Estimates for eigenvalues of weighted Laplacian and weighted p -Laplacian

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ABSTRACT. In this paper, we study two eigenvalue problems of the weighted Laplacian and get the Reilly-type bounds and isoperimetric type bounds for the first nonzero n eigenvalues on hypersurfaces of the Euclidean space. Besides, we give lower bounds for the first eigenvalue of weighted p -Laplacian on submanifolds with locally bounded weighted mean curvature. Meanwhile, several applications of these estimates have also been given.

1. Introduction

The triple $(M, \langle, \rangle, e^{-f} dv)$ is called a smooth metric measure space (SMMS for short), where $(M, \langle, \rangle)$ is an n -dimensional Riemannian manifold with the metric $\langle, \rangle$, f is a smooth real-valued function defined on M , and dv is the Riemannian volume element related to $\langle, \rangle$. On this SMMS, the drifting Laplacian, defined as $\Delta_f := \Delta - \langle \nabla f, \nabla \rangle$, is also called the weighted Laplacian (or the Witten Laplacian). For a given compact SMMS $(M, \langle, \rangle, e^{-f} dv)$ without boundary, one can consider the following closed eigenvalue problem of the weighted Laplacian

$$\Delta_f u + \lambda u = 0, \quad \text{in } M.$$

It is well-known that, in this setting, Δ_f only has discrete spectrum whose elements (also called *eigenvalues*) can be listed non-decreasingly as follows:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty.$$

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Clearly, the first eigenvalue λ_0 has multiplicity one and constant eigenfunction. In recent years, there are some interesting results concerning eigenvalue estimates of the drifting Laplacian and the bi-drifting Laplacian—see, e.g., [9, 11, 12, 15, 18, 21, 22, 23, 30].

When $(M, \langle, \rangle)$ is immersed into the Euclidean N -space $(\mathbb{R}^N, \text{can})$ with the canonical metric can , one can define the weighted mean curvature vector as $\mathbf{H}_f = n\mathbf{H} + (\bar{\nabla}f)^\perp$, where $\mathbf{H}$ is the mean curvature vector of M in $\mathbb{R}^N$, $(\bar{\nabla}f)^\perp$ is the projection of $\bar{\nabla}f$ onto the normal bundle $T^\perp M$ (see also Definition 4 for the precise statement). Wei and Wylie [28] have shown the comparison theorem for the weighted mean curvature. A compact SMMS $(M, \langle, \rangle, e^{-f} dv)$ is said to have the Gaussian density, if $f(x) = a + \frac{c|x|^2}{2}$, where a, c are two constants, and x is the position vector of M in $\mathbb{R}^N$. On the other hand, when $f(x) = a + \frac{c|x|^2}{2}$, it is well known that the one-parameter family of immersions $X(\cdot, t) : M \times [0, T) \hookrightarrow \mathbb{R}^N$ satisfying $\mathbf{H}_f = 0$, that is, $\mathbf{H} = cX^\perp$ is the self-similar solution of the mean curvature flow (MCF for short). When $c < 0$, $X_t(M) = X(M, t)$, called shrinkers, contract to a point under the MCF; when $c = 0$, $X_t(M)$ are the minimal submanifolds; when $c > 0$, $X_t(M)$, called expanders, enlarge under the MCF.

Batista, Cavalcante and Pyo [5] got a Reilly-type inequality for the first non-zero eigenvalue of the drifting Laplacian Δ_f as follows

$$\lambda_1 \leq \frac{\int_M |\mathbf{H}_f - \bar{\nabla}f|^2 d\mu}{n|M|_f} = \frac{\int_M (n^2|\mathbf{H}|^2 + |\nabla f|^2) d\mu}{n|M|_f}, \quad (1)$$

where ∇ is the gradient on M , that is the projection on TM of $\bar{\nabla}$, $|M|_f = \int_M e^{-f} dv$ is the weighted volume of M , and $d\mu = e^{-f} dv$. This is a generalization of the classical Reilly's inequality for the Laplacian (cf. [25]). Domingo-Juan and Miquel [17] gave a remark to discuss the equality case of (1) by using the MCF. Recently, Roth [26] generalized (1) to a Reilly-type inequality for the first non-zero eigenvalue of the prescribed weighted divergence-type operator. Inspired by these work, in this paper, we try to get several Reilly-type inequalities for the weighted Laplacian's different eigenvalue problems. First, we can prove the following:

