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Properties of the Dirichlet type 3 distribution

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ABSTRACT. In this paper, we investigate the Dirichlet type 3 distribution. First, some main properties are elaborated and illustrated. Next, we set forward a representation which allows to compute many functionals in a closed form, making the Dirichlet type 3 distribution an exactly soluble model. Furthermore, we consider the Gibbs version of the Dirichlet type 3 distribution including selection. By using the representation mentioned above, we obtain the moment function of the geometrical average of the random variables according to the new distribution; special types of Bell polynomials are shown to be involved. Finally, we provide a concrete example to illustrate the performance of the Dirichlet type 3 distribution.

1. Introduction

The Dirichlet type 1 distribution or simply the Dirichlet distribution is a basic multivariate continuous distribution in probability and statistics. It arises naturally in a large variety of disciplines such as biology, physics, data sciences, etc. Owing to its easiness of interpretation and interesting mathematical properties, the Dirichlet distribution has been popular and widely studied. The Dirichlet type 3 distribution is becoming an area of interest for research, but has not received the same attention over the popularity of the Dirichlet distribution.

Furthermore, in multiple disciplines, data consist of parts of a whole (i.e. vectors of proportions) and thus are subject to constant-sum and non-negative constraints. These datasets called compositional datasets are widespread in economics, medicine, geology, psychology and environmetrics in particular. Among the most well-known simplex distributions, we mention the Dirichlet. Despite its numerous mathematical and statistical properties, it is unsuitable for modelling most compositional data because of the poor dependence structure it implies. Indeed, in many respects, it can be considered as the standard reference for modelling the strongest independence relations compatible with

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compositional data (see for example [1], [19] and [21]). Since the Dirichlet type 3 distribution has the same simplex of the Dirichlet distribution, it may have the same application interest and will be used to model compositional data. Next, the Dirichlet distribution is commonly used as prior distribution in Bayesian statistics, and is in fact the conjugate prior of the multinomial distribution. Similarly, we shall demonstrate in the next section that the Dirichlet type 3 distribution is the conjugate prior of the multinomial distribution.

In this work, we are basically interested in the Dirichlet type 3 distribution (see [2], [9] and [10]) with positive parameters $\boldsymbol{\theta}_n := (\theta_1, \dots, \theta_n)$ and θ_{n+1}, which is a multivariate generalization of the beta type 3 distribution. In the following, we consider that the random vector $\mathbf{S}_n := (S_1, \dots, S_n)$ is distributed according to Dirichlet type 3 distribution, denoted by $D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$.

This paper is organized as follows: in Section 2, we display some main properties of the Dirichlet type 3 distribution; in particular, we find the residual allocation model (RAM) of $D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$ (in Theorem 1) similar to the one given by Devroye [4] for the Dirichlet distribution. Next, we prove the stability under scaling property of Dirichlet type 3 distribution and we determine the scaling property of Dirichlet distribution from the Dirichlet type 3 distribution. Furthermore, we derive a formula which allows to compute many functionals in a closed form (Theorem 3), making $D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$ an exactly solvable distribution. To illustrate this formula, we exhibit some of its applications and we determine the characteristic function of the Dirichlet type 3 distribution which appears to be new. We also emphasize that a similar formula for many functionals of the Dirichlet distribution is given in the real case (see [12]) and in the matrix case (see [6]). In Section 3, the Gibbs version of the Dirichlet type 3 distribution including selection is subsequently examined in further details. We highlight the following main points: In Theorem 4, we compute the partition function of this Gibbs measure. Using this along with the representation of many spacings functionals of Dirichlet type 3 distribution in terms of simpler functionals of independent gamma random variables, we are able to provide in Theorem 5 the geometrical average of the new vector $\mathbf{S}_{n,\sigma} := (S_{1,\sigma}, \dots, S_{n,\sigma})$. Special types of Bell polynomials are shown to be involved. In the last section, an analysis of real dataset using different measures is presented to illustrate the use of Dirichlet type 3 distribution.

Because of their frequent uses, we recall the definitions of the gamma, the beta type 1 and the beta type 3 distributions:

- If X is a random variable distributed according to the gamma distribution, with shape parameter $\theta > 0$ and scale parameter 1 (say $X \overset{d}{\sim} \gamma_\theta$), then its density function is given by $f_X(x) = (\Gamma(\theta))^{-1} e^{-x} x^{\theta-1}$, $x > 0$ and its moment function is $\mathbf{E}[X^q] = \Gamma(\theta+q)/\Gamma(\theta) =: (\theta)_q$, $q > -\theta$ where $\Gamma(\cdot)$ denotes the Euler gamma function.

- If a random variable A has the beta type 1 distribution with parameters $\alpha, \beta > 0$ (say $A \overset{d}{\sim} \beta^1(\alpha, \beta)$), then its density function is

$$f_A(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1,$$

so that its moment function is $\mathbf{E}[A^q] = (\alpha)_q/(\alpha+\beta)_q$.

- If a random variable B has the beta type 3 distribution with parameters $\alpha, \beta > 0$ (say $B \overset{d}{\sim} \beta^3(\alpha, \beta)$), then its density function is

$$f_B(x) = 2^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}(1+x)^{-(\alpha+\beta)}, \qquad 0 < x < 1,$$

so that its moment function is

$$\mathbf{E}[B^q] = \frac{2^{-\beta}(\alpha)_q}{(\alpha+\beta)_q} {}_2F_1\left(\beta, \alpha+\beta; \alpha+\beta+q; \frac{1}{2}\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function given as

$${}_2F_1(a, b; c; z) = \sum_{j \geq 0} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}.$$

The integral representation of the Gauss hypergeometric function is expressed as ([15, Eq. 3.6(1)]),

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt, \tag{1}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)| < \pi$.

2. Definition and main properties of the Dirichlet type 3 distribution

2.1. The Dirichlet type 3 distribution. Let $X_1, \ldots, X_{n+1}$ be $n+1$ independent random variables with respective gamma distributions $\gamma_{\theta_1}, \ldots, \gamma_{\theta_{n+1}}$, and define

$$\mathbf{S}_n = \left(\frac{X_1}{\sum_{m=1}^n X_m + 2X_{n+1}}, \ldots, \frac{X_n}{\sum_{m=1}^n X_m + 2X_{n+1}} \right).$$

Then, it is known that (see [2]) $\mathbf{S}_n := (S_1, \ldots, S_n)$ is distributed according to the Dirichlet type 3 distribution $D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$ with parameters $\boldsymbol{\theta}_n$ and θ_{n+1} on

$$\Lambda_n = \left\{ 0 < s_m < 1,\, m = 1, \ldots, n,\, \sum_{m=1}^n s_m < 1 \right\},$$

that is,

$$\mathbf{S}_n \overset{d}{\sim} \mu(\mathrm{d}\mathbf{s}_n) = f_{S_1,\dots,S_n}(s_1,\dots,s_n)\prod_{m=1}^{n}\mathrm{d}s_m.$$

Here, $f_{S_1,\dots,S_n}(s_1,\dots,s_n)$ is the joint density function written as

$$f_{S_1,\dots,S_n}(s_1,\dots,s_n) = \frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\prod_{m=1}^{n+1}\Gamma(\theta_m)} \times\left(\prod_{m=1}^{n}s_m^{\theta_m-1}\right)\left(1-\sum_{m=1}^{n}s_m\right)^{\theta_{n+1}-1}\left(1+\sum_{m=1}^{n}s_m\right)^{-\sum_{m=1}^{n+1}\theta_m}. \quad (2)$$

Alternatively, it is easy to show that the law of $\mathbf{S}_n := (S_1,\dots,S_n)$ can be characterized by its joint moment function ($q_m > -\theta_m$, $m=1,\dots,n$)

$$\mathbf{E}\left[\prod_{m=1}^{n}S_m^{q_m}\right] = \frac{2^{-\theta_{n+1}}\prod_{m=1}^{n}(\theta_m)_{q_m}}{(\sum_{m=1}^{n+1}\theta_m)_{\sum_{m=1}^{n}q_m}} \times {}_2F_1\left(\theta_{n+1},\sum_{m=1}^{n+1}\theta_m;\sum_{m=1}^{n}(\theta_m+q_m)+\theta_{n+1};\frac{1}{2}\right). \quad (3)$$

We write $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n;\theta_{n+1})$ if $\mathbf{S}_n$ has the Dirichlet type 3 distribution with parameters $\boldsymbol{\theta}_n$ and θ_{n+1} as stated above. For more details of the Dirichlet type 3 distribution, consult for example [2] and [10].

