}^s - B_{\bar{j}\bar{i}}^s\Gamma_{sk}^r - B_{\bar{j}\bar{i}}^{\bar{s}}\Gamma_{\bar{s}k}^r)\\ &= -\mathscr{H}_{\bar{j}\bar{i}k}{}^{r}.\end{aligned} \tag{2.13}$$

Lemma 2.2 (The first Bianchi identity for the Chern curvature). *For any* $X, Y, Z \in T^{\mathbb{C}}M$,

$$\sum \Omega(X, Y)Z = \sum (T(T(X, Y), Z) + \nabla_X T(Y, Z)),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$\mathscr{R}_{i\bar{j}k}{}^{l} = \mathscr{R}_{k\bar{j}i}{}^{l} - T_{ik}^{\bar{r}}T_{\bar{r}\bar{j}}^{l} + \nabla_{\bar{j}}T_{ki}^{l} = \mathscr{R}_{k\bar{j}i}{}^{l} - B_{ik}^{\bar{r}}B_{\bar{r}\bar{j}}^{l} + \nabla_{\bar{j}}T_{ki}^{l}, \tag{2.14}$$

$$\mathscr{H}_{ijk}{}^{l} = T_{ji}^{\bar{r}}T_{\bar{r}\bar{l}}^{\bar{k}} + \nabla_{\bar{l}}T_{ji}^{\bar{k}} = -B_{ji}^{\bar{r}}T_{\bar{r}\bar{l}}^{\bar{k}} + \nabla_{\bar{l}}T_{ji}^{\bar{k}}, \tag{2.15}$$

where we used that $\mathscr{R}_{ij\bar{k}\bar{l}} = \mathscr{R}_{\bar{i}\bar{j}kl} = \mathscr{H}_{j\bar{l}ik} = \mathscr{H}_{\bar{j}\bar{l}i\bar{k}} = \mathscr{H}_{\bar{l}ijk} = \mathscr{H}_{l\bar{i}\bar{j}\bar{k}} = 0$.

Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to a local g-unitary $(1,0)$-frame, we have $g_{i\bar{j}} = \delta_{ij}$ for any $i, j, k = 1, \ldots, n$, and the Christoffel symbols satisfy

$$\Gamma_{ij}^k = -\Gamma_{i\bar{k}}^{\bar{j}}, \qquad \Gamma_{\bar{i}\bar{j}}^{\bar{k}} = -\Gamma_{\bar{i}k}^{j},$$

since we have

$$\begin{aligned}\Gamma_{ij}^k &= g(\nabla_i Z_j, Z_{\bar{k}}) = Z_i(g_{j\bar{k}}) - g(Z_j, \nabla_i Z_{\bar{k}}) = -\Gamma_{i\bar{k}}^{\bar{j}},\\ \Gamma_{\bar{i}\bar{j}}^{\bar{k}} &= g(Z_k, \nabla_{\bar{i}} Z_{\bar{j}}) = Z_{\bar{i}}(g_{k\bar{j}}) - g(\nabla_{\bar{i}} Z_k, Z_{\bar{j}}) = -\Gamma_{\bar{i}k}^{j}.\end{aligned}$$

We also have

$$\begin{aligned}\mathscr{R}_{i\bar{j}k}{}^{r} &= Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} \\ &= -Z_i(\Gamma^{\bar{k}}_{\bar{j}\bar{r}}) + Z_{\bar{j}}(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{\bar{j}\bar{s}} - \Gamma^{\bar{s}}_{\bar{j}\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{i\bar{j}}\Gamma^{\bar{k}}_{s\bar{r}} - B^{\bar{s}}_{\bar{j}i}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\ &= -\mathscr{R}_{i\bar{j}\bar{r}}{}^{\bar{k}}, \end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathscr{H}_{ijk}{}^{r} &= Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} \\ &= -Z_i(\Gamma^{\bar{k}}_{j\bar{r}}) - Z_j(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{j\bar{s}} - \Gamma^{\bar{s}}_{j\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{ij}\Gamma^{\bar{k}}_{s\bar{r}} + B^{\bar{s}}_{ij}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\ &= -\mathscr{H}_{ij\bar{r}}{}^{\bar{k}} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\overline{\mathscr{R}_{i\bar{j}k}{}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{j\bar{k}}) - Z_j(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{j\bar{k}} - \Gamma^{\bar{r}}_{j\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}j}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} + B^s_{j\bar{i}}\Gamma^{\bar{r}}_{s\bar{k}} \\ &= Z_j(\Gamma^k_{\bar{i}r}) - Z_{\bar{i}}(\Gamma^k_{jr}) + \Gamma^s_{\bar{i}r}\Gamma^k_{js} - \Gamma^s_{jr}\Gamma^k_{\bar{i}s} - B^s_{j\bar{i}}\Gamma^k_{sr} + B^{\bar{s}}_{\bar{i}j}\Gamma^k_{\bar{s}r} \\ &= \mathscr{R}_{j\bar{i}r}{}^{k}, \end{aligned} \tag{2.18}$$

$$\begin{aligned}\overline{\mathscr{H}_{ijk}{}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{\bar{j}\bar{k}}) - Z_{\bar{j}}(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{\bar{j}\bar{k}} - \Gamma^{\bar{r}}_{\bar{j}\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} - B^s_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{s\bar{k}} \\ &= -Z_{\bar{i}}(\Gamma^k_{\bar{j}r}) + Z_{\bar{j}}(\Gamma^k_{\bar{i}r}) + \Gamma^s_{\bar{i}r}\Gamma^k_{\bar{j}s} - \Gamma^s_{\bar{j}r}\Gamma^k_{\bar{i}s} - B^{\bar{s}}_{\bar{j}\bar{i}}\Gamma^k_{\bar{s}r} - B^s_{\bar{j}\bar{i}}\Gamma^k_{sr} \\ &= \mathscr{H}_{\bar{j}\bar{i}r}{}^{k}. \end{aligned} \tag{2.19}$$

Hence we obtain $\mathscr{R}_{i\bar{j}k\bar{r}} = -\mathscr{R}_{i\bar{j}\bar{r}k}$, $\mathscr{H}_{ijk\bar{r}} = -\mathscr{H}_{ij\bar{r}k}$ and $\overline{\mathscr{R}_{i\bar{j}k\bar{r}}} = \mathscr{R}_{j\bar{i}r\bar{k}}$, $\overline{\mathscr{H}_{ijk\bar{r}}} = \mathscr{H}_{\bar{j}\bar{i}r\bar{k}}$ by using a local $(1,0)$-unitary frame with respect to g.

3. Some results for a smooth real-valued function on almost Hermitian manifolds

Let (M, J, g) be an almost Hermitian manifold. In what follows, for any smooth real-valued function φ on M, we shall use the following notations of derivatives with respect to the Chern connection ∇ of g with a local $(1,0)$-frame $\{Z_r\}$ such that

$$\varphi_i = \nabla_i\varphi = \partial\varphi(Z_i) = Z_i(\varphi),$$

$$\varphi_{i\bar{j}} = \nabla_{\bar{j}}\nabla_i\varphi, \qquad \varphi_{i\bar{j}k} = \nabla_k\nabla_{\bar{j}}\nabla_i\varphi, \qquad \varphi_{i\bar{j}k\bar{l}} = \nabla_{\bar{l}}\nabla_k\nabla_{\bar{j}}\nabla_i\varphi.$$

Note that we have that

$$\begin{aligned}\varphi_{\bar{j}i} &= \nabla_i\nabla_{\bar{j}}\varphi \\ &= Z_iZ_{\bar{j}}(\varphi) - \Gamma^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}}\end{aligned}$$

$$= Z_{\bar{j}} Z_i(\varphi) + [Z_i, Z_{\bar{j}}](\varphi) - \Gamma^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}}$$

$$= \nabla_{\bar{j}}\nabla_i\varphi + \Gamma^{s}_{\bar{j}i}\varphi_s + B^{s}_{i\bar{j}}\varphi_s + B^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}} - \Gamma^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}}$$

$$= \nabla_{\bar{j}}\nabla_i\varphi + \Gamma^{s}_{\bar{j}i}\varphi_s - B^{s}_{\bar{j}i}\varphi_s + \Gamma^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}} - \Gamma^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}}$$

$$= \nabla_{\bar{j}}\nabla_i\varphi + \Gamma^{s}_{\bar{j}i}\varphi_s - \Gamma^{s}_{\bar{j}i}\varphi_s$$

$$= \nabla_{\bar{j}}\nabla_i\varphi$$

$$= \varphi_{i\bar{j}},$$

where we used that $B^{\bar{s}}_{i\bar{j}} = \Gamma^{\bar{s}}_{i\bar{j}}$ and $B^{s}_{i\bar{j}} = -B^{s}_{\bar{j}i} = -\Gamma^{s}_{\bar{j}i}$. From the computation above, for instance, we have the following:

$$\nabla_k\varphi_{i\bar{j}} = \nabla_k\varphi_{\bar{j}i}, \qquad \nabla_{\bar{k}}\varphi_{i\bar{j}} = \nabla_{\bar{k}}\varphi_{\bar{j}i}, \qquad \nabla_{\bar{l}}\nabla_k\varphi_{i\bar{j}} = \nabla_{\bar{l}}\nabla_k\varphi_{\bar{j}i}. \tag{3.1}$$

We can choose a local unitary $(1,0)$-frame $\{Z_r\}$ with respect to g around a point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$ (cf. [7]). Then we have $\Gamma^k_{ij}(p_0) = 0$ since $\nabla_i Z_j(p_0) = \Gamma^k_{ij}(p_0) Z_k = 0$, also we obtain that

$$[Z_i, Z_{\bar{j}}](p_0) = \nabla_{Z_i} Z_j(p_0) - \nabla_{Z_j} Z_i(p_0) - T(Z_i, Z_{\bar{j}})(p_0)$$
$$= 0 \qquad \text{for all } i, j = 1, \dots, n.$$

Then we have that $0 = [Z_i, Z_{\bar{j}}](p_0) = B^k_{i\bar{j}}(p_0) Z_k + B^{\bar{k}}_{i\bar{j}}(p_0) Z_{\bar{k}}$, which gives that $B^k_{i\bar{j}}(p_0) = 0$, $B^{\bar{k}}_{i\bar{j}}(p_0) = 0$ for all $i, j, k = 1, \dots, n$. By choosing such a local unitary frame around a point p_0, since $\Gamma^k_{ij}(p_0) = \Gamma^k_{ji}(p_0) = 0$, we have that $T^k_{ij}(p_0) = -B^k_{ij}(p_0)$ for all $i, j, k = 1, \dots, n$. We show the following critical lemma for proving the main result. We choose and fix a local unitary $(1,0)$-frame $\{Z_r\}$ with respect to g around a point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$. Our computations will be done at the point p_0.

Lemma 3.1. *One has for a smooth real-valued function φ on M,*

$$B^{\bar{s}}_{kj} Z_{\bar{i}} Z_{\bar{s}}(\varphi) = T^{\bar{s}}_{kj}(T^{\bar{r}}_{\bar{i}\bar{s}}\varphi_{\bar{r}} + T^{r}_{\bar{i}\bar{s}}\varphi_r), \qquad B^{s}_{\bar{k}\bar{j}} Z_i Z_s(\varphi) = T^{s}_{\bar{k}\bar{j}}(T^{r}_{is}\varphi_r + T^{\bar{r}}_{is}\varphi_{\bar{r}}). \tag{3.2}$$