THEOREM 1. *Let $(M, \langle, \rangle)$ be an n -dimensional ($n \geq 2$) compact submanifold in $\mathbb{R}^{n+k}$ endowed with a density $e^{-f} dv$. Assume that λ_i is the i -th closed eigenvalue of the drifting Laplacian acting on functions on M . Then we have*

$$\frac{1}{n} \sum_{i=1}^n \lambda_i^{1/2} \leq \left(\frac{\int_M |\mathbf{H}_f - \bar{\nabla}f|^2 d\mu}{n|M|_f} \right)^{1/2} = \left(\frac{\int_M (n^2|\mathbf{H}|^2 + |\nabla f|^2) d\mu}{n|M|_f} \right)^{1/2}, \quad (2)$$

where $\mathbf{H}_f$, $\mathbf{H}$ are the weighted mean curvature vector and mean curvature vector of M in $\mathbb{R}^{n+k}$, respectively, and $|M|_f$ is the weighted volume of M . If the equality in (2) holds, then M is a self-shrinker of the MCF and there is a point $p \in \mathbb{R}^{n+k}$ such that $f = a - \frac{\lambda_1}{2} r_p^2$, where r_p is the Euclidean distance to the center of mass p of M . In particular, if $k = 1$, $|\mathbf{H}| > 0$ or if $n = 2$, $k = 1$, and M is embedded, has genus 0, then M is a hypersphere.

REMARK 1. Since $\sum_{i=1}^n \lambda_i^{1/2} \geq n \lambda_1^{1/2}$, then (1) can be obtained directly from (2).

Consider the Wentzell problem of the drifting Laplacian on a compact SMMS $(\Omega, \langle, \rangle, e^{-f} dv)$, with boundary $\partial\Omega$, as follows:

$$\begin{cases} \Delta_f u = 0 & \text{in } \Omega, \\ -\beta \bar{\Delta}_f u + \partial_v u = \lambda u & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where β is a real number, and ∂_v denotes the outward unit normal derivative on the boundary. Clearly, if $f = \text{const.}$ is a constant function, then (3) degenerates into the usual Wentzell problem for the Laplacian, and recently some interesting eigenvalue estimates of this problem have been obtained—see, e.g., [8, 14, 16]. If $\beta = 0$, then (3) becomes the Steklov eigenvalue problem of the weighted Laplacian, and also some recent eigenvalue estimates of this problem can be found—see, e.g., [6, 19].

When $\beta \geq 0$, $\bar{\Delta}_f$ in (3) only has discrete spectrum and all the elements (i.e., eigenvalues) can be listed non-decreasingly as follows:

$$\lambda_{0,\beta} = 0 < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \cdots \uparrow +\infty,$$

with corresponding real orthonormal, in the sense of $L^2(\partial\Omega)$, eigenfunctions $u_0, u_1, u_2, \dots$, where each eigenvalue is repeated according to its multiplicity. Consider the Hilbert space

$$H(\Omega) = \{u \in H^1(\Omega), \text{Tr}_{\partial\Omega}(u) \in H^1(\partial\Omega)\},$$

where $\text{Tr}_{\partial\Omega}$ is the trace operator. Define on $H(\Omega)$ two bilinear forms as follows

$$A_\beta(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\mu + \beta \int_{\partial\Omega} \bar{\nabla} u \cdot \bar{\nabla} v \, d\bar{\mu}, \quad B(u, v) = \int_{\partial\Omega} uv \, d\bar{\mu}, \quad (4)$$

where ∇ and $\bar{\nabla}$ are the gradient operators on Ω and $\partial\Omega$, respectively, and $d\bar{\mu}$ is the restriction of $d\mu$ on the boundary $\partial\Omega$ (i.e., the induced volume density of $\partial\Omega$). Since $\beta \geq 0$, these two bilinear forms are positive and the variational

characterization for the k -th eigenvalue $\lambda_{k,\beta}$ is given by

$$\lambda_{k,\beta} = \min \left\{ \frac{A_\beta(u, u)}{B(u, u)} \mid u \in H(\Omega), u \neq 0, \int_{\partial\Omega} uu_i d\tilde{\mu} = 0, i = 0, \dots, k-1 \right\}. \quad (5)$$

Clearly, when $k = 1$, the minimum of $A_\beta(u, u)/B(u, u)$ is taken over the space of functions orthogonal to constant functions in the sense of $L^2(\partial\Omega)$, i.e., $\int_{\partial\Omega} u d\tilde{\mu} = 0$. We can prove the following isoperimetric type lower bound for the first non-zero n eigenvalues $\lambda_{i,\beta}$, $i = 1, 2, \dots, n$.