In Bayesian probability theory, if the posterior distribution is in the same probability distribution family as the prior probability distribution, the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function. In our case if

$$p(x_1,\dots,x_n,k|s_1,\dots,s_n) = \binom{x_1+\cdots+x_n+k}{x_1,\dots,x_n,k}s_1^{x_1}\cdots s_n^{x_n}\left(1-\sum_{m=1}^{n}s_m\right)^{k},$$

and

$$p(s_1,\dots,s_n) = C\left(\theta_1,\dots,\theta_n,\theta_{n+1},\sum_{m=1}^{n+1}\theta_m\right)\left(\prod_{m=1}^{n}s_m^{\theta_m-1}\right) \times\left(1-\sum_{m=1}^{n}s_m\right)^{\theta_{n+1}-1}\left(1+\sum_{m=1}^{n}s_m\right)^{-\sum_{m=1}^{n+1}\theta_m}$$

$$= \frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\prod_{m=1}^{n+1}\Gamma(\theta_m)}\left(\prod_{m=1}^{n}s_m^{\theta_m-1}\right)$$
$$\times\left(1-\sum_{m=1}^{n}s_m\right)^{\theta_{n+1}-1}\left(1+\sum_{m=1}^{n}s_m\right)^{-\sum_{m=1}^{n+1}\theta_m},$$

where $0 < s_m < 1,\ m = 1,\dots,n,\ \sum_{m=1}^{n} s_m < 1,$ and

$$[C(\theta_1,\dots,\theta_n,\theta_{n+1},\alpha)]^{-1} = \frac{\prod_{m=1}^{n+1}\Gamma(\theta_m)}{\Gamma(\sum_{m=1}^{n+1}\theta_m)}\,{}_2F_1\left(\sum_{m=1}^{n}\theta_m,\alpha;\sum_{m=1}^{n+1}\theta_m;-1\right),$$

then,

$$p(s_1,\dots,s_n|x_1,\dots,x_n,k)$$
$$= C\left(\theta_1+x_1,\dots,\theta_n+x_n,\theta_{n+1}+k,\sum_{m=1}^{n+1}\theta_m\right)$$
$$\times\left(\prod_{m=1}^{n}s_m^{\theta_m+x_m-1}\right)\left(1-\sum_{m=1}^{n}s_m\right)^{\theta_{n+1}+k-1}\left(1+\sum_{m=1}^{n}s_m\right)^{-\sum_{m=1}^{n+1}\theta_m}.$$

Thus, the Dirichlet type 3 distribution is conjugate prior for the multinomial distribution (see [17] for a more general case, when dealing with the multivariate Gauss hypergeometric distribution).

It follows from Eq. (2) that $\tilde{S} := \sum_{m=1}^{n} S_m \overset{d}{\sim} \beta^3(\sum_{m=1}^{n}\theta_m,\theta_{n+1})$ (see [2]). Note that the marginal distribution of S_m is not a beta type 3 distribution and it is easy to notice that its density function is written as (with $0 < s_m < 1$)

$$f_{S_m}(s_m) = \frac{2^{-\theta_{n+1}}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\Gamma(\theta_m)\Gamma(\sum_{m=1}^{n+1}\theta_m-\theta_m)}s_m^{\theta_m-1}(1-s_m)^{\sum_{m=1}^{n+1}\theta_m-\theta_m-1}$$
$$\times\,{}_2F_1\left(\theta_{n+1},\sum_{m=1}^{n+1}\theta_m;\sum_{l=1}^{n+1}\theta_l-\theta_m;\frac{1-s_m}{2}\right).$$

Moreover, Eq. (3) allows to obtain some statistical insight into the geometrical average of the variables $S_m;\ m = 1,\dots,n$. Indeed, putting $q_m = q/n$; $m = 1,\dots,n$, the moment function ($q/n > -\theta_m$, $m = 1,\dots,n$) of $\prod_{m=1}^{n} S_m^{1/n}$ is expressed by

$$\mathbf{E}\left[\left(\prod_{m=1}^{n}S_m^{1/n}\right)^q\right] = \frac{2^{-\theta_{n+1}}\prod_{m=1}^{n}(\theta_m)_{q/n}}{(\sum_{m=1}^{n+1}\theta_m)_q}\,{}_2F_1\left(\theta_{n+1},\sum_{m=1}^{n+1}\theta_m;\sum_{m=1}^{n+1}\theta_m+q;\frac{1}{2}\right).$$

Departing from this, we infer a result concerning the geometrical average of the S_m; $m = 1, \ldots, n$:

LEMMA 1. *Suppose that* $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$ *and that* $\mathbf{Y}_n := (Y_1, \ldots, Y_n)$ *is a random vector distributed according to Dirichlet distribution, denoted by* $D_n(\boldsymbol{\theta}_n)$. *Let* $\tilde{S} := \sum_{m=1}^n S_m \overset{d}{\sim} \beta^3(\sum_{m=1}^n \theta_m, \theta_{n+1})$ *be a random variable independent of* $\prod_{m=1}^n Y_m^{1/n}$. *Then,*

$$\prod_{m=1}^n S_m^{1/n} \overset{d}{=} \prod_{m=1}^n Y_m^{1/n} \cdot \tilde{S}.$$

PROOF. Firstly, suppose that $\mathbf{Y}_n := (Y_1, \ldots, Y_n)$ is distributed according to Dirichlet distribution $D_n(\boldsymbol{\theta}_n)$, with $\boldsymbol{\theta}_n := (\theta_1, \ldots, \theta_n)$. Then, its joint moment function is given by

$$\mathbf{E}\left[\prod_{m=1}^n Y_m^{q_m}\right] = \frac{\prod_{m=1}^n (\theta_m)_{q_m}}{(\sum_{m=1}^n \theta_m)_{\sum_{m=1}^n q_m}}.$$

Putting $q_m = q/n$; $m = 1, \ldots, n$, we have

$$\mathbf{E}\left[\left(\prod_{m=1}^n Y_m^{1/n}\right)^q\right] = \frac{\prod_{m=1}^n (\theta_m)_{q/n}}{(\sum_{m=1}^n \theta_m)_q}.$$

Then,

$$\mathbf{E}\left[\left(\prod_{m=1}^n S_m^{1/n}\right)^q\right] = \frac{2^{-\theta_{n+1}}(\sum_{m=1}^n \theta_m)_q}{(\sum_{m=1}^{n+1} \theta_m)_q} \times {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + q; \frac{1}{2}\right) \cdot \mathbf{E}\left[\left(\prod_{m=1}^n Y_m^{1/n}\right)^q\right].$$

Next, the moment function of $\tilde{S} \overset{d}{\sim} \beta^3(\sum_{m=1}^n \theta_m, \theta_{n+1})$ is given by

$$\mathbf{E}[\tilde{S}^q] = \frac{2^{-\theta_{n+1}}(\sum_{m=1}^n \theta_m)_q}{(\sum_{m=1}^{n+1} \theta_m)_q} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + q; \frac{1}{2}\right).$$

Departing from this, we obtain

$$\mathbf{E}\left[\left(\prod_{m=1}^n Y_m^{1/n}\right)^q\right] \cdot \mathbf{E}[\tilde{S}^q] = \mathbf{E}\left[\left(\prod_{m=1}^n S_m^{1/n}\right)^q\right],$$

and we are done. □

2.2. The RAM structure of the Dirichlet type 3 distribution. The objective of this section is to present a fundamental property of the Dirichlet type 3 distribution $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$. The idea is to give the residual allocation model (RAM) representation of a random vector which can be defined as follows in a general setting (see [20]):

Definition 1. A random vector $(W_1, \ldots, W_n)$ with values on $(0,1)^n$ such that $\sum_{m=1}^n W_m = \tilde{W}$ has a RAM representation if and only if there exist $n-1$ independent $(0,1)$-valued random variables $B_1, \ldots, B_{n-1}$ mutually independent with $\tilde{W}$, such that $W_1 = \tilde{W} \cdot B_1$ and

$$W_m = \tilde{W} \cdot B_m \prod_{k=1}^{m-1} (1 - B_k), \qquad m = 2, \ldots, n-1,$$

$$W_n = \tilde{W} \cdot \prod_{m=1}^{n-1} (1 - B_m).$$

Such a model is also called a stick-breaking model and is used in non-parametric Bayesian statistics (see for instance [13]).

Concerning this, we have the following result:

Theorem 1. *Let $A_1, \ldots, A_{n-1}$ be $(n-1)$ independent random variables with distribution $A_m \overset{d}{\sim} \beta^3(\theta_m, \sum_{l=1}^n \theta_l - \sum_{l=1}^m \theta_l)$, $m = 1, \ldots, n-1$. Let $B_m := 2A_m/(1 + A_m)$; $m = 1, \ldots, n-1$, and $\tilde{S} \overset{d}{\sim} \beta^3(\sum_{m=1}^n \theta_m, \theta_{n+1})$ be a random variable independent of $(B_1, \ldots, B_{n-1})$. With $\prod_{k=1}^0 (1 - B_k) := 1$, define*

$$S_m := \tilde{S} \cdot B_m \prod_{k=1}^{m-1} (1 - B_k), \qquad m = 1, \ldots, n-1, \tag{4}$$

$$S_n = \tilde{S} - \sum_{m=1}^{n-1} S_m = \tilde{S} \cdot \prod_{m=1}^{n-1} (1 - B_m).$$

Then, $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$.