Proof. We compute that from (2.7),

$$\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}})$$

$$= Z_k(\varphi_{j\bar{i}}) - Z_j(\varphi_{k\bar{i}}) - B^{s}_{kj}\varphi_{s\bar{i}} - B^{\bar{s}}_{k\bar{i}}\varphi_{j\bar{s}} + B^{\bar{s}}_{j\bar{i}}\varphi_{k\bar{s}}$$

$$= Z_k(Z_j Z_{\bar{i}}(\varphi) - B^{\bar{s}}_{j\bar{i}}\varphi_{\bar{s}}) - Z_j(Z_k Z_{\bar{i}}(\varphi) - B^{\bar{s}}_{k\bar{i}}\varphi_{\bar{s}}) - B^{s}_{kj}(Z_s Z_{\bar{i}}(\varphi) - B^{\bar{r}}_{s\bar{i}}\varphi_{\bar{r}})$$

$$= Z_k Z_j Z_{\bar{i}}(\varphi) - Z_j Z_k Z_{\bar{i}}(\varphi) - B^{s}_{kj} Z_s Z_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{s}}_{k\bar{i}})\varphi_{\bar{s}}$$

$$= [Z_k, Z_j]Z_{\bar{i}}(\varphi) - B^s_{kj}Z_sZ_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{r}}_{s\bar{i}})\varphi_{\bar{r}}$$

$$= B^{\bar{s}}_{kj}Z_{\bar{s}}Z_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{s}}_{k\bar{i}})\varphi_{\bar{s}}$$

$$= B^{\bar{s}}_{kj}[Z_{\bar{s}}, Z_{\bar{i}}](\varphi) + B^{\bar{s}}_{kj}Z_{\bar{i}}Z_{\bar{s}}(\varphi) - \{Z_k(\Gamma^{\bar{s}}_{j\bar{i}}) - Z_j(\Gamma^{\bar{s}}_{k\bar{i}})\}\varphi_{\bar{s}}$$

$$= B^{\bar{s}}_{kj}B^r_{\bar{s}\bar{i}}\varphi_r + B^{\bar{s}}_{kj}B^{\bar{r}}_{\bar{s}\bar{i}}\varphi_{\bar{r}} + B^{\bar{s}}_{kj}Z_{\bar{i}}Z_{\bar{s}}(\varphi) - \overline{\mathscr{H}_{\bar{k}j\bar{i}}{}^s\varphi_s}, \tag{3.3}$$

where we have used $\Gamma^{\bar{k}}_{i\bar{j}}(p_0) = B^{\bar{k}}_{i\bar{j}}(p_0) = 0$, $\Gamma^k_{\bar{i}\bar{j}}(p_0) = B^k_{\bar{i}\bar{j}}(p_0) = 0$, $\Gamma^k_{ij}(p_0) = 0$ for all $i, j, k = 1, \dots, n$, and that from (2.13),

$$\mathscr{H}_{\bar{k}j\bar{i}}{}^s(p_0) = \{Z_{\bar{k}}(\Gamma^s_{\bar{j}i}) - Z_{\bar{j}}(\Gamma^s_{\bar{k}i}) + \Gamma^s_{\bar{k}r}\Gamma^r_{\bar{j}i} - \Gamma^s_{\bar{j}r}\Gamma^r_{\bar{k}i} - B^r_{\bar{k}\bar{j}}\Gamma^s_{ri} - B^{\bar{r}}_{\bar{k}\bar{j}}\Gamma^s_{\bar{r}i}\}(p_0)$$

$$= Z_{\bar{k}}(\Gamma^s_{\bar{j}i})(p_0) - Z_{\bar{j}}(\Gamma^s_{\bar{k}i})(p_0).$$

We compute that

$$\begin{aligned} B^{\bar{s}}_{kj}Z_{\bar{i}}Z_{\bar{s}}(\varphi) &= Z_{\bar{i}}(B^{\bar{s}}_{kj}Z_{\bar{s}}(\varphi)) - Z_{\bar{i}}(B^{\bar{s}}_{kj})Z_{\bar{s}}(\varphi) \\ &= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - Z_{\bar{i}}(B^{\bar{s}}_{kj})\bar{\partial}\varphi(Z_{\bar{s}}) \\ &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) - Z_{\bar{i}}(B^{\bar{s}}_{kj})\bar{\partial}\varphi(Z_{\bar{s}}), \end{aligned} \tag{3.4}$$

where we used that from (2.5),

$$\begin{aligned} \partial^2\varphi(Z_k, Z_j) &= \partial(\partial\varphi)(Z_k, Z_j) \\ &= d(\partial\varphi)(Z_k, Z_j) \\ &= Z_k(\partial\varphi(Z_j)) - Z_j(\partial\varphi(Z_k)) - \partial\varphi([Z_k, Z_j]) \\ &= Z_kZ_j(\varphi) - Z_jZ_k(\varphi) - B^s_{kj}Z_s(\varphi) \\ &= [Z_k, Z_j](\varphi) - B^s_{kj}Z_s(\varphi) \\ &= B^{\bar{s}}_{kj}\varphi_{\bar{s}}, \end{aligned} \tag{3.5}$$

and from (3.5), since $B^s_{\bar{i}k}(p_0) = 0$,

$$\begin{aligned} \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) &= \bar{\partial}(\partial^2\varphi)(Z_{\bar{i}}, Z_k, Z_j) \\ &= d(\partial^2\varphi)(Z_{\bar{i}}, Z_k, Z_j) \\ &= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - \partial^2\varphi([Z_{\bar{i}}, Z_k], Z_j) + \partial^2\varphi([Z_{\bar{i}}, Z_j], Z_k) \\ &= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - B^s_{\bar{i}k}\partial^2\varphi(Z_s, Z_j) + B^s_{\bar{i}j}\partial^2\varphi(Z_s, Z_k) \end{aligned}$$

$$= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - B_{sj}^{\bar{r}} B_{\bar{i}k}^{s} \varphi_{\bar{r}} + B_{sk}^{\bar{r}} B_{\bar{i}j}^{s} \varphi_{\bar{r}}$$

$$= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)).$$

By combining (3.3) with (3.4), we obtain

$$\begin{aligned}\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}} B_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &\quad + \{B_{kj}^{\bar{r}} B_{\bar{r}\bar{i}}^{\bar{s}} - Z_{\bar{i}}(B_{kj}^{\bar{s}}) - \overline{\mathscr{H}_{\bar{k}\bar{j}i}{}^{s}}\}\bar{\partial}\varphi(Z_{\bar{s}}) \\ &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}} B_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r),\end{aligned}$$

where we have used that from (2.15) and (2.19),

$$\begin{aligned}\overline{\mathscr{H}_{\bar{k}\bar{j}i}{}^{s}} &= \mathscr{H}_{jks}{}^{i} \\ &= -B_{kj}^{\bar{r}} T_{\bar{r}\bar{i}}^{\bar{s}} + \nabla_{\bar{i}} T_{kj}^{\bar{s}} \\ &= B_{kj}^{\bar{r}} B_{\bar{r}\bar{i}}^{\bar{s}} + Z_{\bar{i}}(T_{kj}^{\bar{s}}) - \Gamma_{\bar{i}k}^{r} T_{rj}^{\bar{s}} - \Gamma_{\bar{i}j}^{r} T_{kr}^{\bar{s}} + \Gamma_{\bar{i}\bar{r}}^{\bar{s}} T_{kj}^{\bar{r}} \\ &= B_{kj}^{\bar{r}} B_{\bar{r}\bar{i}}^{\bar{s}} + Z_{\bar{i}}(T_{kj}^{\bar{s}}).\end{aligned} \tag{3.6}$$

By taking the complex conjugate of (3.5), we have that

$$\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}}) = B_{\bar{k}\bar{j}}^{s}\varphi_s. \tag{3.7}$$

We compute by applying (3.7),

$$\begin{aligned}\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}} B_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &= \bar{\partial}(\partial^2\varphi(Z_k, Z_j))(Z_{\bar{i}}) + B_{kj}^{\bar{s}} B_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &= \bar{\partial}(B_{kj}^{\bar{s}}\bar{\partial}\varphi(Z_{\bar{s}}))(Z_{\bar{i}}) + T_{kj}^{\bar{s}} T_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &= \bar{\partial}(B_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) + B_{kj}^{\bar{s}}\bar{\partial}^2\varphi(Z_{\bar{i}}, Z_{\bar{s}}) + T_{kj}^{\bar{s}} T_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &= -\bar{\partial}(T_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) - T_{kj}^{\bar{s}} B_{\bar{i}\bar{s}}^{r} \partial\varphi(Z_r) + T_{kj}^{\bar{s}} T_{\bar{s}\bar{i}}^{r} \partial\varphi(Z_r) \\ &= -Z_{\bar{i}}(T_{kj}^{\bar{s}})\varphi_{\bar{s}} - T_{kj}^{\bar{r}} T_{\bar{r}\bar{i}}^{\bar{s}} \varphi_{\bar{s}},\end{aligned} \tag{3.8}$$

where we have used that $B_{\bar{i}\bar{s}}^{r} = -T_{\bar{i}\bar{s}}^{r} = T_{\bar{s}\bar{i}}^{r}$, and used that

$$\begin{aligned}\bar{\partial}(T_{kj}^{\bar{s}})(Z_{\bar{i}}) &= \bar{\partial}(T^{\bar{s}}(Z_k, Z_j))(Z_{\bar{i}}) \\ &= \bar{\partial}T^{\bar{s}}(Z_{\bar{i}}, Z_k, Z_j) \\ &= dT^{\bar{s}}(Z_{\bar{i}}, Z_k, Z_j)\end{aligned}$$

$$
\begin{aligned}
&= Z_{\bar{i}}(T^{\bar{s}}(Z_k, Z_j)) - Z_k(T^{\bar{s}}(Z_{\bar{i}}, Z_j)) + Z_j(T^{\bar{s}}(Z_{\bar{i}}, Z_k)) \\
&\quad - T^{\bar{s}}([Z_{\bar{i}}, Z_k], Z_j) + T^{\bar{s}}([Z_{\bar{i}}, Z_j], Z_k) - T^{\bar{s}}([Z_k, Z_j], Z_{\bar{i}}) \\
&= Z_{\bar{i}}(T^{\bar{s}}_{kj}) - B^{r}_{\bar{i}k} T^{\bar{s}}_{rj} + B^{r}_{\bar{i}j} T^{\bar{s}}_{rk} - B^{\bar{r}}_{kj} T^{\bar{s}}_{\bar{r}\bar{i}} \\
&= Z_{\bar{i}}(T^{\bar{s}}_{kj}) + T^{\bar{r}}_{kj} T^{\bar{s}}_{\bar{r}\bar{i}}.
\end{aligned}
$$

By combining (3.3) with (3.8), we obtain that

$$
\begin{aligned}
B^{\bar{s}}_{kj} Z_{\bar{i}} Z_{\bar{s}}(\varphi) &= -Z_{\bar{i}}(T^{\bar{s}}_{kj})\varphi_{\bar{s}} - T^{\bar{r}}_{kj} T^{\bar{s}}_{\bar{r}\bar{i}} \varphi_{\bar{s}} + \overline{\mathscr{H}_{\bar{k}\bar{j}i}{}^{s}\varphi_{s}} - B^{\bar{s}}_{kj} B^{r}_{\bar{s}\bar{i}} \varphi_r - B^{\bar{s}}_{kj} B^{\bar{r}}_{\bar{s}\bar{i}} \varphi_{\bar{r}} \\
&= T^{\bar{s}}_{kj}(T^{\bar{r}}_{\bar{i}\bar{s}} \varphi_{\bar{r}} + T^{r}_{\bar{i}\bar{s}} \varphi_r),
\end{aligned}
$$

where we used (3.6).

4. Proof of Theorem 1.1

Let (M, J, ω) be a compact almost Hermitian manifold with g the associated almost Hermitian metric with respect to ω and let ∇ denote the Chern connection of the metric g (see Lemma 2.1) in this whole section. We assume that u is a smooth real function which satisfies (1.1). We first rewrite the equation (1.1) as follows:

$$
F(\{\mathfrak{g}_{i\bar{j}}[u]\}) = f(\lambda(\mathfrak{g}[u])) = -\psi, \tag{4.1}
$$

where $\lambda(\mathfrak{g}[u])$ are the eigenvalues of $\mathfrak{g}[u]$ with respect to ω and f is defined by (1.5).

For a given Hermitian matrix $A = \{a_{i\bar{j}}\}$, we write (cf. [2], [3, Lemma 2.5])

$$
F^{i\bar{j}}(A) = \frac{\partial F}{\partial a_{i\bar{j}}}(A), \qquad F^{i\bar{j},k\bar{l}}(A) = \frac{\partial^2 F}{\partial a_{i\bar{j}} \partial a_{k\bar{l}}}(A).
$$

We have the following lemmas in the almost Hermitian case as well (cf. [2], [3], [5, Proposition 6, Lemma 9], [8]).

Lemma 4.1 (cf. [3, Lemma 2.3], [8, Lemma 2.1]). *For any $\sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)$, there is a $\kappa_0 > 0$ depending on σ such that $\sum_{i=1}^{n} f_i(\lambda) \geq \kappa_0$ for $\lambda \in \partial \Gamma_n^{\sigma} := \{\lambda \in \Gamma_n : f(\lambda) = \sigma\}$.*

Lemma 4.2 (cf. [3, Lemma 2.2], [8, Lemma 2.2]). *In addition to $n \geq 2$, suppose that there exists a $\mathscr{C}$-subsolution $\underline{u} \in C^2(\overline{M})$ in the sense of Definition 1.1, 1.2. Then there exist positive constants R_0, ε with the following property. If $|\lambda| \geq R_0$, then we have either*

(1) $F^{i\bar{j}}(\underline{\mathfrak{g}}_{i\bar{j}} - \mathfrak{g}_{i\bar{j}}) \geq \varepsilon \sum F^{i\bar{j}} g_{i\bar{j}}$ *or*
(2) $F^{i\bar{j}} \geq \varepsilon(F^{p\bar{q}} g_{p\bar{q}}) g^{i\bar{j}}$.