THEOREM 2. *Let Ω be an open bounded connected domain, with smooth boundary $\partial\Omega$, in $\mathbb{R}^n$, and let $\lambda_{i,\beta}$ be the i -th eigenvalue of (3) for $\beta \geq 0$. Then we have*

$$\sum_{i=1}^n \frac{|\Omega|_f}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial\Omega|_f}{\lambda_{i,\beta}} \geq \frac{|\partial\Omega|_f^2}{\int_{\partial\Omega} |\mathbf{H}_f - \bar{\nabla}f|^2 d\tilde{\mu}} = \frac{|\partial\Omega|_f^2}{\int_{\partial\Omega} (n^2 |\mathbf{H}|^2 + |\nabla f|^2) d\tilde{\mu}}, \quad (6)$$

where $\mathbf{H}_f$, $\mathbf{H}$ are the weighted mean curvature vector and mean curvature vector of $\partial\Omega$ in $\mathbb{R}^n$, respectively, and $|\Omega|_f$, $|\partial\Omega|_f$ denote the weighted volume of Ω and the weighted area of $\partial\Omega$, respectively. The equality in (6) holds if and only if $f = \text{const.}$ is a constant function and $\partial\Omega$ is a hypersphere.

For an open bounded connected domain Ω on a given complete SMMS $(M, \langle \cdot, \cdot \rangle, e^{-f} dv)$, the so-called weighted p -Laplace ($1 < p < \infty$) operator $\Delta_{p,f}$ on Ω is given by

$$\Delta_{p,f} u = \text{div}_f(|\nabla u|^{p-2} \nabla u),$$

where $\text{div}_f(\cdot) = e^f \text{div}(e^{-f} \cdot)$ is the weighted divergence. Clearly, if $f = \text{const.}$ is a constant function, then $\Delta_{p,f}$ becomes the usual p -Laplacian $\Delta_p = \text{div}(|\nabla u|^{p-2} \nabla u)$. For the weighted p -Laplacian, one can consider the following Dirichlet problem

$$\begin{cases} \Delta_{p,f} u + \lambda |u|^{p-2} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Similar to the p -Laplacian case, since Ω is bounded, it is not hard to know that $\Delta_{p,f}$ in (7) has discrete spectrum whose elements (i.e., eigenvalues) can be listed non-decreasingly as follows

$$0 < \lambda_{1,p}^f < \lambda_{2,p}^f \leq \lambda_{3,p}^f \leq \dots \uparrow \infty.$$

The strict positivity of the lowest eigenvalue $\lambda_{1,p}^f$ can be attained by using the maximum principle of quasi-linear second-order elliptic PDEs and the Dirichlet boundary condition. Similar to the argument of Courant's nodal domain

theorem for the Laplacian (see, e.g., [7, pp. 19–20]), one knows that $\lambda_{1,p}^f$ has multiplicity one and is strictly less than the second Dirichlet eigenvalue $\lambda_{2,p}^f$. Besides, using the variational method, one can obtain that $\lambda_{1,p}^f$ can be characterized as follows

$$\lambda_{1,p}^f(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p d\mu}{\int_{\Omega} |u|^p d\mu} \mid u \in W_0^{1,p}(\Omega), u \neq 0 \right\}, \quad (8)$$

where $W_0^{1,p}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)$, i.e., smooth functions on Ω with compact support, under the Sobolev norm

$$\|u\|_{1,p}^{d\mu} := \left(\int_{\Omega} |u|^p d\mu + \int_{\Omega} |\nabla u|^p d\mu \right)^{1/p}.$$

Not only Dirichlet eigenvalue problem (7) can be considered for $\Delta_{p,f}$ but also the Neumann version can also be investigated. In fact, there exist some estimates for Neumann eigenvalues of the weighted p -Laplacian on bounded domains—see, e.g., [27].

Similar to the case of the p -Laplacian, by applying the Max-min principle, we can obtain the following result.