Proof. First, using the independence property,

$$\mathbf{E}\left[\prod_{m=1}^n S_m^{q_m}\right] = \left[\prod_{m=1}^{n-1} \mathbf{E}(B_m^{q_m}(1 - B_m)^{\sum_{l=m+1}^n q_l})\right] \mathbf{E}[\tilde{S}^{\sum_{m=1}^n q_m}].$$

Additionally, it is easy to infer that if $A_m \overset{d}{\sim} \beta^3(\theta_m, \sum_{l=1}^n \theta_l - \sum_{l=1}^m \theta_l)$, $m = 1, \ldots, n$, then $B_m \overset{d}{\sim} \beta^1(\theta_m, \sum_{l=1}^n \theta_l - \sum_{l=1}^m \theta_l)$, $m = 1, \ldots, n$ (see for instance [2]). From this, we can check that

$$\mathbf{E}(B_m^{q_m}(1-B_m)^{\sum_{l=m+1}^{n} q_l}) = \frac{(\theta_m)_{q_m}\Gamma(\sum_{l=1}^{n}\theta_l - \sum_{l=1}^{m-1}\theta_l)}{\Gamma(\sum_{l=1}^{n}\theta_l - \sum_{l=1}^{m}\theta_l)} \times \frac{\Gamma(\sum_{l=1}^{n}\theta_l - \sum_{l=1}^{m}\theta_l + \sum_{l=m+1}^{n} q_l)}{\Gamma(\sum_{l=1}^{n}\theta_l - \sum_{l=1}^{m-1}\theta_l + \sum_{l=m}^{n} q_l)}.$$

On the other side, the moment function of $\tilde{S}$ reads

$$\mathbf{E}[\tilde{S}^{\sum_{m=1}^{n} q_m}] = \frac{2^{-\theta_{n+1}}\Gamma(\sum_{m=1}^{n+1}\theta_m)\Gamma(\sum_{m=1}^{n}(\theta_m+q_m))}{\Gamma(\sum_{m=1}^{n}\theta_m)\Gamma(\sum_{m=1}^{n}(\theta_m+q_m)+\theta_{n+1})} \times {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1}\theta_m; \sum_{m=1}^{n}(\theta_m+q_m)+\theta_{n+1}; \frac{1}{2}\right).$$

Therefore, we obtain

$$\mathbf{E}\left[\prod_{m=1}^{n} S_m^{q_m}\right] = \frac{\Gamma(\sum_{m=1}^{n}\theta_m)\prod_{m=1}^{n}(\theta_m)_{q_m}}{\Gamma(\sum_{m=1}^{n}(\theta_m+q_m))} \times \frac{2^{-\theta_{n+1}}\Gamma(\sum_{m=1}^{n+1}\theta_m)\Gamma(\sum_{m=1}^{n}(\theta_m+q_m))}{\Gamma(\sum_{m=1}^{n}\theta_m)\Gamma(\sum_{m=1}^{n}(\theta_m+q_m)+\theta_{n+1})} \times {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1}\theta_m; \sum_{m=1}^{n}(\theta_m+q_m)+\theta_{n+1}; \frac{1}{2}\right).$$

This confirms that $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$. □

REMARK 1. *The variables $\tilde{S}\cdot(1-B_k)$ can be interpreted as residual fractions in a stick-breaking scheme: start with a stick of length $\tilde{S} < 1$. Choose a point on the stick according to distribution $B_1\tilde{S}$, 'break' the stick into two pieces, discard the piece of length $B_1\tilde{S}$ and rescale the remaining half to have length $\tilde{S}$. Repeating this procedure m times, and* (4) *is the fraction broken off at step m relative to the original stick length. Note also that this formula is similar to the RAM structure obtained for the Dirichlet distribution and appears as an exercise (without proof) in the book of Devroye* [4] *on page* 585.

2.3. The gamma distribution, Dirichlet type 3 distribution and Dirichlet distribution. In this section, our central focus is upon the scaling property of the Dirichlet type 3 distribution along with the one of the Dirichlet distribution.

2.3.1. *Scaling property of the Dirichlet type* 3 *distribution.* Let $X_1, \ldots, X_{n+1}$ be $n+1$ independent gamma random variables with respective gamma distributions $\gamma_{\theta_1}, \ldots, \gamma_{\theta_{n+1}}$. For $x > 0$, we define

$$\mathbf{S}_n(x) := \left(\frac{X_1(x)}{\sum_{m=1}^{n} X_m + 2X_{n+1}}, \ldots, \frac{X_n(x)}{\sum_{m=1}^{n} X_m + 2X_{n+1}} \right),$$

where $X_m(x) = xX_m$; $m = 1, \ldots, n$. In this case, the distribution of $\mathbf{S}_n(x) := (S_m(x);\ m = 1, \ldots, n)$ is expressed by (with $0 < \sum_{m=1}^{n} s_m < x$)

$$f_{S_1(x),\ldots,S_n(x)}(s_1, \ldots, s_n) = \frac{2^{\sum_{m=1}^{n}\theta_m} \Gamma\left(\sum_{m=1}^{n+1} \theta_m \right)}{x^{-1} \prod_{m=1}^{n+1} \Gamma(\theta_m)} \left(\prod_{m=1}^{n} s_m^{\theta_m - 1} \right)$$
$$\times \left(x - \sum_{m=1}^{n} s_m \right)^{\theta_{n+1}-1} \left(x + \sum_{m=1}^{n} s_m \right)^{-\sum_{m=1}^{n+1}\theta_m}. \quad (5)$$

Let $\tilde{X} := \sum_{m=1}^{n+1} X_m$ be the sum of $(n+1)$ independent and identically distributed gamma random variables γ_{θ_m}.

Proposition 1. (i) *Let*

$$\mathbf{S}_n(1) := \left(\frac{X_1}{\sum_{m=1}^{n} X_m + 2X_{n+1}}, \ldots, \frac{X_n}{\sum_{m=1}^{n} X_m + 2X_{n+1}} \right).$$

Then, $\mathbf{S}_n(1)$ *and* $\tilde{X}$ *are independent, and it holds that* $\mathbf{S}_n(1) \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$.

(ii) *The scaling property is given by* $xS_m \overset{d}{=} S_m(x)$, $m = 1, \ldots, n$.

Proof. (i) Using the independence of X_m, the joint density function of $X_1, \ldots, X_{n+1}$ is expressed by

$$\frac{e^{-\sum_{m=1}^{n+1} x_m}}{\prod_{m=1}^{n+1} \Gamma(\theta_m)} \prod_{m=1}^{n+1} x_m^{\theta_m - 1}. \quad (6)$$

Making the transformation $s_m = x_m/(x_{n+1} + \tilde{x})$; $m = 1, \ldots, n$, with $\tilde{x} = \sum_{m=1}^{n+1} x_m$ and the Jacobian

$$J(x_1, \ldots, x_{n+1} \to s_1, \ldots, s_n, \tilde{x}) = \frac{(2\tilde{x})^n}{(1 + \sum_{m=1}^{n} s_m)^{n+1}},$$

in Eq. (6), the joint density function of $S_1, \ldots, S_n, \tilde{X}$ is given by

$$e^{-\tilde{x}} \tilde{x}^{\sum_{m=1}^{n+1}\theta_m - 1} \cdot \frac{2^{\sum_{m=1}^{n}\theta_m}}{\prod_{m=1}^{n+1} \Gamma(\theta_m)} \left(\prod_{m=1}^{n} s_m^{\theta_m - 1} \right)$$
$$\times \left(1 - \sum_{m=1}^{n} s_m \right)^{\theta_{n+1}-1} \left(1 + \sum_{m=1}^{n} s_m \right)^{-\sum_{m=1}^{n+1}\theta_m}.$$

Consequently,

$$\mathbf{S}_n(1) = \left(\frac{X_1}{\sum_{m=1}^n X_m + 2X_{n+1}}, \ldots, \frac{X_n}{\sum_{m=1}^n X_m + 2X_{n+1}} \right)$$

and $\tilde{X}$ are independent. The first statement follows.

(ii) It follows that

$$xS_m \overset{d}{=} xS_m(1) = S_m(x); \qquad m = 1, \ldots, n. \qquad \square$$

2.3.2. *Scaling property of Dirichlet distribution from the Dirichlet type* 3 *distribution.* In this section, we address the following question raised in the previous section: what is the scaling property of the Dirichlet distribution using the Dirichlet type 3 distribution? Note that we can find the same result if we use the gamma distribution.

It is known that the Dirichlet distribution $\mathbf{Y}_n := (Y_1, \ldots, Y_n) \overset{d}{\sim} D_n(\boldsymbol{\theta}_n)$, can be generated by $Y_m = X_m/\mathscr{X}$, where $\mathscr{X} := \sum_{m=1}^n X_m$ is the sum of n independent gamma(θ_m) distributed random variables. From this, its joint density function is written as

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \frac{\Gamma(\sum_{m=1}^n \theta_m)}{\prod_{m=1}^n \Gamma(\theta_m)} \prod_{m=1}^n y_m^{\theta_m - 1} \cdot \delta_{(\sum_{m=1}^n y_m = 1)}.$$

We will define $\mathbf{Y}_n \overset{d}{\sim} D_n(\boldsymbol{\theta}_n)$ on Ξ_n and $\mathbf{Y}_{n-1} \overset{d}{\sim} D_{n-1}(\boldsymbol{\theta}_{n-1}; \theta_n)$ on Λ_{n-1} where

$$\Xi_n = \left\{ (y_1, \ldots, y_n) \in (0,1)^n : \sum_{m=1}^n y_m = 1 \right\},$$

$$\Lambda_{n-1} = \left\{ (y_1, \ldots, y_{n-1}) \in (0,1)^{n-1} : \sum_{m=1}^{n-1} y_m < 1 \right\}.$$

It should be noticed that if $\mathbf{Y}_{n-1} = (Y_1, \ldots, Y_{n-1}) \overset{d}{\sim} D_{n-1}(\boldsymbol{\theta}_{n-1}; \theta_n)$ and $Y_n = 1 - \sum_{m=1}^{n-1} Y_m$, then $\mathbf{Y}_n = (Y_1, \ldots, Y_n) \overset{d}{\sim} D_n(\boldsymbol{\theta}_n)$.