Proof (Proof of Theorem 1.1). We define

$$\phi := Ae^{\eta}, \qquad \eta := A\left[\underline{u} - u - \inf_{\overline{M}}(\underline{u} - u)\right],$$

where A is a positive constant which will be determined later. Here we suppose that $e^{\phi}|\nabla u|^2$ achieves its maximum at an interior $p_0 \in M$. We define $W := |\nabla u|^2$. We choose and fix a local unitary $(1,0)$-frame $\{Z_r\}$ with respect to g around the point p_0 such that $g_{i\bar{j}}(p_0) = \delta_{ij}$, $\nabla Z(p_0) = 0$, $\mathfrak{g}_{i\bar{j}}(p_0) = \lambda_i \delta_{ij}$ and $F^{i\bar{j}}(p_0) = f_i \delta_{ij}$ (cf. [4], [7]). Then we have at the point p_0,

$$\frac{W_i}{W} + \phi_i = 0, \qquad \frac{W_{\bar{i}}}{W} + \phi_{\bar{i}} = 0, \qquad \frac{W_{i\bar{i}}}{W} - \frac{|W_i|^2}{W^2} + \phi_{i\bar{i}} \leq 0. \tag{4.2}$$

Our computations will be done at the point p_0 in what follows. By using $\Gamma_{ij}^k(p_0) = 0$, $\Gamma_{i\bar{l}}^{\bar{k}}(p_0) = B_{i\bar{l}}^{\bar{k}}(p_0) = 0$, we compute

$$\begin{aligned}
u_{\bar{l}i\bar{j}} &= \nabla_{\bar{j}} \nabla_i \nabla_{\bar{l}} u \\
&= Z_{\bar{j}} \nabla_i \nabla_{\bar{l}} u - \Gamma_{\bar{j}i}^{k} \nabla_k \nabla_{\bar{l}} u - \Gamma_{\bar{j}\bar{l}}^{\bar{k}} \nabla_i \nabla_{\bar{k}} u \\
&= Z_{\bar{j}}(Z_i \nabla_{\bar{l}} u - \Gamma_{i\bar{l}}^{\bar{k}} \nabla_{\bar{k}} u) \\
&= Z_{\bar{j}} Z_i Z_{\bar{l}}(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}} - \Gamma_{i\bar{l}}^{\bar{k}} Z_{\bar{j}} Z_{\bar{k}}(u) \\
&= Z_{\bar{j}} Z_i Z_{\bar{l}}(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}}.
\end{aligned} \tag{4.3}$$

Since we have $[Z_i, Z_{\bar{l}}] = B_{i\bar{l}}^s Z_s + B_{i\bar{l}}^{\bar{s}} Z_{\bar{s}}$ and $B_{i\bar{l}}^s(p_0) = 0$, we obtain that

$$\begin{aligned}
&Z_{\bar{j}} Z_i Z_{\bar{l}}(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}} \\
&\quad = Z_{\bar{j}}[Z_i, Z_{\bar{l}}](u) + Z_{\bar{j}} Z_{\bar{l}} Z_i(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}} \\
&\quad = Z_{\bar{j}}(B_{i\bar{l}}^s Z_s + B_{i\bar{l}}^{\bar{s}} Z_{\bar{s}})(u) + Z_{\bar{j}} Z_{\bar{l}} Z_i(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}} \\
&\quad = Z_{\bar{j}}(B_{i\bar{l}}^s) u_s + Z_{\bar{j}}(B_{i\bar{l}}^{\bar{s}}) u_{\bar{s}} + Z_{\bar{j}} Z_{\bar{l}} Z_i(u) - Z_{\bar{j}}(\Gamma_{i\bar{l}}^{\bar{k}}) u_{\bar{k}}.
\end{aligned} \tag{4.4}$$

Note that we have by using $\Gamma_{\bar{j}\bar{l}}^{\bar{s}}(p_0) = \Gamma_{\bar{l}\bar{j}}^{\bar{s}}(p_0) = 0$,

$$B_{\bar{j}\bar{l}}^{\bar{s}} = -(\Gamma_{\bar{j}\bar{l}}^{\bar{s}} - \Gamma_{\bar{l}\bar{j}}^{\bar{s}} - B_{\bar{j}\bar{l}}^{\bar{s}}) = -T_{\bar{j}\bar{l}}^{\bar{s}}. \tag{4.5}$$

And we compute that by using $\Gamma^r_{\bar{s}i}(p_0) = 0$,

$$Z_{\bar{s}} Z_i(u) = Z_{\bar{s}} \nabla_i u = \nabla_{\bar{s}} \nabla_i u + \Gamma^r_{\bar{s}i} \nabla_r u = \nabla_{\bar{s}} \nabla_i u = u_{i\bar{s}}. \tag{4.6}$$

By using $[Z_{\bar{j}}, Z_{\bar{l}}] = B^s_{\bar{j}\bar{l}} Z_s + B^{\bar{s}}_{\bar{j}\bar{l}} Z_{\bar{s}}$ and $B^s_{\bar{j}\bar{l}} = -T^s_{\bar{j}\bar{l}}$, $B^{\bar{s}}_{\bar{j}\bar{l}} = -T^{\bar{s}}_{\bar{j}\bar{l}}$, and applying (3.2), (4.5) and (4.6),

$$\begin{aligned}
Z_{\bar{j}} Z_{\bar{l}} Z_i(u) &= [Z_{\bar{j}}, Z_{\bar{l}}] Z_i(u) + Z_{\bar{l}} Z_{\bar{j}} Z_i(u) \\
&= B^s_{\bar{j}\bar{l}} Z_s Z_i(u) + B^{\bar{s}}_{\bar{j}\bar{l}} Z_{\bar{s}} Z_i(u) + Z_{\bar{l}} Z_{\bar{j}} Z_i(u) \\
&= B^s_{\bar{j}\bar{l}} Z_i Z_s(u) + B^s_{\bar{j}\bar{l}} [Z_s, Z_i](u) - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + Z_{\bar{l}} Z_{\bar{j}} Z_i(u) \\
&= T^s_{\bar{j}\bar{l}} (T^r_{is} u_r + T^{\bar{r}}_{is} u_{\bar{r}}) - T^s_{\bar{j}\bar{l}} (B^r_{si} Z_r + B^{\bar{r}}_{si} Z_{\bar{r}})(u) - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + Z_{\bar{l}} Z_{\bar{j}} Z_i(u) \\
&= -T^s_{\bar{j}\bar{l}} T^r_{si} u_r - T^s_{\bar{j}\bar{l}} T^{\bar{r}}_{si} u_{\bar{r}} + T^s_{\bar{j}\bar{l}} T^r_{si} u_r + T^s_{\bar{j}\bar{l}} T^{\bar{r}}_{si} u_{\bar{r}} - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + Z_{\bar{l}} Z_{\bar{j}} Z_i(u) \\
&= -T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + Z_{\bar{l}} Z_{\bar{j}} Z_i(u).
\end{aligned} \tag{4.7}$$

Using $\Gamma^{\bar{s}}_{\bar{l}\bar{j}}(p_0) = 0$, $\Gamma^s_{\bar{l}i}(p_0) = 0$, we obtain that

$$\begin{aligned}
Z_{\bar{l}} Z_{\bar{j}} Z_i(u) &= Z_{\bar{l}} Z_{\bar{j}} \nabla_i u \\
&= Z_{\bar{l}} (\nabla_{\bar{j}} \nabla_i u + \Gamma^s_{\bar{j}i} Z_s(u)) \\
&= \nabla_{\bar{l}} \nabla_{\bar{j}} \nabla_i u + \Gamma^{\bar{s}}_{\bar{l}\bar{j}} \nabla_{\bar{s}} \nabla_i u + \Gamma^s_{\bar{l}i} \nabla_{\bar{j}} \nabla_s u + Z_{\bar{l}}(\Gamma^s_{\bar{j}i}) Z_s(u) + \Gamma^s_{\bar{j}i} Z_{\bar{l}} Z_s(u) \\
&= u_{i\bar{j}\bar{l}} + Z_{\bar{l}}(\Gamma^s_{\bar{j}i}) u_s.
\end{aligned} \tag{4.8}$$

Combining (4.3)–(4.4) with (4.6)–(4.8), and using $B^{\bar{s}}_{i\bar{l}} = \Gamma^{\bar{s}}_{i\bar{l}}$, $B^s_{i\bar{l}} = -B^s_{\bar{l}i} = -\Gamma^s_{\bar{l}i}$,

$$\begin{aligned}
u_{\bar{l}i\bar{j}} &= Z_{\bar{j}}(B^s_{i\bar{l}}) u_s + Z_{\bar{j}}(B^{\bar{s}}_{i\bar{l}}) u_{\bar{s}} - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + u_{i\bar{j}\bar{l}} + Z_{\bar{l}}(\Gamma^s_{\bar{j}i}) u_s - Z_{\bar{j}}(\Gamma^{\bar{s}}_{i\bar{l}}) u_{\bar{s}} \\
&= u_{i\bar{j}\bar{l}} - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + \{Z_{\bar{l}}(\Gamma^s_{\bar{j}i}) - Z_{\bar{j}}(\Gamma^s_{\bar{l}i})\} u_s \\
&= u_{i\bar{j}\bar{l}} - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + \mathscr{H}_{\bar{l}\bar{j}i}{}^s u_s \\
&= u_{i\bar{j}\bar{l}} - T^{\bar{s}}_{\bar{j}\bar{l}} u_{i\bar{s}} + O(|\nabla u|),
\end{aligned} \tag{4.9}$$

where we used that from (2.13), and $\Gamma^s_{ki}(p_0) = \Gamma^s_{\bar{k}i}(p_0) = 0$,

$$\begin{aligned}
\mathscr{H}_{\bar{l}\bar{j}i}{}^s(p_0) &= (Z_{\bar{l}}(\Gamma^s_{\bar{j}i}) - Z_{\bar{j}}(\Gamma^s_{\bar{l}i}) + \Gamma^r_{\bar{l}k} \Gamma^k_{\bar{j}i} - \Gamma^s_{\bar{j}k} \Gamma^k_{\bar{l}i} - B^k_{\bar{l}\bar{j}} \Gamma^s_{ki} - B^{\bar{k}}_{\bar{l}\bar{j}} \Gamma^s_{\bar{k}i})(p_0) \\
&= Z_{\bar{l}}(\Gamma^s_{\bar{j}i})(p_0) - Z_{\bar{j}}(\Gamma^s_{\bar{l}i})(p_0),
\end{aligned}$$

and $O(|\nabla u|)$ is the set of all terms including ∇u and whose norm $|\cdot|$ can be estimated by $C|\nabla u|$ for some positive constant C. By taking the complex conjugate of (4.9), we have that

$$\begin{aligned} u_{l\bar{i}\bar{j}} &= u_{\bar{i}\bar{j}l} - T^s_{jl} u_{s\bar{i}} + \overline{\mathscr{H}_{\bar{l}\bar{j}\bar{i}}{}^s u_s} \\ &= u_{j\bar{i}l} - T^s_{jl} u_{s\bar{i}} + O(|\nabla u|), \end{aligned} \tag{4.10}$$

where we used that $u_{\bar{i}jl} = u_{j\bar{i}l}$ from (3.1).