LEMMA 1. *Domain monotonicity of eigenvalues with vanishing Dirichlet data also holds for the Dirichlet eigenvalues of the weighted p -Laplacian on a complete SMMS $(M, \langle, \rangle, e^{-f} dv)$.*

PROOF. This conclusion can be obtained by replacing the density dv by the weighted density $e^{-f} dv = d\mu$ in the proof of [13, Lemma 1.1].

For a complete noncompact SMMS $(M, \langle, \rangle, e^{-f} dv)$, let $B_M(q, r)$ be the geodesic ball, with center q and radius r , on M . By Lemma 1, we know that $\lambda_{1,p}^f(B_M(q, r))$ decreases as r increases and then it has a limit independent of the choice of the center q . Then the first eigenvalue of the weighted p -Laplacian $\Delta_{p,f}$ on M can be defined by

$$\lambda_{1,p}^f(M) := \lim_{r \rightarrow \infty} \lambda_{1,p}^f(B_M(q, r)). \quad (9)$$

At the end of this paper, we will give several bounds for the first eigenvalue of the weighted p -Laplacian—see Section 3 for details.

2. Reilly-type and isoperimetric type estimates for eigenvalues of the weighted Laplacian

Using some ideas of [14], we can give the proof of Theorem 1 as follows.

PROOF (Proof of Theorem 1). Let $\{u_i\}_{i=0}^{+\infty}$ be the normalized eigenfunctions corresponding to the eigenvalues λ_i , $i \in \mathbb{N} \cup \{0\}$, of the drifting Laplacian Δ_f on M , that is,

$$\Delta u_i = -\lambda_i u_i, \quad \int_M u_i u_j d\mu = \delta_{ij}. \quad (10)$$

We have $u_0 = 1/\sqrt{|M|_f}$ and for each $i \in \mathbb{N}$,

$$\lambda_i = \min_{u \neq 0, \int_M u u_j d\mu = 0, j=0, \dots, i-1} \frac{\int_M |\nabla u|^2 d\mu}{\int_M u^2 d\mu}. \quad (11)$$

Let $x_1, \dots, x_{n+k}$ be the coordinate functions on $\mathbb{R}^{n+k}$. Replacing if necessary x_i by $x_i - \frac{\int_M x_i d\mu}{|M|_f}$, we can assume

$$\int_M x_i d\mu = 0, \quad i = 1, \dots, n+k. \quad (12)$$

Next, we would like to show that a suitable rotation of axes can be made such that

$$\int_M x_j u_i d\mu = 0, \quad (13)$$

for $j = 2, 3, \dots, n+k$ and $i = 1, \dots, j-1$. To see this, we define an $(n+k) \times (n+k)$ matrix $Q = (q_{ji})$, where $q_{ji} = \int_M x_j u_i$, for $i, j = 1, 2, \dots, n+k$. Using the orthogonalization of Gram and Schmidt (i.e., QR-factorization theorem), there exist an upper triangle matrix $T = (T_{ji})$ and an orthogonal matrix $U = (a_{ji})$ such that $T = UQ$, i.e.,

$$T_{ji} = \sum_{k=1}^{n+k} a_{jk} q_{ki} = \int_M \sum_{k=1}^{n+k} a_{jk} x_k u_i d\mu = 0, \quad 1 \leq i < j \leq n+k.$$

Letting $y_j = \sum_{k=1}^{n+k} a_{jk} x_k$, we get

$$\int_M y_j u_i d\mu = \int_M \sum_{k=1}^{n+k} a_{jk} x_k u_i d\mu = 0, \quad 1 \leq i < j \leq n+k. \quad (14)$$

Since U is an orthogonal matrix, $y_1, y_2, \dots, y_{n+k}$ are also coordinate functions on $\mathbb{R}^{n+k}$. Therefore, denote these coordinate functions still by $x_1, x_2, \dots, x_{n+k}$, and then (13) can be obtained directly from (14).