More generally, we consider $\mathbf{Y}_n(x) := (X_1(x)/\mathscr{X}, \ldots, X_n(x)/\mathscr{X})$, where $X_m(x) = xX_m$; $m = 1, \ldots, n$ and $x > 0$. Then, the distribution of $\mathbf{Y}_n(x) := (Y_m(x); m = 1, \ldots, n)$ is found to be

$$f_{Y_1(x),\ldots,Y_n(x)}(y_1, \ldots, y_n) = \frac{\Gamma(\sum_{m=1}^n \theta_m)}{\prod_{m=1}^n \Gamma(\theta_m)} \frac{\prod_{m=1}^n y_m^{\theta_m - 1}}{x^{\sum_{m=1}^n \theta_m - 1}} \cdot \delta_{(\sum_{m=1}^n y_m = x)}. \tag{7}$$

Furthermore, we obtain the following result:

PROPOSITION 2. (i) *With* $\tilde{S} = \sum_{m=1}^n S_m$, *let* $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$ *be the Dirichlet type* 3 *distribution. Consider the Dirichlet distribution* $\mathbf{Y}_{n-1} \overset{d}{\sim}$

$D_{n-1}(\boldsymbol{\theta}_{n-1};\theta_n)$. *Then, it can also be defined conditionally as*

$$\mathbf{Y}_{n-1} \stackrel{d}{=} \left(\frac{S_1}{\tilde{S}}, \dots, \frac{S_{n-1}}{\tilde{S}} \,\middle|\, \tilde{S} = s\right), \qquad 0 < s < 1.$$

(ii) *The scaling property of the Dirichlet distribution is given by* $xY_m \stackrel{d}{=} Y_m(x)$, $m = 1, \dots, n-1$.

Proof. (i) Recall that the Dirichlet distribution $\mathbf{Y}_{n-1} \stackrel{d}{\sim} D_{n-1}(\boldsymbol{\theta}_{n-1};\theta_n)$ is defined by

$$f_{Y_1,\dots,Y_{n-1}}(y_1,\dots,y_{n-1}) = \frac{\Gamma(\sum_{m=1}^{n}\theta_m)}{\prod_{m=1}^{n}\Gamma(\theta_m)}\left(\prod_{m=1}^{n-1} y_m^{\theta_m-1}\right)\left(1-\sum_{m=1}^{n-1} y_m\right)^{\theta_n-1}\cdot\delta_{(\sum_{m=1}^{n-1} y_m<1)}.$$

Additionally, making the transformation $s_m = y_m \cdot \tilde{s}$; $m = 1, \dots, n-1$ with $\tilde{s} = \sum_{m=1}^{n} s_m$ and the Jacobian

$$J(s_1,\dots,s_n \to y_1,\dots,y_{n-1},\tilde{s}) = \tilde{s}^{n-1}$$

in Eq. (2), the joint density function of $Y_1,\dots,Y_{n-1},\tilde{S}$ is expressed by

$$\frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\prod_{m=1}^{n+1}\Gamma(\theta_m)} s^{\sum_{m=1}^{n}\theta_m-1}(1-s)^{\theta_{n+1}-1}(1+s)^{-\sum_{m=1}^{n+1}\theta_m}$$
$$\times\left(\prod_{m=1}^{n-1} y_m^{\theta_m-1}\right)\left(1-\sum_{m=1}^{n-1} y_m\right)^{\theta_n-1}\cdot\delta_{(\sum_{m=1}^{n-1} y_m<1)}.$$

Therefore, $\mathbf{Y}_{n-1} = (S_1/\tilde{S},\dots,S_{n-1}/\tilde{S})$ and $\tilde{S}$ are independent. Consequently, the density of $(S_1/\tilde{S},\dots,S_{n-1}/\tilde{S})$ conditioned to $\tilde{S} = s$ is written as

$$f^{\tilde{S}=s}_{S_1/\tilde{S},\dots,S_{n-1}/\tilde{S}}(s_1,\dots,s_{n-1}) = \frac{\Gamma(\sum_{m=1}^{n}\theta_m)}{\prod_{m=1}^{n}\Gamma(\theta_m)}\prod_{m=1}^{n-1} s_m^{\theta_m-1}$$
$$\times\left(1-\sum_{m=1}^{n-1} s_m\right)^{\theta_n-1}\cdot\delta_{(\sum_{m=1}^{n-1} s_m<1)},$$

and so we conclude that

$$\mathbf{Y}_{n-1} \stackrel{d}{=} \left(\frac{S_1}{\tilde{S}}, \dots, \frac{S_{n-1}}{\tilde{S}} \,\middle|\, \tilde{S} = s\right), \qquad 0 < s < 1.$$

(ii) For $x > 0$, we define

$$\mathbf{Y}_{n-1}(x) := \left(\frac{xS_1}{\tilde{S}}, \dots, \frac{xS_{n-1}}{\tilde{S}}\right).$$

In this case, the distribution of $\mathbf{Y}_{n-1}(x) := (Y_m(x);\, m = 1, \dots, n-1)$ is given by

$$\begin{aligned} & f_{Y_1(x),\dots,Y_{n-1}(x)}(y_1, \dots, y_{n-1}) \\ &= \frac{\Gamma(\sum_{m=1}^{n} \theta_m)}{\prod_{m=1}^{n} \Gamma(\theta_m)} \cdot \frac{\prod_{m=1}^{n-1} y_m^{\theta_m - 1}(x - \sum_{m=1}^{n-1} y_m)^{\theta_n - 1}}{x^{\sum_{m=1}^{n} \theta_m - 1}} \cdot \delta_{(\sum_{m=1}^{n-1} y_m < x)}. \end{aligned}$$

One can check that

$$\left(\frac{xS_1}{\tilde{S}}, \dots, \frac{xS_{n-1}}{\tilde{S}} \,\middle|\, x\tilde{S} = xs\right) \stackrel{d}{=} \mathbf{Y}_{n-1}(x).$$

Thus, we obtain the scaling property $xY_m \stackrel{d}{=} Y_m(x)$, $m = 1, \dots, n-1$. □

2.4. A constructive formula for computing with Dirichlet type 3. The core result of this paper is displayed in this section. Our main objective is to elaborate a formula which allows to compute many functionals of Dirichlet type 3 distributions in terms of simpler functionals of independent gamma random variables. At this stage of analysis, we need the following definition:

DEFINITION 2. Let $\mathbf{s}_n := (s_1, \dots, s_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. If $f(x\mathbf{s}_n) = x^d f(\mathbf{s}_n)$ for $x > 0$, then f is said to be homogeneous of degree d.

In addition, the following result which allows to compute many functionals of Dirichlet distributions in terms of simpler functionals of independent gamma random variables should be required (see [12]).

THEOREM 2. *Consider the Dirichlet distribution* $\mathbf{Y}_n \stackrel{d}{\sim} D_n(\boldsymbol{\theta}_n)$. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be any Borel-measurable function for which*

$$\int_0^\infty \mathbf{E}(|f(\mathbf{Y}_n(x))|) x^{\sum_{m=1}^{n} \theta_m - 1} e^{-yx}\, dx < \infty.$$

Then, with $\mathbf{X}_n(y) := (X_m(y);\, m = 1, \dots, n)$, *n independent random variables defined by* $X_m(y) = (1/y)X_m$, $y > 0$, $m = 1, \dots, n$, *where* $X_m \stackrel{d}{\sim} \gamma_{\theta_m}$, *we have*

$$\int_0^\infty \mathbf{E}(f(\mathbf{Y}_n(x))) x^{\sum_{m=1}^{n} \theta_m - 1} e^{-yx}\, dx = \Gamma\left(\sum_{m=1}^{n} \theta_m\right) y^{-\sum_{m=1}^{n} \theta_m} \mathbf{E}(f(\mathbf{X}_n(y))). \quad (8)$$

Consequently, one has

Theorem 3. *Consider the Dirichlet type* 3 *distribution* $\mathbf{S}_n \overset{d}{\sim} D_n^3(\boldsymbol{\theta}_n; \theta_{n+1})$. (i) *With* $0 < s < 1$, *let* $\mathbf{X}_n(y/(1-s)) = (X_m(y/(1-s)); m = 1, \ldots, n)$, *be* n *independent random variables defined as above in Theorem* 2. *Then, for* $p > 0$ *and a Borel-measurable function* $f : \mathbb{R}^n \to \mathbb{R}$ *satisfying*

$$\int_0^1 \mathbf{E}\left(\left| f\left(\mathbf{X}_n\left(\frac{y}{1-s}\right)\right)\right|\right) \times s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^n \theta_m - 1}\left(1 - \frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds < \infty,$$

one has

$$\mathbf{E}(f(\mathbf{S}_n(p))) = \frac{2^{-\theta_{n+1}}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{p^{\sum_{m=1}^n \theta_m - 1}\Gamma(\theta_{n+1})} \times \frac{1}{2\pi i}\int_L e^{py} y^{-\sum_{m=1}^n \theta_m} \int_0^1 \mathbf{E}\left(f\left(\mathbf{X}_n\left(\frac{y}{1-s}\right)\right)\right) \times s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^n \theta_m - 1}\left(1 - \frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} dsdy, \tag{9}$$

where L *denotes any path in the complex* t*-plane originating at* $-\infty$ *encircling in the positive direction all finite singularities of the integrand and returning to* $-\infty$.