We compute that

$$\begin{aligned} u_{pq\bar{j}} &= \nabla_{\bar{j}} \nabla_q \nabla_p u \\ &= Z_{\bar{j}} \nabla_q \nabla_p u - \Gamma^k_{\bar{j}q} \nabla_k \nabla_p u - \Gamma^k_{\bar{j}p} \nabla_q \nabla_k u \\ &= Z_{\bar{j}} (Z_q \nabla_p u - \Gamma^k_{qp} \nabla_k u) \\ &= Z_{\bar{j}} Z_q Z_p(u) - Z_{\bar{j}}(\Gamma^k_{qp}) u_k - \Gamma^k_{qp} Z_{\bar{j}} Z_k(u) \\ &= Z_{\bar{j}} Z_q Z_p(u) - Z_{\bar{j}}(\Gamma^k_{qp}) u_k. \end{aligned} \tag{4.11}$$

Using $[Z_{\bar{j}}, Z_q](p_0) = \Gamma^s_{pq}(p_0) = \Gamma^{\bar{k}}_{q\bar{j}}(p_0) = B^k_{\bar{j}p}(p_0) = B^{\bar{k}}_{\bar{j}p}(p_0) = 0$, and applying (3.2),

$$\begin{aligned} Z_{\bar{j}} Z_q Z_p(u) &= Z_q Z_{\bar{j}} Z_p(u) + [Z_{\bar{j}}, Z_q] Z_p(u) \\ &= Z_q Z_p Z_{\bar{j}}(u) + Z_q [Z_{\bar{j}}, Z_p](u) \\ &= [Z_q, Z_p] Z_{\bar{j}}(u) + Z_p Z_q Z_{\bar{j}}(u) + Z_q \{ B^k_{\bar{j}p} Z_k(u) + B^{\bar{k}}_{\bar{j}p} Z_{\bar{k}}(u) \} \\ &= Z_p \{ \nabla_q \nabla_{\bar{j}} u + \Gamma^{\bar{k}}_{q\bar{j}} u_{\bar{k}} \} + B^k_{qp} Z_k Z_{\bar{j}}(u) + B^{\bar{k}}_{qp} Z_{\bar{k}} Z_{\bar{j}}(u) \\ &\quad + Z_q(B^k_{\bar{j}p}) u_k + Z_q(B^{\bar{k}}_{\bar{j}p}) u_{\bar{k}} \\ &= \nabla_p \nabla_q \nabla_{\bar{j}} u + \Gamma^s_{pq} \nabla_s \nabla_{\bar{j}} u + \Gamma^{\bar{s}}_{p\bar{j}} \nabla_q \nabla_{\bar{s}} u + Z_p(\Gamma^{\bar{k}}_{q\bar{j}}) u_{\bar{k}} \\ &\quad + \Gamma^{\bar{k}}_{q\bar{j}} u_{\bar{k}p} - T^k_{qp} Z_{\bar{j}} Z_k(u) + B^{\bar{k}}_{qp} [Z_{\bar{k}}, Z_{\bar{j}}](u) + B^{\bar{k}}_{qp} Z_{\bar{j}} Z_{\bar{k}}(u) \\ &\quad + Z_q(\Gamma^k_{\bar{j}p}) u_k - Z_q(B^{\bar{k}}_{p\bar{j}}) u_{\bar{k}} \\ &= u_{\bar{j}qp} - T^k_{qp} u_{k\bar{j}} - T^{\bar{k}}_{qp} \{ B^s_{\bar{k}\bar{j}} Z_s(u) + B^{\bar{s}}_{\bar{k}\bar{j}} Z_{\bar{s}}(u) \} \\ &\quad + T^{\bar{k}}_{qp} (T^{\bar{s}}_{\bar{j}\bar{k}} u_{\bar{s}} + T^s_{\bar{j}\bar{k}} u_s) + Z_q(\Gamma^k_{\bar{j}p}) u_k + \{ Z_p(\Gamma^{\bar{k}}_{q\bar{j}}) - Z_q(\Gamma^{\bar{k}}_{p\bar{j}}) \} u_{\bar{k}} \end{aligned}$$

$$= u_{\bar{j}qp} - T^k_{qp}u_{k\bar{j}} + T^{\bar{k}}_{qp}T^s_{\bar{k}\bar{j}}u_s + T^{\bar{k}}_{qp}T^{\bar{s}}_{\bar{k}\bar{j}}u_{\bar{s}} - T^{\bar{k}}_{qp}T^{\bar{s}}_{\bar{k}\bar{j}}u_{\bar{s}} - T^{\bar{k}}_{qp}T^s_{\bar{k}\bar{j}}u_s$$

$$+ Z_q(\Gamma^k_{\bar{j}p})u_k + \{\overline{Z_{\bar{p}}(\Gamma^k_{\bar{q}j}) - Z_{\bar{q}}(\Gamma^k_{\bar{p}j})}\}u_{\bar{k}}$$

$$= u_{\bar{j}qp} - T^k_{qp}u_{k\bar{j}} + Z_q(\Gamma^k_{\bar{j}p})u_k + \overline{\mathscr{H}_{\bar{p}\bar{q}j}{}^k}u_{\bar{k}}, \tag{4.12}$$

where we used that from $\Gamma^k_{sj}(p_0) = \Gamma^k_{\bar{s}j}(p_0) = 0$,

$$\mathscr{H}_{\bar{p}\bar{q}j}{}^k(p_0) = \{Z_{\bar{p}}(\Gamma^k_{\bar{q}j}) - Z_{\bar{q}}(\Gamma^k_{\bar{p}j}) + \Gamma^k_{\bar{p}s}\Gamma^s_{\bar{q}j} - \Gamma^k_{\bar{q}s}\Gamma^s_{\bar{p}j} - B^s_{\bar{p}\bar{q}}\Gamma^k_{sj} - B^{\bar{s}}_{\bar{p}\bar{q}}\Gamma^k_{\bar{s}j}\}(p_0)$$

$$= Z_{\bar{p}}(\Gamma^k_{\bar{q}j})(p_0) - Z_{\bar{q}}(\Gamma^k_{\bar{p}j})(p_0).$$

By combining (4.11) with (4.12), we obtain that using $u_{\bar{j}qp} = u_{q\bar{j}p}$ from (3.1),

$$u_{pq\bar{j}} = u_{\bar{j}qp} - T^k_{qp}u_{k\bar{j}} + \{Z_q(\Gamma^k_{\bar{j}p}) - Z_{\bar{j}}(\Gamma^k_{qp})\}u_k + \overline{\mathscr{H}_{\bar{p}\bar{q}j}{}^k}u_{\bar{k}}$$

$$= u_{q\bar{j}p} - T^k_{qp}u_{k\bar{j}} + \mathscr{R}_{q\bar{j}p}{}^k u_k + \overline{\mathscr{H}_{\bar{p}\bar{q}j}{}^k}u_{\bar{k}}$$

$$= u_{q\bar{j}p} - T^k_{qp}u_{k\bar{j}} + O(|\nabla u|), \tag{4.13}$$

where we used that from $\Gamma^k_{sp}(p_0) = \Gamma^k_{\bar{s}p}(p_0) = 0$,

$$\mathscr{R}_{q\bar{j}p}{}^k(p_0) = \{Z_q(\Gamma^k_{\bar{j}p}) - Z_{\bar{j}}(\Gamma^k_{qp}) + \Gamma^k_{qs}\Gamma^s_{\bar{j}p} - \Gamma^k_{\bar{j}s}\Gamma^s_{qp} - B^s_{q\bar{j}}\Gamma^k_{sp} + B^{\bar{s}}_{\bar{j}q}\Gamma^k_{\bar{s}p}\}(p_0)$$

$$= Z_q(\Gamma^k_{\bar{j}p})(p_0) - Z_{\bar{j}}(\Gamma^k_{qp})(p_0).$$

By applying (4.9) for $j = i$, $l = k$ and (3.1), we obtain that

$$u_{i\bar{k}\bar{i}} = \nabla_{\bar{i}}\nabla_{\bar{k}}\nabla_i u$$

$$= \nabla_{\bar{i}}\nabla_i\nabla_{\bar{k}} u$$

$$= u_{i\bar{i}\bar{k}} - T^{\bar{s}}_{\bar{i}\bar{k}}u_{i\bar{s}} + O(|\nabla u|). \tag{4.14}$$

We have that

$$W_i = \sum_k (u_{ki}u_{\bar{k}} + u_k u_{\bar{k}i}),$$

and using $u_{ki\bar{i}} = u_{i\bar{i}k} - T^s_{ik}u_{s\bar{i}} + O(|\nabla u|)$ from (4.13) for $p = k$, $q = i$, $j = i$, and (3.1),

$$W_{i\bar{i}} = u_{ki\bar{i}}u_{\bar{k}} + u_{ki}u_{\bar{k}\bar{i}} + u_{k\bar{i}}u_{\bar{k}i} + u_k u_{\bar{k}i\bar{i}}$$

$$= u_{ki\bar{i}}u_{\bar{k}} + u_{ki}u_{\bar{k}\bar{i}} + u_{k\bar{i}}u_{i\bar{k}} + u_k u_{i\bar{k}\bar{i}}$$

$$= \sum_k |u_{ki}|^2 + \sum_k |u_{i\bar{k}}|^2 - T^s_{ik} u_{s\bar{i}} + u_{i\bar{i}k} u_{\bar{k}} - T^{\bar{s}}_{\bar{i}\bar{k}} u_{i\bar{s}} u_k + u_{i\bar{i}\bar{k}} u_k + O(|\nabla u|^2)$$

$$= \sum_k |u_{ki}|^2 + \sum_k |u_{i\bar{k}}|^2 + 2\sum_k \mathrm{Re}(u_{i\bar{i}k} u_{\bar{k}}) - 2\sum_k \mathrm{Re}\{T^s_{ik} u_{s\bar{i}} u_{\bar{k}}\} + O(|\nabla u|^2)$$

$$= \sum_k |u_{ki}|^2 + 2\sum_k \mathrm{Re}(u_{i\bar{i}k} u_{\bar{k}})$$

$$+ \sum_k \left| u_{i\bar{k}} - \sum_l T^k_{il} u_{\bar{l}} \right|^2 - \sum_k \left| \sum_l T^k_{il} u_{\bar{l}} \right|^2 + O(|\nabla u|^2), \tag{4.15}$$

where $O(|\nabla u|^2)$ is the set of all terms including $\nabla u \overline{\nabla} u$ and whose norm $|\cdot|$ can be estimated by $C|\nabla u|^2$ for some positive constant C.

Differentiating the equation (4.1), we have

$$F^{i\bar{i}}(u_{i\bar{i}k} + \chi_{i\bar{i},\zeta_\alpha} u_{\alpha k} + \chi_{i\bar{i},\bar{\zeta}_\alpha} u_{\bar{\alpha}k}) = -\psi_k - F^{i\bar{i}} \chi_{i\bar{i},k}.$$

We define the following linearized operator of the equation (4.1) for $u \in C^2(M)$:

$$\mathscr{L}(u) := F^{i\bar{j}} u_{i\bar{j}} + F^{i\bar{j}} \chi_{i\bar{i},\zeta_\alpha} u_\alpha + F^{i\bar{j}} \chi_{i\bar{i},\bar{\zeta}_\alpha} u_{\bar{\alpha}}.$$

Since we have assumed (1.4) and since we have from (4.2)

$$|W_i|^2 \leq |\nabla u|^2 \sum_k |u_{ki}|^2 - 2|\nabla u|^2 \sum_k \mathrm{Re}(u_k u_{i\bar{k}} \phi_{\bar{i}}), \tag{4.16}$$

we then get by combining with (4.15),

$$\mathscr{L}(W) \geq F^{i\bar{i}} |u_{ki}|^2 - C|\nabla u| \left(1 + \sum_i F^{i\bar{i}}\right) - C|\nabla u|^2 \sum_i F^{i\bar{i}}. \tag{4.17}$$

By applying $\phi_i = \phi \eta_i$, the condition (1.3) and the Cauchy-Schwarz inequality, we have that

$$2\phi^{-1}\,\mathrm{Re}\{u_k u_{i\bar{k}} \phi_{\bar{i}}\} \geq 2\,\mathrm{Re}\{g_{i\bar{k}} u_k \eta_{\bar{i}}\} - \frac{1}{2}|\nabla u|^2 |\eta_i|^2 - C|\nabla u|^2,$$

and from $\phi_{i\bar{i}} = \phi(|\eta_i|^2 + \eta_{i\bar{i}})$, we obtain

$$\mathscr{L}\phi = \phi \mathscr{L}\eta + \phi F^{i\bar{i}} |\eta_i|^2. \tag{4.18}$$

By combining (4.16)–(4.18), then we get that

$$|\nabla u|^2\left(\frac{1}{2}F^{i\bar{i}}|\eta_i|^2 + \mathscr{L}\eta\right) - \frac{C}{\phi}|\nabla u|\left(1+\sum_i F^{i\bar{i}}\right) - \left(C|\nabla u|^2 + \frac{C}{\phi}|\nabla u|\right)\sum_i F^{i\bar{i}}$$

$$\begin{aligned} &\leq -2\,\mathrm{Re}\{F^{i\bar{i}}\mathfrak{g}_{i\bar{i}}u_i\eta_{\bar{i}}\} \\ &\leq \frac{1}{4}|\nabla u|^2 F^{i\bar{i}}|\eta_i|^2 + 4\sum_i F^{i\bar{i}}\mathfrak{g}_{i\bar{i}}^2, \end{aligned} \tag{4.19}$$

where we have used the Cauchy-Schwarz inequality in the last line. As in the claim of [8, (3.5)], we can prove that there exists a positive constant c_0 such that

$$\sum_i F^{i\bar{i}}\mathfrak{g}_{i\bar{i}}^2 \leq c_0\left(1+\sum_i F^{i\bar{i}}\right). \tag{4.20}$$

The rest of the proof is similar to the one in [8] given by applying Lemma 4.1 and 4.2 (1), (2). For readers convenience, we introduce the argument: From Lemma 4.1, we have that $\sum F^{i\bar{i}} \geq \kappa_0$ for some $\kappa_0 > 0$. Assume that $|\lambda| \geq R_0$, where R_0 is the constant appears in Lemma 4.2. In the case of Lemma 4.2 (1), we obtain that

$$\frac{A\varepsilon\kappa_0}{1+\kappa_0}\left(1+\sum_i F^{i\bar{i}}\right) - \left(\frac{C}{\phi|\nabla u|} + \frac{4c_0}{|\nabla u|^2}\right)\left(1+\sum_i F^{i\bar{i}}\right) - C\left(1+\frac{1}{\phi}\right)\sum_i F^{i\bar{i}} \leq 0,$$

which implies that we can get a bound $|\nabla u| \leq C$ by choosing A large enough. On the other hand, in the case of Lemma 4.2 (2), we have $F^{i\bar{i}} \geq \varepsilon \sum F^{k\bar{k}} \geq \varepsilon\kappa_0$. Then by $\mathfrak{g}^{i\bar{i}}\psi \geq F^{i\bar{i}}$, we obtain that $\mathfrak{g}_{i\bar{i}} \leq (\varepsilon\kappa_0)^{-1}\psi$ and there exists a positive constant C_{R_0} such that $\sum F^{i\bar{i}}\mathfrak{g}_{i\bar{i}} \leq C_{R_0}$. Hence, from the inequality (4.19), we can have a bound $|\nabla u| \leq C$. When $|\lambda| < R_0$, the proof goes in a similar manner as in the case of Lemma 4.2 (2).