So, we have

$$\lambda_i \int_M x_i^2 d\mu \leq \int_M |\nabla x_i|^2 d\mu, \quad i = 1, \dots, n+k, \quad (15)$$

with equality if and only if

$$A_f x_i = -\lambda_i x_i.$$

Since

$$|\nabla x_i|^2 \leq 1, \quad \sum_{i=1}^{n+k} |\nabla x_i|^2 = n,$$

we have

$$\begin{aligned} \sum_{i=1}^{n+k} \lambda_i^{1/2} |\nabla x_i|^2 &= \sum_{i=1}^n \lambda_i^{1/2} |\nabla x_i|^2 + \sum_{j=1}^k \lambda_{n+j}^{1/2} |\nabla x_{n+j}|^2 \\ &\geq \sum_{i=1}^n \lambda_i^{1/2} |\nabla x_i|^2 + \lambda_n^{1/2} \left(n - \sum_{i=1}^n |\nabla x_i|^2 \right) \\ &\geq \sum_{i=1}^n \lambda_i^{1/2} |\nabla x_i|^2 + \sum_{i=1}^n \lambda_i^{1/2} (1 - |\nabla x_i|^2) \\ &= \sum_{i=1}^n \lambda_i^{1/2}, \end{aligned} \tag{16}$$

which gives

$$\sum_{i=1}^{n+k} \lambda_i^{1/2} \int_M |\nabla x_i|^2 d\mu \geq |M|_f \sum_{i=1}^n \lambda_i^{1/2}. \tag{17}$$

For any positive constant δ , since $\sum_{i=1}^{n+k} |A_f x_i|^2 = |\mathbf{H}_f - \bar{\nabla} f|^2$, we have from the Schwarz inequality and (15) that

$$\begin{aligned} \sum_{i=1}^{n+k} \lambda_i^{1/2} \int_M |\nabla x_i|^2 d\mu &= \sum_{i=1}^{n+k} \lambda_i^{1/2} \int_M (-x_i A_f x_i) d\mu \\ &\leq \frac{1}{2} \sum_{i=1}^{n+k} \left\{ \delta \lambda_i \int_M x_i^2 d\mu + \frac{1}{\delta} \int_M (A_f x_i)^2 d\mu \right\} \\ &\leq \frac{1}{2} \left\{ \delta n |M|_f + \frac{1}{\delta} \int_M |\mathbf{H}_f - \bar{\nabla} f|^2 d\mu \right\}. \end{aligned} \tag{18}$$

Taking

$$\delta = \left\{ \frac{\int_M |\mathbf{H}_f - \bar{\nabla} f|^2 d\mu}{n |M|_f} \right\}^{1/2}$$

in (18) and using (17), we get (2).

If the equality holds in (2), then inequalities (15) and (16) become equalities, which means that $\Delta_f x_i = -\lambda_i x_i$, $i = 1, \dots, n+k$ and $\lambda_1 = \lambda_2 = \dots = \lambda_{n+k}$. Hence, we have

$$\Delta_f x = (\Delta_f x_1, \dots, \Delta_f x_{n+k}) = -\lambda_1(x_1, \dots, x_{n+k}) = -\lambda_1 x. \quad (19)$$

On the other hand, it holds that

$$\begin{aligned} \Delta_f x &= (\Delta_f x_1, \dots, \Delta_f x_{n+k}) \\ &= (\Delta x_1, \dots, \Delta x_{n+k}) - (\langle \nabla f, \nabla x_1 \rangle, \dots, \langle \nabla f, \nabla x_{n+k} \rangle) \\ &= n\mathbf{H} - \nabla f, \end{aligned} \quad (20)$$

where $\mathbf{H}$ is the mean curvature vector of M in $\mathbb{R}^{n+k}$. Combining (19) and (20), we have $n\mathbf{H} - \nabla f = -\lambda_1 x$, which implies that $n\mathbf{H} = -\lambda_1 x^\perp$ and $\nabla f = \lambda_1 x^\top$. Here $x^\perp$ and $x^\top$ denote the normal part and tangential part of x , respectively. From $n\mathbf{H} = -\lambda_1 x^\perp$ and $\lambda_1 > 0$, we know that M is a self-shrinker of the MCF in $\mathbb{R}^{n+k}$. Since $x^\top = \frac{1}{2}\nabla|x|^2$, we infer from $\nabla f = \lambda_1 x^\top$ that $\nabla\left(f + \frac{\lambda_1}{2}|x|^2\right) = 0$. Therefore, $f|_M = a - \frac{\lambda_1}{2}|x|^2$ for a constant a . In particular, if $k = 1$, $\mathbf{H} > 0$ or if $n = 2$, $k = 1$, and M is embedded, has genus 0, we see from [4, 20] that M has to be a hypersphere. This completes the proof of Theorem 1.

Now, we give the proof of Theorem 2 in detail.