(ii) *If* f *is homogeneous of degree* d, *and if* $\mathbf{E}(|f(\mathbf{X}_n)|) < \infty$, *then, with* $\mathbf{X}_n := (X_1, \ldots, X_n)$,

$$\mathbf{E}(f(\mathbf{S}_n)) = \frac{2^{-\theta_{n+1}}}{(\sum_{m=1}^{n+1}\theta_m)_d}\, {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1}\theta_m; \sum_{m=1}^{n+1}\theta_m + d; \frac{1}{2}\right)\mathbf{E}(f(\mathbf{X}_n)). \tag{10}$$

Proof. (i) Firstly, using the scaling property $pS_m \overset{d}{=} S_m(p)$, $m = 1, \ldots, n$, one can check that $\mathbf{S}_n(p) = (S_1(p), \ldots, S_n(p))$ given $p\tilde{S} := \sum_{m=1}^n S_m(p) = x$ has the same distribution as $\mathbf{Y}_n(x)$. Indeed, the density of $\mathbf{S}_n(p)$ conditioned to $p\tilde{S} = x$ is expressed as

$$f_{S_1(p),\ldots,S_n(p)}^{p\tilde{S}=x}(s_1, \ldots, s_n) = \frac{f_{pS_1,\ldots,pS_n}(s_1, \ldots, s_n)}{f_{p\tilde{S}}(x)} \cdot \delta_{(\sum_{m=1}^n s_m = x)}. \tag{11}$$

It is easy to see that the density of $p\tilde{S}$ is given by

$$f_{p\tilde{S}}(x) = \frac{2^{\sum_{m=1}^n \theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)p}{\Gamma(\sum_{m=1}^n \theta_m)\Gamma(\theta_{n+1})} \times x^{\sum_{m=1}^n \theta_m - 1}(p-x)^{\theta_{n+1}-1}(p+x)^{-\sum_{m=1}^{n+1}\theta_m}, \qquad 0 < x < p.$$

Along with Eq. (2), we deduce that Eq. (11) can be written as

$$f^{p\tilde{S}=x}_{S_1(p),\dots,S_n(p)}(s_1,\dots,s_n) = \frac{\Gamma(\sum_{m=1}^{n}\theta_m)}{\prod_{m=1}^{n}\Gamma(\theta_m)}\frac{\prod_{m=1}^{n} s_m^{\theta_m-1}}{x^{\sum_{m=1}^{n}\theta_m-1}}\cdot\delta_{(\sum_{m=1}^{n}s_m=x)},$$

which coincides with Eq. (7). Therefore, we obtain $\mathbf{Y}_n(x) \overset{d}{=} (\mathbf{S}_n(p) \mid p\tilde{S} = x)$ for $0 < x < p$.

Hence, with f as in the statement of Theorem 3, we find

$$\mathbf{E}(f(\mathbf{Y}_n(x))) = \mathbf{E}(f(\mathbf{S}_n(p) \mid p\tilde{S} = x)).$$

Multiplying both sides of this identity by the density of $p\tilde{S}$ and integrating with respect to dx yield

$$\mathbf{E}(f(\mathbf{S}_n(p))) = \frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)p}{\Gamma(\sum_{m=1}^{n}\theta_m)\Gamma(\theta_{n+1})} \times \int_0^p \mathbf{E}(f(\mathbf{Y}_n(x)))\frac{(p-x)^{\theta_{n+1}-1}(p+x)^{-\sum_{m=1}^{n+1}\theta_m}}{x^{-(\sum_{m=1}^{n}\theta_m-1)}}\,dx.$$

After the change of variable $x = pu$, we get

$$\mathbf{E}(f(\mathbf{S}_n(p))) = \frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\Gamma(\sum_{m=1}^{n}\theta_m)\Gamma(\theta_{n+1})} \times \int_0^1 \mathbf{E}(f(\mathbf{Y}_n(pu)))\frac{(1-u)^{\theta_{n+1}-1}(1+u)^{-\sum_{m=1}^{n+1}\theta_m}}{u^{-(\sum_{m=1}^{n}\theta_m-1)}}\,du.$$

Multiplying both sides of this identity by $p^{\sum_{m=1}^{n}\theta_m-1}e^{-py}$ and integrating them with respect to dp yield

$$\begin{aligned}\int_0^\infty \mathbf{E}(f(\mathbf{S}_n(p)))p^{\sum_{m=1}^{n}\theta_m-1}e^{-py}\,dp &= \frac{2^{\sum_{m=1}^{n}\theta_m}\Gamma(\sum_{m=1}^{n+1}\theta_m)}{\Gamma(\sum_{m=1}^{n}\theta_m)\Gamma(\theta_{n+1})} \\ &\quad\times \int_0^1 u^{\sum_{m=1}^{n}\theta_m-1}(1-u)^{\theta_{n+1}-1}(1+u)^{-\sum_{m=1}^{n+1}\theta_m} \\ &\quad\times \int_0^\infty \mathbf{E}(f(\mathbf{Y}_n(pu)))p^{\sum_{m=1}^{n}\theta_m-1}e^{-py}\,dpdu. \qquad (12)\end{aligned}$$

Furthermore, from Theorem 2 and after a change of variable $x = up$, Eq. (8) becomes

$$\int_0^\infty \mathbf{E}(f(\mathbf{Y}_n(pu)))p^{\sum_{m=1}^{n}\theta_m-1}e^{-py}\,dp = \Gamma\left(\sum_{m=1}^{n}\theta_m\right)y^{-\sum_{m=1}^{n}\theta_m}\mathbf{E}\left(f\left(\mathbf{X}_n\left(\frac{y}{u}\right)\right)\right).$$

Substituting this identity into Eq. (12) gives

$$\int_0^\infty \mathbf{E}(f(\mathbf{S}_n(p)))p^{\sum_{m=1}^n \theta_m - 1} e^{-py}\, dp$$

$$= \frac{2^{\sum_{m=1}^n \theta_m} \Gamma(\sum_{m=1}^{n+1} \theta_m) y^{-\sum_{m=1}^n \theta_m}}{\Gamma(\theta_{n+1})}$$

$$\times \int_0^1 \mathbf{E}\left(f\left(\mathbf{X}_n\left(\frac{y}{u}\right)\right)\right) u^{\sum_{m=1}^n \theta_m - 1}(1-u)^{\theta_{n+1}-1}(1+u)^{-\sum_{m=1}^{n+1}\theta_m} du.$$

Making the change of variable $s = 1 - u \in (0, 1)$, we obtain

$$\int_0^\infty \mathbf{E}(f(\mathbf{S}_n(p)))p^{\sum_{m=1}^n \theta_m - 1} e^{-py}\, dp$$

$$= \frac{2^{-\theta_{n+1}} \Gamma(\sum_{m=1}^{n+1} \theta_m) y^{-\sum_{m=1}^n \theta_m}}{\Gamma(\theta_{n+1})}$$

$$\times \int_0^1 \mathbf{E}\left(f\left(\mathbf{X}_n\left(\frac{y}{1-s}\right)\right)\right) s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^n \theta_m - 1}\left(1 - \frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds.$$

It is clear that $\int_0^\infty \mathbf{E}(f(\mathbf{S}_n(p)))p^{\sum_{m=1}^n \theta_m - 1} e^{-py}\, dp$ is the Laplace transform in the variable p of $\mathbf{E}(f(\mathbf{S}_n(p)))p^{\sum_{m=1}^n \theta_m - 1}$. By inverting this Laplace transform, the result follows.

(ii) Recall that $S_m(p) \overset{d}{=} pS_m$ and $X_m(y/(1-s)) = ((1-s)/y)X_m$. Using the fact that f is homogeneous of degree d, Eq. (9) becomes

$$\mathbf{E}(f(\mathbf{S}_n)) = \mathbf{E}(f(\mathbf{X}_n)) \frac{2^{-\theta_{n+1}} \Gamma(\sum_{m=1}^{n+1} \theta_m)}{\Gamma(\theta_{n+1}) p^{d + \sum_{m=1}^n \theta_m - 1}} \cdot \frac{1}{2\pi i} \int_L e^{py} y^{-(\sum_{m=1}^n \theta_m + d)}$$

$$\times \int_0^1 s^{\theta_{n+1}-1}(1-s)^{d+\sum_{m=1}^n \theta_m - 1}\left(1 - \frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds dy.$$

Finally, using Eq. (1) and the fact that

$$\frac{1}{2\pi i} \int_L e^{py} y^{-k}\, dy = \frac{p^{k-1}}{\Gamma(k)}, \tag{13}$$

we get the desired result. □

There are some direct applications of Theorem 3.