5. Proof of Theorem 1.2

Proof (Proof of Theorem 1.2). Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix $A = \{A^i_j\} = \{g^{i\bar{q}}\mathfrak{g}_{j\bar{q}}\}$, where $\mathfrak{g}_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ at each point, and g is the almost Hermitian metric. We put $H := \lambda_1 e^{\phi}$ for the largest eigenvalue $\lambda_1 : \overline{M} \to \mathbb{R}$, where ϕ is the test function chosen later. Suppose that H achieves its maximum at an interior point $p_0 \in M$. As in the proof of Theorem 1.1, we choose and fix a local unitary $(1,0)$-frame $\{Z_r\}$ with respect to g around the point p_0 such that $g_{i\bar{j}}(p_0) = \delta_{ij}$, $\nabla Z(p_0) = 0$, $\mathfrak{g}_{i\bar{j}}(p_0) = \lambda_i\delta_{ij}$ and $F^{i\bar{j}}(p_0) = f_i\delta_{ij}$. Let B be a diagonal matrix B^p_q with real entries satisfying $B^1_1 = 0$, $B^n_n > 2B^2_2$, and such that $B^n_n < B^{n-1}_{n-1} < \cdots < B^2_2 < 0$ are small. Define the matrix $\tilde{A} := A + B$ with eigenvalues $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$. At the point p_0, we have $\tilde{\lambda}_1 = \lambda_1 = \mathfrak{g}_{1\bar{1}}$, $\tilde{\lambda}_i = \lambda_i + B_i^i$ if $i \geq 2$ and the eigenvalues of $\tilde{A}$ define C^2-functions near the point p_0. We note that $\tilde{H} := \tilde{\lambda}_1 e^{\phi}$ achieves its maximum at the same point p_0. We may assume that $\lambda_1(p_0) = \tilde{\lambda}_1(p_0) > 1$. In what follows, all computations are going to be done at the point p_0. We have

$$\frac{\tilde{\lambda}_{1,k}}{\lambda_1} + \phi_k = 0, \tag{5.1}$$

$$\frac{\tilde{\lambda}_{1,k\bar{k}}}{\lambda_1} - \frac{|\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \phi_{k\bar{k}} \leq 0. \tag{5.2}$$

We obtain that (cf. [3, Lemma 2.4], [8])

$$\tilde{\lambda}_{1,k} = \mathfrak{g}_{1\bar{1}k} - (B_1^1)_k \tag{5.3}$$

$$\begin{aligned}\tilde{\lambda}_{1,k\bar{k}} = \mathfrak{g}_{1\bar{1}k\bar{k}} &+ \sum_{p>1} \frac{|\mathfrak{g}_{p\bar{1}k}|^2 + |\mathfrak{g}_{1\bar{p}k}|^2}{\tilde{\lambda}_1 - \tilde{\lambda}_p} - (B_1^1)_{k\bar{k}} \\ &+ 2\,\mathrm{Re}\left\{\sum_{p>1} \frac{\mathfrak{g}_{p\bar{1}k}(B_p^1)_{\bar{k}} + \mathfrak{g}_{1\bar{p}k}(B_1^p)_{\bar{k}}}{\tilde{\lambda}_1 - \tilde{\lambda}_p}\right\} + \tilde{\lambda}_1^{pq,rs}(B_q^p)_k (B_s^r)_{\bar{k}},\end{aligned} \tag{5.4}$$

where

$$\tilde{\lambda}_1^{pq,rs} = (1 - \delta_{1p}) \frac{\delta_{1q}\delta_{1r}\delta_{ps}}{\tilde{\lambda}_1 - \tilde{\lambda}_p} + (1 - \delta_{1r}) \frac{\delta_{1s}\delta_{1p}\delta_{rq}}{\tilde{\lambda}_1 - \tilde{\lambda}_r}.$$

By choosing a sufficiently small B, we may assume that $\sum_i \tilde{\lambda}_i > 0$. Hence we have $|\tilde{\lambda}_i| \leq (n-1)\tilde{\lambda}_1$ for all i and then we get $(\tilde{\lambda}_1 - \tilde{\lambda}_p)^{-1} \geq (n\tilde{\lambda}_1)^{-1}$. As we see in [5], [8], we have that

$$\tilde{\lambda}_{1,k\bar{k}} \geq \mathfrak{g}_{1\bar{1}k\bar{k}} + \frac{1}{2n\lambda_1} \sum_{p>1} (|\mathfrak{g}_{p\bar{1}k}|^2 + |\mathfrak{g}_{1\bar{p}k}|^2) - C_0.$$

Assume that $\lambda_1 \geq R_0$, where R_0 is the constant in Lemma 4.2. Since if for all $i = 1, \ldots, n$,

$$\begin{aligned} f_i = F^{i\bar{i}} &\geq \varepsilon \sum_k F^{k\bar{k}} \\ &\geq \varepsilon \kappa_0, \end{aligned} \tag{5.5}$$

then which gives a bound $\lambda_1 \leq C$, the only case we need to consider is the following:

$$\begin{aligned} F^{i\bar{j}}(\underline{\mathfrak{g}}_{i\bar{j}} - \mathfrak{g}_{i\bar{j}}) &\geq \varepsilon F^{k\bar{l}} g_{k\bar{l}} \\ &\geq \frac{\kappa_0 \varepsilon}{1 + \kappa_0}(1 + F^{k\bar{l}} g_{k\bar{l}}). \end{aligned} \tag{5.6}$$

We compute

$$\chi_{i\bar{j}k} = \chi_{i\bar{j},k} + \chi_{i\bar{j},\zeta_\alpha} u_{\alpha k} + \chi_{i\bar{j},\bar{\zeta}_\alpha} u_{\bar{\alpha}k}, \tag{5.7}$$

and from the assumption (1.3), we have that by using $u_{\alpha\bar{k}} = \mathfrak{g}_{\alpha\bar{k}} - \chi_{\alpha\bar{k}}$,

$$\begin{aligned}
\chi_{i\bar{i}k\bar{k}} &= (\chi_{i\bar{i},k})_{\bar{k}} + (\chi_{i\bar{i},\zeta_\alpha})_{\bar{k}} u_{\alpha k} + (\chi_{i\bar{i},\bar{\zeta}_\alpha})_{\bar{k}} u_{\bar{\alpha}k} + \chi_{i\bar{i},\zeta_\alpha} u_{\alpha k\bar{k}} + \chi_{i\bar{i},\bar{\zeta}_\alpha} u_{\bar{\alpha}k\bar{k}} \\
&= \chi_{i\bar{i},k\bar{k}} + \chi_{i\bar{i},k\zeta_\alpha} u_{\alpha\bar{k}} + \chi_{i\bar{i},k\bar{\zeta}_\alpha} u_{\bar{\alpha}\bar{k}} + \chi_{i\bar{i},\zeta_\alpha\bar{k}} u_{\alpha k} + \chi_{i\bar{i},\bar{\zeta}_\alpha\bar{k}} u_{\bar{\alpha}k} + \chi_{i\bar{i},\zeta_\alpha} u_{\alpha k\bar{k}} + \chi_{i\bar{i},\bar{\zeta}_\alpha} u_{k\bar{\alpha}\bar{k}} \\
&= \chi_{i\bar{i},k\bar{k}} + 2\mathfrak{g}_{k\bar{k}} \operatorname{Re}(\chi_{i\bar{i},k\zeta_k}) - 2\operatorname{Re}(\chi_{i\bar{i},k\zeta_\alpha}\chi_{\alpha\bar{k}}) + 2\operatorname{Re}(\chi_{i\bar{i},\zeta_\alpha\bar{k}} u_{\alpha k}) \\
&\quad + 2\mathfrak{g}_{k\bar{k}} \operatorname{Re}(\chi_{i\bar{i},\zeta_\alpha} T^k_{\alpha k}) - 2\operatorname{Re}(\chi_{i\bar{i},\zeta_\alpha} T^s_{\alpha k}\chi_{s\bar{k}}) + O(|\nabla u|) + 2\operatorname{Re}(\chi_{i\bar{i},\zeta_\alpha}\mathfrak{g}_{k\bar{k}\alpha}) \\
&\quad - 2\operatorname{Re}(\chi_{i\bar{1},\zeta_\alpha}\chi_{k\bar{k},\alpha} + \chi_{i\bar{i},\zeta_\alpha}\chi_{k\bar{k},\zeta_\beta} u_{\beta\alpha} + \chi_{i\bar{i},\zeta_\alpha}\chi_{k\bar{k},\bar{\zeta}_\beta} u_{\alpha\bar{\beta}}),
\end{aligned} \tag{5.8}$$

where we used that $u_{\bar{\alpha}k\bar{k}} = u_{k\bar{\alpha}\bar{k}}$ from (3.1), $u_{\alpha k\bar{k}} = u_{k\bar{k}\alpha} - T^s_{k\alpha} u_{s\bar{k}} + O(|\nabla u|)$ from (4.13) for $p = \alpha$, $q = k$, $j = k$, $u_{k\bar{\alpha}\bar{k}} = u_{k\bar{k}\bar{\alpha}} - T^{\bar{s}}_{\bar{k}\bar{\alpha}} u_{k\bar{s}} + O(|\nabla u|)$ from the formula (4.14) for $i = k$, $k = \alpha$, and used (5.7). At p_0, we have by differentiating the equation (4.1),

$$F^{k\bar{k}}\mathfrak{g}_{k\bar{k}l} = -\psi_l, \tag{5.9}$$

$$F^{k\bar{k}}\mathfrak{g}_{k\bar{k}1\bar{1}} = -F^{i\bar{j},l\bar{m}}\mathfrak{g}_{i\bar{j}1}\mathfrak{g}_{l\bar{m}\bar{1}} - \psi_{1\bar{1}}. \tag{5.10}$$

Since it can be verified that we have

$$f_{11} + \frac{f_1}{\lambda_1} \le 0, \qquad \frac{f_1 - f_i}{\lambda_i - \lambda_1} \ge \frac{f_i}{\lambda_1} \tag{5.11}$$

for all $i > 1$, we obtain that

$$\begin{aligned}
-F^{i\bar{j},l\bar{m}}\mathfrak{g}_{i\bar{j}1}\mathfrak{g}_{l\bar{m}\bar{1}} &\ge \sum_{i>1} \frac{f_1 - f_i}{\lambda_i - \lambda_1} |\mathfrak{g}_{i\bar{1}1}|^2 \\
&\ge \sum_{k>1} \frac{1}{\lambda_1} F^{k\bar{k}} |\mathfrak{g}_{k\bar{1}1}|^2.
\end{aligned} \tag{5.12}$$