PROOF (Proof of Theorem 2). Let u_i ($i \in \mathbb{N} \cup \{0\}$) be the orthonormal eigenfunctions, in the sense of $L^2(\partial\Omega)$ norm, corresponding to eigenvalues $0 = \lambda_{0,\beta} < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \dots$, of the problem (3), that is,

$$\begin{cases} \Delta_f u_i = 0 & \text{in } \Omega, \\ -\beta \bar{\Delta}_f u + \partial_\nu u_i = \lambda_i u_i & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_i u_j = \delta_{ij}. \end{cases}$$

The eigenvalues $\lambda_{i,\beta}$, $i = 1, 2, \dots$, are characterized by

$$\lambda_{i,\beta} = \min_{\substack{u \in H(\Omega) \setminus \{0\} \\ u \perp \text{span}\{u_0, \dots, u_{i-1}\}}} \frac{\int_{\Omega} |\nabla u|^2 d\mu + \beta \int_{\partial\Omega} |\bar{\nabla} u|^2 d\tilde{\mu}}{\int_{\partial\Omega} u^2 d\tilde{\mu}}. \quad (21)$$

As in the proof of Theorem 1, by suitably rotating axes, the n coordinate functions $x_1, \dots, x_n$ of $\mathbb{R}^n$ satisfy

$$\int_{\partial\Omega} x_j u_i d\tilde{\mu} = 0, \quad (22)$$

for $j = 2, 3, \dots, n$ and $i = 1, \dots, j-1$.

Then we infer from (22) that for each fixed $i = 1, \dots, n$,

$$\lambda_{i,\beta} \int_{\partial\Omega} x_i^2 d\tilde{\mu} \leq \int_{\Omega} |\nabla x_i|^2 d\tilde{\mu} + \beta \int_{\partial\Omega} |\bar{\nabla} x_i|^2 d\tilde{\mu} = |\Omega|_f + \beta \int_{\partial\Omega} |\nabla x_i|^2 d\tilde{\mu}, \quad (23)$$

with equality if and only if

$$\beta \Delta x_i + \partial_\nu x_i = -\lambda_{i,\beta} x_i, \quad \text{on } \partial\Omega. \quad (24)$$

Hence,

$$\int_{\partial\Omega} x_i^2 d\tilde{\mu} \leq \frac{|\Omega|_f}{\lambda_{i,\beta}} + \frac{\beta}{\lambda_{i,\beta}} \int_{\partial\Omega} |\nabla x_i|^2 d\tilde{\mu}. \quad (25)$$

Observing

$$\sum_{i=1}^n |\nabla x_i|^2 = n - 1, \quad |\nabla x_i|^2 \leq 1,$$

we get

$$\begin{aligned} \sum_{i=1}^n \frac{|\nabla x_i|^2}{\lambda_{i,\beta}} &= \sum_{i=1}^{n-1} \frac{|\nabla x_i|^2}{\lambda_{i,\beta}} + \frac{|\nabla x_n|^2}{\lambda_{n,\beta}} \\ &= \sum_{i=1}^{n-1} \frac{|\nabla x_i|^2}{\lambda_{i,\beta}} + \frac{1}{\lambda_{n,\beta}} \sum_{i=1}^{n-1} (1 - |\nabla x_i|^2) \\ &\leq \sum_{i=1}^{n-1} \frac{|\nabla x_i|^2}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{1}{\lambda_{i,\beta}} (1 - |\nabla x_i|^2) \\ &= \sum_{i=1}^{n-1} \frac{1}{\lambda_{i,\beta}}. \end{aligned} \quad (26)$$

Combining (25) and (26), we have

$$\sum_{i=1}^n \int_{\partial\Omega} x_i^2 d\tilde{\mu} \leq \sum_{i=1}^n \frac{|\Omega|_f}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial\Omega|_f}{\lambda_{i,\beta}}. \quad (27)$$

Multiplying both sides of (27) by $\int_{\partial\Omega} |(n-1)\mathbf{H} - \nabla f|^2 d\tilde{\mu}$, we have

$$\begin{aligned} &\int_{\partial\Omega} |(n-1)\mathbf{H} - \nabla f|^2 d\tilde{\mu} \int_{\partial\Omega} |x|^2 d\tilde{\mu} \\ &\leq \left(\sum_{i=1}^n \frac{|\Omega|_f}{\lambda_{i,\beta}} + \sum_{i=1}^{n-1} \frac{\beta |\partial\Omega|_f}{\lambda_{i,\beta}} \right) \int_{\partial\Omega} |(n-1)\mathbf{H} - \nabla f|^2 d\tilde{\mu}. \end{aligned} \quad (28)$$