1. The importance of the statement (i) of Theorem 3 can be detected when used to compute the characteristic function of the Dirichlet type 3 distribution which is an important statistical quantity that was not given until today unlike the one given for the Dirichlet distribution and for the Dirichlet inverse distribution (or Dirichlet type 2 distribution).

Let $f(S_1, \ldots, S_n) = \prod_{m=1}^{n} e^{it_m S_m} = e^{i\sum_{m=1}^{n} t_m S_m}$. Then, $\mathbf{E}(f(S_1, \ldots, S_n))$ is the characteristic function of the Dirichlet type 3 distribution. Using the independence of $X_m(y)$; $m = 1, \ldots, n$, we obtain

$$\mathbf{E}\left(\prod_{m=1}^{n} e^{it_m X_m(y/(1-s))}\right) = \prod_{m=1}^{n} \mathbf{E}(e^{it_m((1-s)/y)X_m}) = \prod_{m=1}^{n}\left(1 - it_m \frac{(1-s)}{y}\right)^{-\theta_m}.$$

Applying the statement (i) of Theorem 3, we observe that

$$\int_0^1 \mathbf{E}\left(f\left(\mathbf{X}_n\left(\frac{y}{1-s}\right)\right)\right) s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^{n}\theta_m - 1}\left(1 - \frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds$$

is equal to

$$\begin{aligned}
&\int_0^1 \prod_{m=1}^{n}\left(1 - it_m\frac{(1-s)}{y}\right)^{-\theta_m} s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^{n}\theta_m - 1}\left(1-\frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds\\
&\quad= \sum_{k_1\geq 0,\ldots,k_n\geq 0}\prod_{m=1}^{n}\frac{(\theta_m)_{k_m}}{k_m!}\left(\frac{it_m}{y}\right)^{k_m}\\
&\qquad\times\int_0^1 s^{\theta_{n+1}-1}(1-s)^{\sum_{m=1}^{n}(\theta_m+k_m)-1}\left(1-\frac{s}{2}\right)^{-\sum_{m=1}^{n+1}\theta_m} ds\\
&\quad= \sum_{k_1\geq 0,\ldots,k_n\geq 0}\left(\prod_{m=1}^{n}\frac{(\theta_m)_{k_m}}{k_m!}\left(\frac{it_m}{y}\right)^{k_m}\right)\frac{\Gamma(\theta_{n+1})\Gamma(\sum_{m=1}^{n}(\theta_m+k_m))}{\Gamma(\sum_{m=1}^{n+1}\theta_m + \sum_{m=1}^{n}k_m)}\\
&\qquad\times {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1}\theta_m; \sum_{m=1}^{n+1}\theta_m + \sum_{m=1}^{n}k_m; \frac{1}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}(e^{ip\sum_{m=1}^{n}t_m S_m})\\
&\quad= \frac{2^{-\theta_{n+1}}}{p^{\sum_{m=1}^{n}\theta_m - 1}}\sum_{k_1\geq 0,\ldots,k_n\geq 0}\left(\prod_{m=1}^{n}\frac{(\theta_m)_{k_m}}{k_m!}(it_m)^{k_m}\right)\frac{\Gamma(\sum_{m=1}^{n}(\theta_m+k_m))}{(\sum_{m=1}^{n+1}\theta_m)_{\sum_{m=1}^{n}k_m}}\\
&\qquad\times {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1}\theta_m; \sum_{m=1}^{n+1}\theta_m + \sum_{m=1}^{n}k_m; \frac{1}{2}\right)\cdot\frac{1}{2\pi i}\int_L e^{py} y^{-\sum_{m=1}^{n}(\theta_m+k_m)} dy.
\end{aligned}$$

Using Eq. (13) and putting $p = 1$, the above expression can be rewritten as

$$\mathbf{E}(e^{i\sum_{m=1}^{n} t_m S_m}) = 2^{-\theta_{n+1}} \sum_{k_1 \geq 0, \ldots, k_n \geq 0} \prod_{m=1}^{n} \frac{(\theta_m)_{k_m}}{k_m!} \frac{(it_m)^{k_m}}{(\sum_{m=1}^{n+1} \theta_m)_{\sum_{m=1}^{n} k_m}}$$

$$\times\, {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + \sum_{m=1}^{n} k_m; \frac{1}{2}\right).$$

2. From part (ii) of Theorem 3, any homogeneous functional of Dirichlet type 3 distribution can be directly computed from the simpler one of independent gamma variables, each with parameter θ_m, leading to considerable simplification.

- Considering the function $f(S_1, \ldots, S_n) = \prod_{m=1}^{n} S_m^{q_m}$, we obtain that f is homogeneous of degree $d = \sum_{m=1}^{n} q_m$. Application of (ii) to this functional provides Eq. (3).
- If we consider the function $f(S_1, \ldots, S_n) = (\sum_{m=1}^{n} S_m)^q$, we observe that f is homogeneous of degree $d = q$. Applying part (ii) of Theorem 3 to this particular case provides

$$\mathbf{E}\left[\left(\sum_{m=1}^{n} S_m\right)^q\right] = \frac{2^{-\theta_{n+1}} (\sum_{m=1}^{n} \theta_m)_q}{(\sum_{m=1}^{n+1} \theta_m)_q} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + q; \frac{1}{2}\right),$$

demonstrating that $\sum_{m=1}^{n} S_m \overset{d}{\sim} \beta^3(\sum_{m=1}^{n} \theta_m, \theta_{n+1})$.

- Consider the random variables

$$\tilde{S}_m := S_m \Big/ \sum_{l=1}^{n} S_l, \qquad m = 1, \ldots, k < n.$$

They constitute a partition of the unit interval. This is the Dirichlet distribution and

$$(\tilde{S}_m; m = 1, \ldots, k) \overset{d}{\sim} D_k\left(\theta_1, \ldots, \theta_k, \sum_{m=k+1}^{n} \theta_m\right),$$

$$(\tilde{S}_m; m = 1, \ldots, k) \text{ and } \sum_{m=1}^{n} S_m \text{ are independent.}$$

To prove this, we observe that $f(S_1, \ldots, S_n) := (\sum_{m=1}^{n} S_m)^{q_0} \prod_{m=1}^{k} \tilde{S}_m^{q_m}$ is homogeneous with degree q_0, resulting in

$$\mathbf{E}(f(S_1, \ldots, S_n)) = \frac{2^{-\theta_{n+1}}}{(\sum_{m=1}^{n+1} \theta_m)_{q_0}} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + q_0; \frac{1}{2}\right)$$

$$\times\, \mathbf{E}\left[\left(\sum_{m=1}^{n} X_m\right)^{q_0} \prod_{m=1}^{k} \tilde{X}_m^{q_m}\right]$$

$$= \frac{2^{-\theta_{n+1}} (\sum_{m=1}^{n} \theta_m)_{q_0}}{(\sum_{m=1}^{n+1} \theta_m)_{q_0}} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + q_0; \frac{1}{2}\right)$$

$$\times \frac{\prod_{m=1}^{k} (\theta_m)_{q_m}}{(\sum_{m=1}^{n} \theta_m)_{\sum_{m=1}^{k} q_m}}$$

because $\tilde{X}_m := X_m / \sum_{l=1}^{n} X_l$, $X_m \overset{d}{\sim} \gamma_{\theta_m}$, $m = 1, \dots, k$ are independent of $\sum_{m=1}^{n} X_m$ and $(\tilde{X}_m; m = 1, \dots, k) \overset{d}{\sim} D_k(\theta_1, \dots, \theta_k, \sum_{m=k+1}^{n} \theta_m)$.

• The distribution of the partition function $\sum_{m=1}^{n} S_m^\alpha$ is sometimes of interest. In particular, its mean value $\mathbf{E}(\sum_{m=1}^{n} S_m^\alpha)$, as well as its full moment function $\mathbf{E}((\sum_{m=1}^{n} S_m^\alpha)^\lambda)$ with $\lambda \in \mathbb{N}$, is worth being considered. For general partitions, these quantities are hardly computable. Nevertheless, when considering the Dirichlet type 3 distribution, significant simplifications are expected since the spacing functional

$$f(S_1, \dots, S_n) = \left(\sum_{m=1}^{n} S_m^\alpha\right)^\lambda,$$

which corresponds to quasi-arithmetic or Kolmogorov–Nagumo mean for $\alpha = 1/\lambda$, is homogeneous of degree $d = \alpha\lambda$ and so,

$$\mathbf{E}\left(\left(\sum_{m=1}^{n} S_m^\alpha\right)^\lambda\right)$$

$$= \frac{2^{-\theta_{n+1}}}{(\sum_{m=1}^{n+1} \theta_m)_{\alpha\lambda}} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + \alpha\lambda; \frac{1}{2}\right) \mathbf{E}\left(\left(\sum_{m=1}^{n} X_m^\alpha\right)^\lambda\right).$$