Then we get that from (5.8), (5.10) and (5.12),

$$\begin{aligned}
F^{k\bar{k}}\mathfrak{g}_{1\bar{1}k\bar{k}} &= F^{k\bar{k}}\mathfrak{g}_{k\bar{k}1\bar{1}} + F^{k\bar{k}}(u_{1\bar{1}k\bar{k}} - u_{k\bar{k}1\bar{1}}) + F^{k\bar{k}}(\chi_{1\bar{1}k\bar{k}} - \chi_{k\bar{k}1\bar{1}}) \\
&\ge \sum_{k>1} \frac{1}{\lambda_1} F^{k\bar{k}} |\mathfrak{g}_{k\bar{1}1}|^2 + F^{k\bar{k}}(u_{1\bar{1}k\bar{k}} - u_{k\bar{k}1\bar{1}}) - 2\operatorname{Re}(F^{k\bar{k}}\chi_{k\bar{k},\zeta_\alpha}\mathfrak{g}_{1\bar{1}\alpha}) \\
&\quad - 2\operatorname{Re}(F^{k\bar{k}}\chi_{k\bar{k},\zeta_\alpha\bar{1}} u_{\alpha 1}) + 2\operatorname{Re}(F^{k\bar{k}}\chi_{1\bar{1},\zeta_\alpha\bar{k}} u_{\alpha k}) \\
&\quad - C - C\lambda_1 \sum_k F^{k\bar{k}}.
\end{aligned} \tag{5.13}$$

Note that we have by using $\Gamma^{\bar{s}}_{\bar{r}\bar{k}}(p_0) = 0$,

$$
\begin{aligned}
Z_{\bar{r}} Z_{\bar{k}}(u) &= Z_{\bar{r}} \nabla_{\bar{k}} u \\
&= \nabla_{\bar{r}} \nabla_{\bar{k}} u + \Gamma^{\bar{s}}_{\bar{r}\bar{k}} \nabla_{\bar{s}} u \\
&= \nabla_{\bar{r}} \nabla_{\bar{k}} u \\
&= u_{\bar{k}\bar{r}}.
\end{aligned} \tag{5.14}
$$

Similarly, we have that by using $\Gamma^{\bar{s}}_{r\bar{k}}(p_0) = 0$,

$$
\begin{aligned}
Z_r Z_{\bar{k}}(u) &= Z_r \nabla_{\bar{k}} u \\
&= \nabla_r \nabla_{\bar{k}} u + \Gamma^{\bar{s}}_{r\bar{k}} u_{\bar{s}} \\
&= \nabla_r \nabla_{\bar{k}} u \\
&= u_{\bar{k}r}.
\end{aligned} \tag{5.15}
$$

We compute at p_0, since we have $\Gamma^{s}_{\bar{r}k}(p_0) = 0$,

$$
\begin{aligned}
Z_1 Z_{\bar{r}} Z_k(u) &= Z_1 \{\nabla_{\bar{r}} \nabla_k u + \Gamma^{s}_{\bar{r}k} u_s\} \\
&= \nabla_1 \nabla_{\bar{r}} \nabla_k u + Z_1(\Gamma^{s}_{\bar{r}k}) u_s,
\end{aligned} \tag{5.16}
$$

similarly, we have

$$
Z_{\bar{1}} Z_{\bar{k}} Z_r(u) = \nabla_{\bar{1}} \nabla_{\bar{k}} \nabla_r u + Z_{\bar{1}}(\Gamma^{s}_{\bar{k}r}) u_s. \tag{5.17}
$$

And since we have $\Gamma^{k}_{\bar{j}i}(p_0) = \Gamma^{\bar{k}}_{p\bar{j}}(p_0) = \Gamma^{k}_{pi}(p_0) = 0$, we compute that

$$
\begin{aligned}
Z_{\bar{q}} Z_p Z_{\bar{j}} Z_i(u) &= Z_{\bar{q}} Z_p \{\nabla_{\bar{j}} \nabla_i u + \Gamma^{k}_{\bar{j}i} u_k\} \\
&= Z_{\bar{q}} \{\nabla_p \nabla_{\bar{j}} \nabla_i u + \Gamma^{\bar{k}}_{p\bar{j}} \nabla_{\bar{k}} \nabla_i u + \Gamma^{k}_{pi} \nabla_{\bar{j}} \nabla_k u + Z_p(\Gamma^{k}_{\bar{j}i}) u_k \\
&\qquad + \Gamma^{k}_{\bar{j}i} \nabla_p \nabla_k u + \Gamma^{k}_{\bar{j}i} \Gamma^{r}_{pk} u_r\} \\
&= \nabla_{\bar{q}} \nabla_p \nabla_{\bar{j}} \nabla_i u + \Gamma^{s}_{\bar{q}p} \nabla_s \nabla_{\bar{j}} \nabla_i u + \Gamma^{\bar{s}}_{\bar{q}\bar{j}} \nabla_p \nabla_{\bar{s}} \nabla_i u + \Gamma^{s}_{\bar{q}i} \nabla_p \nabla_{\bar{j}} \nabla_s u \\
&\quad + Z_{\bar{q}}(\Gamma^{\bar{k}}_{p\bar{j}}) u_{i\bar{k}} + \Gamma^{\bar{k}}_{p\bar{j}} Z_{\bar{q}}(u_{i\bar{k}}) + Z_{\bar{q}}(\Gamma^{k}_{pi}) u_{k\bar{j}} + \Gamma^{k}_{pi} Z_{\bar{q}}(u_{k\bar{j}}) \\
&\quad + Z_{\bar{q}} Z_p(\Gamma^{k}_{\bar{j}i}) u_k + Z_p(\Gamma^{k}_{\bar{j}i}) u_{k\bar{q}} + Z_{\bar{q}}(\Gamma^{k}_{\bar{j}i}) u_{kp} + \Gamma^{k}_{\bar{j}i} Z_{\bar{q}}(u_{kp}) \\
&\quad + Z_{\bar{q}}(\Gamma^{k}_{\bar{j}i}) \Gamma^{r}_{pk} u_r + \Gamma^{k}_{\bar{j}i} Z_{\bar{q}}(\Gamma^{r}_{pk}) u_r + \Gamma^{k}_{\bar{j}i} \Gamma^{r}_{pk} u_{r\bar{q}} \\
&= \nabla_{\bar{q}} \nabla_p \nabla_{\bar{j}} \nabla_i u + Z_{\bar{q}}(\Gamma^{\bar{k}}_{p\bar{j}}) u_{i\bar{k}} + Z_{\bar{q}}(\Gamma^{k}_{pi}) u_{k\bar{j}} + Z_{\bar{q}} Z_p(\Gamma^{k}_{\bar{j}i}) u_k \\
&\quad + Z_p(\Gamma^{k}_{\bar{j}i}) u_{k\bar{q}} + Z_{\bar{q}}(\Gamma^{k}_{\bar{j}i}) u_{kp}.
\end{aligned} \tag{5.18}
$$

We also compute that

$$
\begin{aligned}
Z_k Z_{\bar r} Z_1(u) &= Z_{\bar r} Z_k Z_1(u) + [Z_k, Z_{\bar r}] Z_1(u) \\
&= Z_1 Z_{\bar r} Z_k(u) + [Z_k, Z_{\bar r}] Z_1(u) + [Z_{\bar r}, Z_1] Z_k(u) + Z_{\bar r}[Z_k, Z_1](u) \\
&= Z_1 Z_{\bar r} Z_k(u) + Z_{\bar r}(B^s_{k1} u_s + B^{\bar s}_{k1} u_{\bar s}) \\
&= Z_1 Z_{\bar r} Z_k(u) + Z_{\bar r}(B^s_{k1}) u_s + Z_{\bar r}(B^{\bar s}_{k1}) u_{\bar s} - T^s_{k1} u_{s\bar r} - T^{\bar s}_{k1} u_{\bar s \bar r}. \qquad (5.19)
\end{aligned}
$$

By applying $B^s_{\bar k 1}(p_0) = B^{\bar s}_{\bar k 1}(p_0) = \Gamma^r_{ks}(p_0) = \Gamma^r_{\bar 1 s}(p_0) = \Gamma^{\bar r}_{k\bar s}(p_0) = 0$, we compute that

$$
\begin{aligned}
Z_k Z_{\bar 1} Z_{\bar k} Z_1(u) &= Z_k Z_{\bar 1}[Z_{\bar k}, Z_1](u) + Z_k Z_{\bar 1} Z_1 Z_{\bar k}(u) \\
&= Z_k Z_{\bar 1}\{B^s_{\bar k 1} Z_s + B^{\bar s}_{\bar k 1} Z_{\bar s}\}(u) + [Z_k, Z_{\bar 1}] Z_1 Z_{\bar k}(u) + Z_{\bar 1} Z_k Z_1 Z_{\bar k}(u) \\
&= Z_k\{Z_{\bar 1}(B^s_{\bar k 1}) u_s + B^s_{\bar k 1}(u_{s\bar 1} + \Gamma^r_{\bar 1 s} u_r) + Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{\bar s} \\
&\qquad + B^{\bar s}_{\bar k 1}(u_{\bar s \bar 1} + \Gamma^{\bar r}_{\bar 1 \bar s} u_{\bar r})\} + Z_{\bar 1} Z_k Z_1 Z_{\bar k}(u) \\
&= Z_k Z_{\bar 1}(B^s_{\bar k 1}) u_s + Z_{\bar 1}(B^s_{\bar k 1})(u_{sk} + \Gamma^r_{ks} u_r) + Z_k(B^s_{\bar k 1})(u_{s\bar 1} + \Gamma^r_{\bar 1 s} u_r) \\
&\quad + B^s_{\bar k 1} Z_k(u_{s\bar 1} + \Gamma^r_{\bar 1 s} u_r) + Z_k Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{\bar s} + Z_{\bar 1}(B^{\bar s}_{\bar k 1})(u_{\bar s k} + \Gamma^{\bar r}_{k \bar s} u_{\bar r}) \\
&\quad + Z_k(B^{\bar s}_{\bar k 1})(u_{\bar s \bar 1} + \Gamma^{\bar r}_{\bar 1 \bar s} u_{\bar r}) + B^{\bar s}_{\bar k 1} Z_k(u_{\bar s \bar 1} + \Gamma^{\bar r}_{\bar 1 \bar s} u_{\bar r}) + Z_{\bar 1} Z_k Z_1 Z_{\bar k}(u) \\
&= Z_k Z_{\bar 1}(B^s_{\bar k 1}) u_s + Z_{\bar 1}(B^s_{\bar k 1}) u_{sk} + Z_k(B^s_{\bar k 1}) u_{s\bar 1} + Z_k Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{\bar s} \\
&\quad + Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{k\bar s} + Z_k(B^{\bar s}_{\bar k 1}) u_{\bar s \bar 1} + Z_{\bar 1} Z_k Z_1 Z_{\bar k}(u), \qquad (5.20)
\end{aligned}
$$

where we used that $u_{\bar s k} = u_{k\bar s}$ from (3.1). Also we compute that

$$
\begin{aligned}
Z_{\bar 1} Z_k Z_1 Z_{\bar k}(u) &= Z_{\bar 1}\{[Z_k, Z_1] Z_{\bar k}(u) + Z_1 Z_k Z_{\bar k}(u)\} \\
&= Z_{\bar 1}\{B^r_{k1} Z_r Z_{\bar k}(u) + B^{\bar r}_{k1} Z_{\bar r} Z_{\bar k}(u) + Z_1 Z_k Z_{\bar k}(u)\}. \qquad (5.21)
\end{aligned}
$$

By applying (3.2), we have that

$$
\begin{aligned}
B^r_{\bar k \bar 1} Z_r Z_1(u) &= B^r_{\bar k \bar 1} Z_1 Z_r(u) + B^r_{\bar k \bar 1}[Z_r, Z_1](u) \\
&= T^r_{\bar k \bar 1}(T^s_{1r} u_s + T^{\bar s}_{1r} u_{\bar s}) + B^r_{\bar k \bar 1}\{B^s_{r1} Z_s + B^{\bar s}_{r1} Z_{\bar s}\}(u) \\
&= -T^r_{\bar k \bar 1} T^s_{r1} u_s - T^r_{\bar k \bar 1} T^{\bar s}_{r1} u_{\bar s} - T^r_{\bar k \bar 1} B^s_{r1} u_s + T^r_{\bar k \bar 1} T^{\bar s}_{r1} u_{\bar s} \\
&= T^r_{\bar k \bar 1}(\Gamma^s_{1r} - \Gamma^s_{r1}) u_s, \qquad (5.22)
\end{aligned}
$$

similarly,

$$\begin{aligned}
B^{\bar r}_{k1} Z_{\bar r} Z_{\bar k}(u) &= B^{\bar r}_{k1} Z_{\bar k} Z_{\bar r}(u) + B^{\bar r}_{k1}[Z_{\bar r}, Z_{\bar k}](u) \\
&= T^{\bar r}_{k1}(T^{\bar s}_{\bar k \bar r} u_{\bar s} + T^{s}_{\bar k \bar r} u_s) + B^{\bar r}_{k1}\{B^{s}_{\bar r \bar k} Z_s + B^{\bar s}_{\bar r \bar k} Z_{\bar s}\}(u) \\
&= -T^{\bar r}_{k1} T^{\bar s}_{\bar r \bar k} u_{\bar s} - T^{\bar r}_{k1} T^{s}_{\bar r \bar k} u_s + T^{\bar r}_{k1} T^{s}_{\bar r \bar k} u_s - T^{\bar r}_{k1} B^{\bar s}_{\bar r \bar k} u_{\bar s} \\
&= T^{\bar r}_{k1}(\Gamma^{\bar s}_{\bar k \bar r} - \Gamma^{\bar s}_{\bar r \bar k}) u_{\bar s}. \qquad (5.23)
\end{aligned}$$