On the other hand, by the weighted Hsiung-Minkowski inequality for the Δ_f and the Schwarz inequality, we have

$$\begin{aligned} \int_{\partial\Omega} |x|^2 d\tilde{\mu} \int_{\partial\Omega} |(n-1)\mathbf{H} - \nabla f|^2 d\tilde{\mu} &\geq \left(\int_{\partial\Omega} -\langle x, (n-1)\mathbf{H} - \nabla f \rangle d\tilde{\mu} \right)^2 \\ &= |\partial\Omega|_f^2. \end{aligned} \quad (29)$$

The equality in (29) holds if and only if $(n-1)\mathbf{H} - \nabla f = Cx$ is valid on $\partial\Omega$ for a non-zero constant C . Clearly, substituting (28) into (26), one gets (6).

If the equality in (6) holds, then the inequality in (26) must take equality sign and (24) holds. It is easy to check from the equality case of (26) that at any point of $\partial\Omega$, one has

$$\lambda_{1,\beta} = \lambda_{2,\beta} = \cdots = \lambda_{n,\beta}.$$

It then follows that the position vector $x = (x_1, \dots, x_n)$ when restricted on $\partial\Omega$ satisfies

$$\begin{aligned} \Delta_f x &:= (\Delta_f x_1, \dots, \Delta_f x_n) \\ &= -\frac{1}{\beta} v - \frac{\lambda_{1,\beta}}{\beta} (x_1, \dots, x_n). \end{aligned} \quad (30)$$

On the other hand, as (20), one can easily get

$$\Delta_f x = (n-1)\mathbf{H} - \nabla f, \quad (31)$$

where $\mathbf{H}$ is the mean curvature vector of $\partial\Omega$ in $\mathbb{R}^n$. Combining (30) and (31), we have

$$(n-1)\mathbf{H} - \nabla f = -\frac{1}{\beta} v - \frac{\lambda_{1,\beta}}{\beta} x. \quad (32)$$

Combining $(n-1)\mathbf{H} - \nabla f = Cx$, we have

$$\left(C + \frac{\lambda_{1,\beta}}{\beta} \right) x = -\frac{1}{\beta} v. \quad (33)$$

Since x is in the normal direction, consider the function $g = |x|^2 : M \rightarrow \mathbb{R}$, and it is easy to see from (32) that

$$Zg = 2\langle Z, x \rangle = 0, \quad \forall Z \in \mathfrak{X}(\partial\Omega),$$

where $\mathfrak{X}(\partial\Omega)$ stands for the set of tangent vector fields on $\partial\Omega$. Thus g is a constant function and so $\partial\Omega$ is a hypersphere. On the other hand, since $\Delta_f x_i = \Delta x_i - \langle \nabla f, \nabla x_i \rangle = \langle \nabla f, \nabla x_i \rangle = 0$, $i = 1, 2, \dots, n$, in Ω , we have $\nabla f = 0$,

which implies that $f = \text{const.}$ is a constant function. The proof of Theorem 2 is finished.

3. Eigenvalue estimates for the weighted p -Laplacian

In this section, several estimates for the first eigenvalue of the weighted p -Laplacian will be shown. However, first we need the following notion.

DEFINITION 3. Let $\Omega \subset M$ be a domain with compact closure in a complete SMMS $(M, \langle \cdot, \cdot \rangle, e^{-f} dv)$. Let $\mathcal{X}(\Omega)$ be the set of all smooth vector fields X on Ω with $\|X\|_\infty := \sup_\Omega \|X\| < \infty$ and $\inf \operatorname{div}_f X > 0$ with $\operatorname{div}_f(\cdot) = e^f \operatorname{div}(e^{-f} \cdot)$ the weighted divergence operator on $(M, \langle \cdot, \cdot \rangle, e^{-f} dv)$. Define $\tilde{c}(\Omega)$ by

$$\tilde{c}(\Omega) := \sup \left\{ \frac{\inf \operatorname{div}_f X}{\|X\|_\infty} \mid X \in \mathcal{X}(\Omega) \right\}. \quad (34)$$