Since

$$\mathbf{E}\left(\left(\sum_{m=1}^{n} X_m^\alpha\right)^\lambda\right) = \sum_{\substack{\lambda_1, \dots, \lambda_n \geq 0 \\ \sum_{m=1}^{n} \lambda_m = \lambda}} \frac{\lambda!}{\prod_{m=1}^{n} \lambda_m!} \mathbf{E}\left(\prod_{m=1}^{n} X_m^{\alpha\lambda_m}\right),$$

we have

$$\mathbf{E}\left(\left(\sum_{m=1}^{n} S_m^\alpha\right)^\lambda\right) = \frac{2^{-\theta_{n+1}}}{(\sum_{m=1}^{n+1} \theta_m)_{\alpha\lambda}} {}_2F_1\left(\theta_{n+1}, \sum_{m=1}^{n+1} \theta_m; \sum_{m=1}^{n+1} \theta_m + \alpha\lambda; \frac{1}{2}\right)$$

$$\times \sum_{\substack{\lambda_1, \dots, \lambda_n \geq 0 \\ \sum_{m=1}^{n} \lambda_m = \lambda}} \frac{\lambda!}{\prod_{m=1}^{n} \lambda_m!} \prod_{m=1}^{n} (\theta_m)_{\alpha\lambda_m}. \tag{14}$$

3. The Gibbs version of the Dirichlet type 3 distribution including selection

In this section, we shall consider that $\mathbf{S}_n = (S_1, \dots, S_n)$ is distributed according to the Dirichlet type 3 distribution in the symmetric case (i.e., when $\theta_1 = \cdots = \theta_n = \theta_{n+1} = \theta$) on the simplex Λ_n, that is to say,

$$\mathbf{S}_n \overset{d}{\sim} \mu(\mathrm{d}\mathbf{s}_n) = \frac{2^{n\theta}\Gamma((n+1)\theta)}{\Gamma(\theta)^{n+1}} \left(1 - \sum_{m=1}^{n} s_m\right)^{\theta-1} \times \left(1 + \sum_{m=1}^{n} s_m\right)^{-(n+1)\theta} \cdot \prod_{m=1}^{n} (s_m^{\theta-1}\, \mathrm{d}s_m).$$

Let $\sigma \in \mathbb{R}$ be a "selection" parameter. For $\alpha > 1$, consider the Dirichlet type 3 distribution for $\mathbf{S}_{n,\sigma} := (S_{1,\sigma}, \dots, S_{n,\sigma})$ on the simplex Λ_n with selection, namely,

$$\mathbf{S}_{n,\sigma} \overset{d}{\sim} \mu_\sigma(\mathrm{d}\mathbf{s}_n) = \frac{e^{-\sigma\phi_\alpha(\mathbf{s}_n)}}{Z_n(\sigma)} \mu(\mathrm{d}\mathbf{s}_n), \tag{15}$$

where $\phi_\alpha(\mathbf{s}_n) := \sum_{m=1}^{n} s_m^\alpha$ and

$$Z_n(\sigma) := \mathbf{E}(e^{-\sigma\phi_\alpha(\mathbf{S}_n)}) = \int_{\Lambda_n} e^{-\sigma\sum_{m=1}^{n} s_m^\alpha} \mu(\mathrm{d}\mathbf{s}_n), \qquad \text{with } Z_n(0) = 1$$

is the partition function of the Gibbs measure μ_σ with $\mu_0 = \mu$. Moreover, for all non negative measurable function h on the simplex Λ_n, one can check that

$$\mathbf{E}_\sigma(h(\mathbf{S}_{n,\sigma})) = \int_{\Lambda_n} h(\mathbf{s}_n)\mu_\sigma(\mathrm{d}\mathbf{s}_n) = \frac{1}{Z_n(\sigma)} \mathbf{E}(e^{-\sigma\phi_\alpha(\mathbf{S}_n)} h(\mathbf{S}_n)).$$

The function $\phi_\alpha(\mathbf{s}_n) := \sum_{m=1}^{n} s_m^\alpha$, $\alpha > 0$ that appears in this part is of great interest in the population genetics, and it is called population homozygosity (the case $\alpha = 2$ is mostly considered). For more details, see for example [5], [7], [8], [11] and [14].

Let us now investigate some properties of the Dirichlet type 3 distribution with selection model as defined by Eq. (15). We see that

$$\frac{\mathrm{d}\mu_\sigma}{\mathrm{d}\mu}(\mathbf{s}_n) = \frac{e^{-\sigma\phi_\alpha(\mathbf{s}_n)}}{Z_n(\sigma)}$$

is the Radon-Nykodym derivative (or likelihood ratio) for the measure under selection with respect to the measure under neutrality.

Next, the log-likelihood ratio under selection

$$\mathscr{K}_\sigma(\mathbf{S}_{n,\sigma}) := -\log \mathrm{d}\mu_\sigma/\mathrm{d}\mu(\mathbf{S}_{n,\sigma})$$

reads

$$\mathscr{K}_\sigma(\mathbf{S}_{n,\sigma}) = \sigma\phi_\alpha(\mathbf{S}_{n,\sigma}) - F_n(\sigma),$$

where $F_n(\sigma) := -\log Z_n(\sigma)$ is pressure. In addition, the Shannon entropy of Dirichlet type 3 distribution including selection $\mathbf{E}_\sigma(\mathscr{K}_\sigma(\mathbf{S}_{n,\sigma}))$ is expressed by

$$\mathbf{E}_\sigma(\mathscr{K}_\sigma(\mathbf{S}_{n,\sigma})) = \sigma\mathbf{E}_\sigma(\phi_\alpha(\mathbf{S}_{n,\sigma})) - F_n(\sigma) = \sigma F_n'(\sigma) - F_n(\sigma).$$

Now, if we take the log-likelihood ratio under the neutral model $(\sigma = 0)$

$$\mathscr{K}_\sigma(\mathbf{S}_n) := -\log \mathrm{d}\mu_\sigma/\mathrm{d}\mu(\mathbf{S}_n),$$

then we get

PROPOSITION 3. *The expectation of* $\mathscr{K}_\sigma(\mathbf{S}_n)$ *is given by*

$$\mathbf{E}(\mathscr{K}_\sigma(\mathbf{S}_n)) = \frac{2^{-\theta}\sigma(\theta)_\alpha^n}{((n+1)\theta)_\alpha}{}_2F_1\left(\theta, (n+1)\theta; (n+1)\theta+\alpha; \frac{1}{2}\right) - F_n(\sigma).$$

PROOF. We have

$$\mathbf{E}(\mathscr{K}_\sigma(\mathbf{S}_n)) = \sigma\mathbf{E}\left(\sum_{m=1}^{n} S_m^\alpha\right) - F_n(\sigma).$$

Using Eq. (14) for $\theta_m = \theta_{n+1} = \theta$ and for $\lambda = 1$, the result follows. □

The Bell polynomials (see [3], pages 144–147, Tome 1) in the variables $(x_1, x_2, \ldots)$ are defined by

$$B_{\lambda,l}(x_1, x_2, \ldots) = \sum \frac{\lambda!}{\prod_{i=1}^{\lambda} i!^{a_i} a_i!} \prod_{i=1}^{\lambda} x_i^{a_i},$$

where the summation runs over the integers $a_i \geq 0$, $i = 1, \ldots, \lambda$ satisfying $\sum_{i=1}^{\lambda} i a_i = \lambda$ and $\sum_{i=1}^{\lambda} a_i = l$.

We now would like to compute the partition function $Z_n(\sigma)$.

THEOREM 4. *Let* $(\theta)_{\alpha\bullet} := (\theta)_\alpha, (\theta)_{2\alpha}, \ldots, (\theta)_{n\alpha}, \ldots$ *Then, it follows that*

$$Z_n(\sigma) = 1 + \sum_{l\geq 1}(n)_l \sum_{\lambda\geq l} \frac{(-\sigma)^\lambda}{\lambda!} \frac{2^{-\theta} B_{\lambda,l}((\theta)_{\alpha\bullet})}{((n+1)\theta)_{\alpha\lambda}} \times {}_2F_1\left(\theta, (n+1)\theta; (n+1)\theta + \alpha\lambda; \frac{1}{2}\right),$$

where $B_{\lambda,l}((\theta)_{\alpha\bullet})$ *is a Bell polynomial in the variable* $(\theta)_{\alpha\bullet}$.

PROOF. By expanding the exponential function, we find that

$$Z_n(\sigma) = \mathbf{E}(e^{-\sigma\sum_{m=1}^n S_m^\alpha}) = 1 + \sum_{\lambda\geq 1}\frac{(-\sigma)^\lambda}{\lambda!}\mathbf{E}\left(\left(\sum_{m=1}^n S_m^\alpha\right)^\lambda\right).$$

Since $(\sum_{m=1}^n S_m^\alpha)^\lambda$ is homogeneous with degree $d = \alpha\lambda$, we use Eq. (14) (with $\theta_m = \theta_{n+1} = \theta$), to obtain

$$\mathbf{E}\left(\left(\sum_{m=1}^n S_m^\alpha\right)^\lambda\right) = \frac{2^{-\theta}P_{\lambda,n}((\theta)_{\alpha\bullet})}{((n+1)\theta)_{\alpha\lambda}}\,{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+\alpha\lambda;\frac{1}{2}\right),$$

where

$$P_{\lambda,n}((\theta)_{\alpha\bullet}) := \sum_{\substack{\lambda_1,\ldots,\lambda_n\geq 0\\ \sum_{m=1}^n\lambda_m=\lambda}}\frac{\lambda!}{\prod_{m=1}^n\lambda_m!}\prod_{m=1}^n(\theta)_{\alpha\lambda_m}$$

is a potential polynomial. One can check that

$$\left(1+\sum_{\lambda\geq 1}(\theta)_{\alpha\lambda}\frac{t^\lambda}{\lambda!}\right)^n = 1 + \sum_{\lambda\geq 1}P_{\lambda,n}((\theta)_{\alpha\bullet})\frac{t^\lambda}{\lambda!},$$

which identifies $P_{\lambda,n}((\theta)_{\alpha\bullet})$ to a potential polynomial in the variables $(\theta)_{\alpha\bullet}$.