By combining (5.14)–(5.23), we compute that

$$\begin{aligned}
u_{1\bar 1 k \bar k} &= \nabla_{\bar k} \nabla_k \nabla_{\bar 1} \nabla_1 u \\
&= Z_k Z_{\bar k} Z_{\bar 1} Z_1(u) + [Z_{\bar k}, Z_k] Z_{\bar 1} Z_1(u) \\
&\quad - Z_{\bar k}(\Gamma^{\bar s}_{k \bar 1}) u_{1\bar s} - Z_{\bar k}(\Gamma^{s}_{k1}) u_{s\bar 1} - Z_{\bar k} Z_k(\Gamma^{s}_{\bar 1 1}) u_s - Z_k(\Gamma^{s}_{\bar 1 1}) u_{s\bar k} - Z_{\bar k}(\Gamma^{s}_{\bar 1 1}) u_{sk} \\
&= Z_k\{[Z_{\bar k}, Z_{\bar 1}] Z_1(u) + Z_{\bar 1} Z_{\bar k} Z_1(u)\} \\
&\quad - Z_{\bar k}(\Gamma^{\bar s}_{k \bar 1}) u_{1\bar s} - Z_{\bar k}(\Gamma^{s}_{k1}) u_{s\bar 1} - Z_{\bar k} Z_k(\Gamma^{s}_{\bar 1 1}) u_s - Z_k(\Gamma^{s}_{\bar 1 1}) u_{s\bar k} - Z_{\bar k}(\Gamma^{s}_{\bar 1 1}) u_{sk} \\
&= Z_k\{B^{r}_{\bar k \bar 1} Z_r Z_1(u) + B^{\bar r}_{\bar k \bar 1} Z_{\bar r} Z_1(u)\} + Z_k Z_{\bar 1} Z_{\bar k} Z_1(u) \\
&\quad - Z_{\bar k}(\Gamma^{\bar s}_{k \bar 1}) u_{1\bar s} - Z_{\bar k}(\Gamma^{s}_{k1}) u_{s\bar 1} - Z_{\bar k} Z_k(\Gamma^{s}_{\bar 1 1}) u_s - Z_k(\Gamma^{s}_{\bar 1 1}) u_{s\bar k} - Z_{\bar k}(\Gamma^{s}_{\bar 1 1}) u_{sk} \\
&= Z_k\{T^{r}_{\bar k \bar 1}(\Gamma^{s}_{1r} - \Gamma^{s}_{r1}) u_s\} + Z_k\{B^{\bar r}_{\bar k \bar 1} Z_{\bar r} Z_1(u)\} \\
&\quad + Z_{\bar 1}\{B^{r}_{k1} Z_r Z_{\bar k}(u) + B^{\bar r}_{k1} Z_{\bar r} Z_{\bar k}(u) + Z_1 Z_k Z_{\bar k}(u)\} \\
&\quad + Z_k Z_{\bar 1}(B^{s}_{\bar k 1}) u_s + Z_{\bar 1}(B^{s}_{\bar k 1}) u_{sk} + Z_k(B^{s}_{\bar k 1}) u_{s\bar 1} + Z_k Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{\bar s} \\
&\quad + Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{k\bar s} + Z_k(B^{\bar s}_{\bar k 1}) u_{\bar s \bar 1} - Z_{\bar k}(\Gamma^{\bar s}_{k \bar 1}) u_{1\bar s} - Z_{\bar k}(\Gamma^{s}_{k1}) u_{s\bar 1} - Z_{\bar k} Z_k(\Gamma^{s}_{\bar 1 1}) u_s \\
&= -Z_k(\Gamma^{s}_{\bar 1 1}) u_{s\bar k} - Z_{\bar k}(\Gamma^{s}_{\bar 1 1}) u_{sk} + Z_k\{T^{r}_{\bar k \bar 1}(\Gamma^{s}_{1r} - \Gamma^{s}_{r1})\} u_s + Z_k(B^{\bar r}_{\bar k \bar 1}) u_{1\bar r} \\
&\quad + B^{\bar r}_{\bar k \bar 1}\{Z_1 Z_{\bar r} Z_k(u) + Z_{\bar r}(B^{s}_{k1}) u_s + Z_{\bar r}(B^{\bar s}_{k1}) u_{\bar s} - T^{s}_{k1} u_{s\bar r} - T^{\bar s}_{k1} u_{\bar s \bar r}\} \\
&\quad + Z_{\bar 1}(B^{r}_{k1}) u_{r\bar k} + B^{r}_{k1} Z_{\bar 1} Z_{\bar k} Z_r(u) + B^{r}_{k1} Z_{\bar 1}\{B^{s}_{r\bar k} u_s + B^{\bar s}_{r\bar k} u_{\bar s}\} \\
&\quad + Z_{\bar 1}\{T^{\bar r}_{k1}(\Gamma^{\bar s}_{\bar k \bar r} - \Gamma^{\bar s}_{\bar r \bar k}) u_{\bar s}\} + Z_{\bar 1} Z_1\{B^{s}_{k\bar k} u_s + B^{\bar s}_{k\bar k} u_{\bar s}\} + Z_{\bar 1} Z_1 Z_{\bar k} Z_k(u) \\
&\quad + Z_k Z_{\bar 1}(B^{s}_{\bar k 1}) u_s + Z_{\bar 1}(B^{s}_{\bar k 1}) u_{sk} + Z_k(B^{s}_{\bar k 1}) u_{s\bar 1} + Z_k Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{\bar s} \\
&\quad + Z_{\bar 1}(B^{\bar s}_{\bar k 1}) u_{k\bar s} + Z_k(B^{\bar s}_{\bar k 1}) u_{\bar s \bar 1} - Z_{\bar k}(\Gamma^{\bar s}_{k \bar 1}) u_{1\bar s} - Z_{\bar k}(\Gamma^{s}_{k1}) u_{s\bar 1} - Z_{\bar k} Z_k(\Gamma^{s}_{\bar 1 1}) u_s \\
&\quad - Z_k(\Gamma^{s}_{\bar 1 1}) u_{s\bar k} - Z_{\bar k}(\Gamma^{s}_{\bar 1 1}) u_{sk} \\
&= Z_k(B^{\bar r}_{\bar k \bar 1}) u_{1\bar r} - T^{\bar r}_{\bar k \bar 1} u_{k\bar r 1} - T^{\bar r}_{\bar k \bar 1} Z_1(\Gamma^{s}_{\bar r k}) u_s - T^{\bar r}_{\bar k \bar 1} Z_{\bar r}(B^{s}_{k1}) u_s - T^{\bar r}_{\bar k \bar 1} Z_{\bar r}(B^{\bar s}_{k1}) u_{\bar s}
\end{aligned}$$

$$
\begin{aligned}
&+ T_{\bar{k}\bar{1}}^{\bar{r}} T_{k1}^{s} u_{s\bar{r}} + T_{\bar{k}\bar{1}}^{\bar{r}} T_{k1}^{\bar{s}} u_{\bar{s}\bar{r}} + Z_{\bar{1}}(B_{k1}^{r}) u_{r\bar{k}} - T_{k1}^{r} u_{r\bar{k}\bar{1}} - T_{k1}^{r} Z_{\bar{1}}(\Gamma_{\bar{k}r}^{s}) u_{s} \\
&- T_{k1}^{r} Z_{\bar{1}}(B_{r\bar{k}}^{s}) u_{s} - T_{k1}^{r} Z_{\bar{1}}(B_{r\bar{k}}^{\bar{s}}) u_{\bar{s}} + Z_{k}\{T_{\bar{k}\bar{1}}^{r}(\Gamma_{1r}^{s} - \Gamma_{r1}^{s})\} u_{s} \\
&+ Z_{\bar{1}}\{T_{k1}^{\bar{r}}(\Gamma_{\bar{k}\bar{r}}^{\bar{s}} - \Gamma_{\bar{r}\bar{k}}^{\bar{s}})\} u_{\bar{s}} + Z_{\bar{1}} Z_{1}(B_{k\bar{k}}^{s}) u_{s} + Z_{1}(B_{k\bar{k}}^{s}) u_{s\bar{1}} + Z_{\bar{1}}(B_{k\bar{k}}^{s}) u_{s1} \\
&+ Z_{\bar{1}} Z_{1}(B_{k\bar{k}}^{\bar{s}}) u_{\bar{s}} + Z_{1}(B_{k\bar{k}}^{\bar{s}}) u_{\bar{s}\bar{1}} + Z_{\bar{1}}(B_{k\bar{k}}^{\bar{s}}) u_{\bar{s}1} + Z_{k} Z_{\bar{1}}(B_{\bar{k}1}^{s}) u_{s} + Z_{\bar{1}}(B_{\bar{k}1}^{s}) u_{sk} \\
&+ Z_{k}(B_{\bar{k}1}^{s}) u_{s\bar{1}} + Z_{k} Z_{\bar{1}}(B_{\bar{k}1}^{\bar{s}}) u_{\bar{s}} + Z_{\bar{1}}(B_{\bar{k}1}^{\bar{s}}) u_{k\bar{s}} + Z_{k}(B_{\bar{k}1}^{\bar{s}}) u_{\bar{s}\bar{1}} + \nabla_{\bar{1}} \nabla_{1} \nabla_{\bar{k}} \nabla_{k} u \\
&- Z_{\bar{k}}(\Gamma_{k\bar{1}}^{\bar{s}}) u_{1\bar{s}} - Z_{\bar{k}}(\Gamma_{k1}^{s}) u_{s\bar{1}} - Z_{\bar{k}} Z_{k}(\Gamma_{\bar{1}1}^{s}) u_{s} - Z_{k}(\Gamma_{\bar{1}1}^{s}) u_{s\bar{k}} - Z_{\bar{k}}(\Gamma_{\bar{1}1}^{s}) u_{sk} \\
&+ Z_{\bar{1}}(\Gamma_{1\bar{k}}^{\bar{s}}) u_{k\bar{s}} + Z_{\bar{1}}(\Gamma_{1k}^{s}) u_{s\bar{k}} + Z_{\bar{1}} Z_{1}(\Gamma_{\bar{k}k}^{s}) u_{s} + Z_{1}(\Gamma_{\bar{k}k}^{s}) u_{s\bar{1}} + Z_{\bar{1}}(\Gamma_{\bar{k}k}^{s}) u_{s1} \\
={}& u_{k\bar{k}1\bar{1}} + 2\,\mathrm{Re}\{T_{\bar{1}\bar{k}}^{\bar{r}} u_{k\bar{r}1}\} + O(|\nabla u|) + O(|\nabla\nabla u|) + O(|\nabla\overline{\nabla} u|) \\
={}& u_{k\bar{k}1\bar{1}} + 2\,\mathrm{Re}\{T_{\bar{1}\bar{k}}^{\bar{r}} u_{1\bar{r}k}\} + O(|\nabla u|) + O(|\nabla\nabla u|) + O(|\nabla\overline{\nabla} u|), \qquad (5.24)
\end{aligned}
$$

where we used that $u_{r\bar{k}\bar{1}} = u_{\bar{k}r\bar{1}}$ from (3.1), $u_{k\bar{r}1} = u_{1\bar{r}k} + T_{1k}^{s} u_{s\bar{r}} + O(|\nabla u|)$ from (4.10) for $l = k$, $i = r$ and $j = 1$, and $O(|\nabla\nabla u|)$ (resp. $O(|\nabla\overline{\nabla} u|)$) is the set of all terms including $\nabla\nabla u$ (resp. $\nabla\overline{\nabla} u$) and whose norm $|\cdot|$ can be estimated by $C|\nabla\nabla u|$ (resp. $C|\nabla\overline{\nabla} u|$) for some positive constant C.

In what follows, the positive constants C are different from each other line by line.