As a consequence of the Faa di Bruno formula (see [3], page 152), with monomials x_i to be taken with $x_i = (\theta)_{i\alpha}$, we have

$$P_{\lambda,n}((\theta)_{\alpha\bullet}) = \sum_{l=1}^{\lambda}(n)_l B_{\lambda,l}((\theta)_{\alpha\bullet}).$$

Thus,

$$Z_n(\sigma) = 1 + \sum_{l\geq 1}(n)_l\sum_{\lambda\geq l}\frac{(-\sigma)^\lambda}{\lambda!}\frac{2^{-\theta}B_{\lambda,l}((\theta)_{\alpha\bullet})}{((n+1)\theta)_{\alpha\lambda}}$$
$$\times\,{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+\alpha\lambda;\frac{1}{2}\right). \qquad \square$$

Finally, we shed light on the moment function $\mathbf{E}_\sigma[(\prod_{m=1}^n S_{m,\sigma}^{1/n})^q]$ of the geometrical average of $\mathbf{S}_{n,\sigma} := (S_{1,\sigma},\ldots,S_{n,\sigma})$. One gets

THEOREM 5. *Let* $(\theta+q/n)_{\alpha\bullet} := (\theta+q/n)_\alpha, (\theta+q/n)_{2\alpha},\ldots,(\theta+q/n)_{n\alpha},\ldots.$ *The moment function of* $\prod_{m=1}^n S_{m,\sigma}^{1/n}$ *is expressed by*

$$\mathbf{E}_\sigma\left[\left(\prod_{m=1}^n S_{m,\sigma}^{1/n}\right)^q\right] = \frac{2^{-\theta}((\theta)_{q/n})^n}{Z_n(\sigma)((n+1)\theta)_q}\left[{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+q;\frac{1}{2}\right)\right.$$
$$+\sum_{l\geq 1}(n)_l\sum_{\lambda\geq l}\frac{(-\sigma)^\lambda}{\lambda!}\frac{\Gamma((n+1)\theta+q)}{\Gamma((n+1)\theta+\alpha\lambda+q)}$$
$$\left.\times\,{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+\alpha\lambda+q;\frac{1}{2}\right)\cdot B_{\lambda,l}((\theta+q/n)_{\alpha\bullet})\right],$$

where $Z_n(\sigma)$ *is given in Theorem* 4.

PROOF. We need to compute

$$\mathbf{E}_\sigma\left[\left(\prod_{m=1}^n S_{m,\sigma}^{1/n}\right)^q\right] = \frac{\mathbf{E}(e^{-\sigma\phi_\alpha(\mathbf{S}_n)}\prod_{m=1}^n S_m^{q/n})}{Z_n(\sigma)}.$$

Since the denominator is under control, it remains to examine the numerator $\mathbf{E}(e^{-\sigma\phi_\alpha(\mathbf{S}_n)}\prod_{m=1}^n S_m^{q/n})$. The function $f(S_1,\dots,S_n)=(\sum_{m=1}^n S_m^\alpha)^\lambda\prod_{m=1}^n S_m^{q/n}$ is homogeneous of degree $d=\alpha\lambda+q$ and the statement (ii) of the Theorem 3 for $\theta_m=\theta_{n+1}=\theta$ gives

$$\mathbf{E}\left(e^{-\sigma\phi_\alpha(\mathbf{S}_n)}\prod_{m=1}^n S_m^{q/n}\right)$$
$$=\mathbf{E}\left(\prod_{m=1}^n S_m^{q/n}\right)+\sum_{\lambda\geq 1}\frac{(-\sigma)^\lambda}{\lambda!}\frac{2^{-\theta}}{((n+1)\theta)_{\alpha\lambda+q}}$$
$$\times\,{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+\alpha\lambda+q;\frac{1}{2}\right)\mathbf{E}\left(\left(\sum_{m=1}^n X_m^\alpha\right)^\lambda\prod_{m=1}^n X_m^{q/n}\right),$$

where $(X_m;\, m=1,\dots,n)$ are independent random variables with $X_m \overset{d}{\sim} \gamma_\theta$. Thus,

$$\mathbf{E}\left(\left(\sum_{m=1}^n X_m^\alpha\right)^\lambda\prod_{m=1}^n X_m^{q/n}\right) = \sum_{\substack{\lambda_1,\dots,\lambda_n\geq 0\\ \sum_{m=1}^n\lambda_m=\lambda}}\frac{\lambda!}{\prod_{m=1}^n\lambda_m!}\mathbf{E}\left(\prod_{m=1}^n X_m^{\alpha\lambda_m+q/n}\right)$$
$$=\sum_{\substack{\lambda_1,\dots,\lambda_n\geq 0\\ \sum_{m=1}^n\lambda_m=\lambda}}\frac{\lambda!}{\prod_{m=1}^n\lambda_m!}\prod_{m=1}^n(\theta)_{\alpha\lambda_m+q/n}$$
$$=((\theta)_{q/n})^n\cdot P_{\lambda,n}((\theta+q/n)_{\alpha\bullet}),$$

where

$$P_{\lambda,n}((\theta+q/n)_{\alpha\bullet}) := \sum_{\substack{\lambda_1,\dots,\lambda_n\geq 0\\ \sum_{m=1}^{n}\lambda_m=\lambda}} \frac{\lambda!}{\prod_{m=1}^{n}\lambda_m!}\prod_{m=1}^{n}(\theta+q/n)_{\alpha\lambda_m},$$

and $(\theta+q/n)_{\alpha\bullet} := (\theta+q/n)_{\alpha}, (\theta+q/n)_{2\alpha}, \dots, (\theta+q/n)_{n\alpha}, \dots$.

Now, one can check that

$$\left(1+\sum_{\lambda\geq 1}(\theta+q/n)_{\alpha\lambda}t^{\lambda}/\lambda!\right)^{n} = 1+\sum_{\lambda\geq 1}P_{\lambda,n}((\theta+q/n)_{\alpha\bullet})t^{\lambda}/\lambda!,$$

identifying $P_{\lambda,n}((\theta+q/n)_{\alpha\bullet})$ to a potential polynomial in the variables $(\theta+q/n)_{\alpha\bullet}$.

Finally, we get

$$\begin{aligned}
&\mathbf{E}\left(e^{-\sigma\phi_\alpha(\mathbf{S}_n)}\prod_{m=1}^{n}S_m^{q/n}\right)\\
&\quad = \frac{2^{-\theta}((\theta)_{q/n})^{n}}{((n+1)\theta)_q}\Bigg[{}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+q;\frac{1}{2}\right)\\
&\quad + \sum_{l\geq 1}(n)_l\sum_{\lambda\geq l}\frac{(-\sigma)^{\lambda}}{\lambda!}\frac{\Gamma((n+1)\theta+q)}{\Gamma((n+1)\theta+\alpha\lambda+q)}\\
&\quad \times {}_2F_1\left(\theta,(n+1)\theta;(n+1)\theta+\alpha\lambda+q;\frac{1}{2}\right)\cdot B_{\lambda,l}((\theta+q/n)_{\alpha\bullet})\Bigg].
\end{aligned}$$

Normalizing by $Z_n(\sigma)$ whose expression is provided by Theorem 4, yields the desired result. □

4. Real data application

To corroborate the performance of the Dirichlet type 3 distribution, we present a concrete example. In many cases, biologists are interested in exploring proportions because of practical difficulties in measuring actual numbers. We consider the data of blood serum proportions (pre-albumin, albumin and glubulin) in 3-week-old white Pekin ducklings. We use 23 sets of data, such that each set corresponds to a different diet. The measurements of the blood serum proportions are displayed in [16].

Let X_m be a random variable whose value is proportional to the particular type of blood serum level. Then, $S_m = X_m/(\sum_{l=1}^{2}X_l + 2X_3)$ corresponds to the proportions. Besides, we suppose that X_m can be thought of as being inde-

Table 1. Estimated values and variance-covariance matrix of the estimated parameters.

Estimated values of θ_1, θ_2 and θ_3.

$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
3.6112	23.2517	12.6186

Variance-covariance matrix of the estimated parameters.

	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
$\hat{\theta}_1$	0.5796		
$\hat{\theta}_2$	3.2148	24.3069	
$\hat{\theta}_3$	1.7131	12.4250	7.1480

pendent of each other and $\sum_{l=1}^{2} X_l + 2X_3$. Therefore, the vector $\mathbf{S}_2 = (S_1, S_2)$ would follow the Dirichlet type 3 distribution with parameters θ_1, θ_2 and θ_3. These unknown parameters are estimated by using the maximum likelihood method, and by implementing Fisher scoring method (see [18] and [22]). The estimated values of parameters and their corresponding variance-covariance matrix are outlined in Table 1.

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