From (5.1), (5.2), (5.3), (5.13) and (5.24), we have that as in [8, (4.7)],

$$
\begin{aligned}
0 \geq{}& \mathscr{L}(\phi) - F^{k\bar{k}} \frac{|\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \sum_{k>1} F^{k\bar{k}} \frac{|\mathfrak{g}_{k\bar{1}1}|^2}{\lambda_1^2} + 2\,\mathrm{Re}\left\{ F^{k\bar{k}} \overline{T_{1k}^{1}} \frac{\tilde{\lambda}_{1,k}}{\lambda_1} \right\} \\
&+ \frac{2}{\lambda_1} \mathrm{Re}\{F^{i\bar{i}} \chi_{1\bar{1},\zeta_\alpha \bar{i}} u_{\alpha i}\} - \frac{1}{8} F^{i\bar{i}} (|u_{ki}|^2 + |u_{k\bar{i}}|^2) \\
&- \frac{C}{\lambda_1} |u_{\alpha 1}| \sum_i F^{i\bar{i}} - C\left(1 + \sum_i F^{i\bar{i}}\right), \qquad (5.25)
\end{aligned}
$$

where we have used that from (3.1),

$$
u_{1\bar{r}k} = u_{\bar{r}1k} = \mathfrak{g}_{1\bar{r}k} - \chi_{1\bar{r}k},
$$

and $\overline{T_{1k}^{r}} \mathfrak{g}_{1\bar{r}k} = \overline{T_{1k}^{1}} \mathfrak{g}_{1\bar{1}k} = \overline{T_{1k}^{1}} (\tilde{\lambda}_{1,k} + (B_1^1)_k)$.

We compute that by using

$$
\begin{aligned}
u_{\bar{1}1k} = u_{1\bar{1}k} &= u_{k\bar{1}1} - T_{k1}^{s} u_{s\bar{1}} + O(|\nabla u|) \\
&= u_{\bar{1}k1} - T_{k1}^{s} u_{s\bar{1}} + O(|\nabla u|)
\end{aligned}
$$

from (3.1), and also applying the formula (4.10) for $l = 1$, $i = 1$, $j = k$,

$$\begin{aligned}\tilde{\lambda}_{1,k} &= \mathfrak{g}_{1\bar{1}k} - (B_1^1)_k \\ &= \chi_{1\bar{1},k} + \chi_{1\bar{1},\zeta_\alpha} u_{\alpha k} + \chi_{1\bar{1},\bar{\zeta}_\alpha} u_{\bar{\alpha}k} - \chi_{k\bar{1},1} - \chi_{k\bar{1},\zeta_\alpha} u_{\alpha 1} - \chi_{k\bar{1},\bar{\zeta}_\alpha} u_{\bar{\alpha}1} \\ &\quad + \mathfrak{g}_{k\bar{1}1} - (B_1^1)_k - T_{k1}^s u_{s\bar{1}} + O(|\nabla u|).\end{aligned}$$

Hence we have that

$$\mathfrak{g}_{k\bar{1}1} = \tilde{\lambda}_{1,k} + \tau_k - \chi_{1\bar{1},\zeta_\alpha} u_{\alpha k} + \chi_{k\bar{1},\zeta_\alpha} u_{\alpha 1}, \tag{5.26}$$

where we put

$$\tau_k := \chi_{k\bar{1},1} - \chi_{1\bar{1},k} + \chi_{k\bar{1},\bar{\zeta}_\alpha} u_{\bar{\alpha}1} - \chi_{1\bar{1},\bar{\zeta}_\alpha} u_{\bar{\alpha}k} + T_{k1}^s u_{s\bar{1}} + \mathcal{O}(|\nabla u|) + (B_1^1)_k. \tag{5.27}$$

Hence we have that for $k \geq 2$,

$$\begin{aligned}|\mathfrak{g}_{k\bar{1}1}|^2 &\geq |\tilde{\lambda}_{1,k}|^2 - |\tau_k - \chi_{1\bar{1},\zeta_\alpha} u_{\alpha k}|^2 + 2\,\mathrm{Re}\{\tilde{\lambda}_{1,\bar{k}}(\tau_k - \chi_{1\bar{1},\zeta_\alpha} u_{\alpha k})\} \\ &\quad + \frac{1}{2}\left|\sum_\beta \chi_{k\bar{1},\zeta_\beta} u_{\beta 1}\right|^2 + 2\,\mathrm{Re}\sum_\beta\{\tilde{\lambda}_{1,\bar{k}} \chi_{k\bar{1},\zeta_\beta} u_{\beta 1}\}.\end{aligned} \tag{5.28}$$

We define

$$\phi := A_2|\nabla u|^2 + \Psi(u - \underline{u}) \tag{5.29}$$

for a small constant $0 < A_2 \leq \frac{1}{12n} K^{-1}$, where

$$K := 1 + \sup_M |\nabla u|^2 + \sup_M |\nabla(u - \underline{u})|^2,$$

and for Ψ satisfies that $\Psi' < 0$ and $\Psi'' > 0$. By applying [8, (4.8)–(4.10)] and (5.25), we obtain

$$\begin{aligned}0 &\geq \frac{1}{4} A_2 F^{k\bar{k}}(|u_{ik}|^2 + |u_{\bar{i}k}|^2) + \Psi' \mathcal{L}(u - \underline{u}) + \Psi'' F^{i\bar{i}} |(u - \underline{u})_i|^2 \\ &\quad - C F^{k\bar{k}} \left|\frac{\tilde{\lambda}_{1,k}}{\lambda_1}\right| - C\frac{|u_{\alpha 1}|}{\lambda_1} \sum_i F^{i\bar{i}} - C\left(1 + \sum_i F^{i\bar{i}}\right) \\ &\quad - \frac{2\Psi'}{\lambda_1} \sum_{k>1} \sum_\beta F^{k\bar{k}}\,\mathrm{Re}\{\chi_{k\bar{1},\zeta_\beta} u_{\beta 1}(u - \underline{u})_k\} \\ &\quad + \frac{1}{4\lambda_1^2} \sum_{k>1} F^{k\bar{k}} \left|\sum_\beta \chi_{k\bar{1},\zeta_\beta} u_{\beta 1}\right|^2,\end{aligned} \tag{5.30}$$

where we have used that $0 < f_1 \le \frac{\Psi}{\lambda_1}$ and A_2 is the positive constant chosen to be small enough in order to control $-F^{1\bar{1}}\frac{|\tilde{\lambda}_{1,1}|^2}{\lambda_1^2}$. Then by the same argument in [8], using the assumption (1.4), we can control the terms containing $\sum_\beta \chi_{k\bar{1},\zeta_\beta} u_{\beta 1}$. For readers convenience, we introduce the argument below. Assuming that $\sum_\beta |\Psi' u_{\beta 1}| \le \lambda_1$, we have that

$$-\frac{2\Psi'}{\lambda_1}\sum_\beta \sum_{k>1} F^{k\bar{k}} \operatorname{Re}\{\chi_{k\bar{1},\zeta_\beta} u_{\beta 1}(u-\underline{u})_k\} \ge -C\sum_i F^{i\bar{i}}. \tag{5.31}$$

Suppose that $\sum_\beta |\Psi' u_{\beta 1}| \ge \lambda_1$. If the assumption (1.4) holds, we have that provided $\lambda_1 \gg 1$,

$$\begin{aligned}
&\frac{1}{12}A_2 F^{k\bar{k}}(|u_{ik}|^2 + |u_{\bar{i}k}|^2) - \frac{2\Psi'}{\lambda_1}\sum_\beta \sum_{k>1} F^{k\bar{k}} \operatorname{Re}\{\chi_{k\bar{1},\zeta_\beta} u_{\beta 1}(u-\underline{u})_k\} \\
&\ge \frac{A_2}{12}F^{\beta\bar{\beta}}|u_{1\beta}|^2 - C|\Psi'|\rho(\lambda)\sum_\beta F^{\beta\bar{\beta}}|u_{1\beta}| - C|\Psi'|\rho(\lambda)\sum_i F^{i\bar{i}} \\
&\ge -\sum_i F^{i\bar{i}}.
\end{aligned} \tag{5.32}$$

Plugging (5.31)–(5.32) into (5.30), we obtain that

$$\begin{aligned}
0 \ge{}& \frac{1}{6}A_2 F^{k\bar{k}}(|u_{ik}|^2 + |u_{\bar{i}k}|^2) + \Psi'\mathscr{L}(u-\underline{u}) + \Psi'' F^{i\bar{i}}|(u-\underline{u})_i|^2 \\
&- CF^{k\bar{k}}\left|\frac{\tilde{\lambda}_{1,k}}{\lambda_1}\right| - C\frac{|u_{\alpha 1}|}{\lambda_1}\sum_i F^{i\bar{i}} - C\left(1 + \sum_i F^{i\bar{i}}\right).
\end{aligned}$$

From (5.1) and (5.29), we estimate that

$$\begin{aligned}
F^{k\bar{k}}\left|\frac{\tilde{\lambda}_{1,k}}{\lambda_1}\right| &= F^{k\bar{k}}|A_2(u_{ik}u_{\bar{i}} + u_{\bar{i}k}u_i) + \Psi'(u-\underline{u})_k| \\
&\le A_2\sqrt{K}F^{k\bar{k}}(|u_{ik}| + |u_{\bar{i}k}|) + |\Psi'|F^{k\bar{k}}|(u-\underline{u})_k|.
\end{aligned}$$

Since we have for $a > 0$, $ax^2 - bx \ge -\frac{b^2}{4a}$, we have that

$$\Psi'' F^{i\bar{i}}|(u-\underline{u})_i|^2 - C|\Psi'|F^{i\bar{i}}|(u-\underline{u})_i| \ge -\frac{C^2\Psi'^2}{4\Psi''}\sum_i F^{i\bar{i}}.$$

Then, by taking $0 < \varepsilon \ll 1$ and λ_1 large, we obtain that

$$0 \geq \frac{1}{8} A_2 F^{k\bar{k}} (|u_{ik}|^2 + |u_{\bar{i}k}|^2) - C_I \frac{|u_{\alpha 1}|}{\lambda_1} \sum_i F^{i\bar{i}}$$
$$+ \Psi' \mathscr{L}(u - \underline{u}) - \frac{C^2 \Psi'^2}{4\Psi''} \sum_i F^{i\bar{i}} - C_2 \left(1 + \sum_i F^{i\bar{i}} \right). \tag{5.33}$$

Now, let us choose the function $\Psi : [\inf_{\overline{M}} (u - \underline{u}), +\infty) \to \mathbb{R}$ such that

$$\Psi(x) := \frac{A_1}{(1 + x - \inf_{\overline{M}} (u - \underline{u}))^N},$$

where $A_1 \geq 1$, $N \in \mathbb{N}$. We take $N \in \mathbb{N}$ large enough so that $0 < \frac{C^2 |\Psi'|}{4\Psi''} \ll \frac{1}{2} \frac{\varepsilon\kappa_0}{1+\kappa_0}$. Then we obtain by applying the condition (1.6), the following estimate (cf. [8, (4.13)]):

$$0 \geq \frac{\sum_\alpha |u_{\alpha 1}|}{\lambda_1} \left(\frac{A_2}{16n^2} \frac{\rho_0}{\lambda_1} \sum_\alpha |u_{\alpha 1}| - C_I \right) \sum_i F^{i\bar{i}}$$
$$+ \left(\frac{\varepsilon\kappa_0}{2(1+\kappa_0)} |\Psi'| - C_2 \right) \left(1 + \sum_i F^{i\bar{i}} \right). \tag{5.34}$$

We consider the following two cases: $\sum |u_{\alpha 1}| > \frac{16n^2 C_I}{\rho_0 A_2} \lambda_1$ and $\sum |u_{\alpha 1}| \leq \frac{16n^2 C_I}{\rho_0 A_2} \lambda_1$, where C_I is the constant in (5.33). When $\sum |u_{\alpha 1}| > \frac{16n^2 C_I}{\rho_0 A_2} \lambda_1$, by taking $A_1 \gg 1$ so that

$$\frac{\varepsilon\kappa_0}{4(1+\kappa_0)} |\Psi'| - 100 C_2 - \frac{160 n^2 C_I^2}{\rho_0 A_2} > 0, \tag{5.35}$$

the right-hand side of the inequality (5.34) is positive, which is a contradiction. On the other hand, in the case of $\sum |u_{\alpha 1}| \leq \frac{16n^2 C_I}{\rho_0 A_2} \lambda_1$, fixing constants A_1, A_2 and N as in the previous case, by (5.34) and (5.35), we obtain that

$$0 \geq \frac{\varepsilon\kappa_0}{2(1+\kappa_0)} |\Psi'| \left(1 + \sum_i F^{i\bar{i}} \right) - \frac{16n^2 C_I^2}{\rho_0 A_2} \sum_i F^{i\bar{i}} - C_2 \left(1 + \sum_i F^{i\bar{i}} \right) > 0,$$

which also gives a contradiction. Therefore, we conclude that $\tilde{H}$ cannot achieve its maximal value at any interior point. Therefore, we complete the proof of Theorem 1.2.

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