REMARK 2. Clearly, if $f = \text{const.}$ is a constant function, then $\operatorname{div}_f(\cdot)$ degenerates into the usual divergence operator $\operatorname{div}(\cdot)$ and correspondingly, the quantity $\tilde{c}(\Omega)$ becomes exactly $c(\Omega)$ introduced in [1]. Besides, we claim that $\mathcal{X}(\Omega)$ is not empty. Since the boundary value problem (BVP for short)

$$\begin{cases} \Delta_f u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

always has a solution on the bounded domain Ω , and then at least one can choose $X = \nabla u$, which implies that $\operatorname{div}_f(X) = 1$ and $\|X\| < \infty$. So, at least we see that $X = \nabla u$ belongs to $\mathcal{X}(\Omega)$, which means that our claim is true.

By applying the quantity $\tilde{c}(\Omega)$, one can get the following Cheeger-type lower bound for the first Dirichlet eigenvalue of the weighted p -Laplacian.

LEMMA 2. Let $\Omega \subset M$ be a domain with compact closure and nonempty piecewise smooth boundary $\partial\Omega$ in a SMMS $(M, \langle \cdot, \cdot \rangle, e^{-f} dv)$. Then we have

$$\lambda_{1,p}^f(\Omega) \geq \left(\frac{\tilde{c}(\Omega)}{p} \right)^p > 0, \quad (35)$$

where, as before, $\lambda_{1,p}^f(\Omega)$ is the first Dirichlet eigenvalue of the weighted p -Laplacian $\Delta_{p,f}$, and $\tilde{c}(\Omega)$ is defined as (34).

PROOF. Choose a function $\phi \in C_0^\infty(\Omega)$, with $C_0^\infty(\Omega)$ the set of compactly supported smooth functions defined on Ω , and then, clearly, for $X \in \mathcal{X}(\Omega)$, one knows that $|\phi|^p X$ has compact support on Ω . By a straightforward calculation, we have

$$\begin{aligned}\operatorname{div}_f(|\phi|^p X) &= |\phi|^p \operatorname{div}_f X + \langle \nabla |\phi|^p, X \rangle \\ &\geq \inf_{\Omega} \operatorname{div}_f X |\phi|^p - p |\phi|^{p-1} |\nabla \phi| \sup_{\Omega} |X|.\end{aligned}\quad (36)$$

For any $\varepsilon > 0$, we infer from Young's inequality that

$$|\phi|^{p-1} |\nabla \phi| \leq \frac{1}{p} \left(\frac{|\nabla \phi|}{\varepsilon} \right)^p + \frac{p-1}{p} (\varepsilon |\phi|^{p-1})^{p/(p-1)}.\quad (37)$$

Substituting (37) into (36), and taking $\varepsilon = \left(\frac{\inf_{\Omega} \operatorname{div}_f X}{p \sup_{\Omega} |X|} \right)^{(p-1)/p}$, we have

$$\operatorname{div}_f(|\phi|^p X) \geq \frac{1}{p} \left(\inf_{\Omega} \operatorname{div}_f X \right) |\phi|^p - |\nabla \phi|^p \left(\frac{p \sup_{\Omega} |X|}{\inf_{\Omega} \operatorname{div}_f X} \right)^{p-1} \sup_{\Omega} |X|.$$

Integrating both sides of the above inequality and noticing $\int_{\Omega} \operatorname{div}_f(|\phi|^p X) d\mu = 0$, we have

$$\int_{\Omega} |\nabla \phi|^p d\mu \geq \left(\frac{\inf_{\Omega} \operatorname{div}_f X}{p \sup_{\Omega} |X|} \right)^p \int_{\Omega} |\phi|^p d\mu.\quad (38)$$

By the characterization (8) and by taking the supremum over all vector $X \in \mathcal{X}(\Omega)$, we can obtain (35) from (38). This completes the proof.

REMARK 3. Clearly, if $f = \text{const.}$ is a constant function, then the eigenvalue estimate (35) here degenerates into

$$\lambda_{1,p}(\Omega) \geq \left(\frac{c(\Omega)}{p} \right)^p > 0,$$

which is exactly the Cheeger-type lower bound for the first Dirichlet eigenvalue of p -Laplacian given in [24, Lemma 2.3]. As pointed out in [24, Remark 2.4 (2)], $c(\Omega) \leq h(\Omega)$ with $h(\Omega) := \inf_{A \subset \Omega} \frac{\operatorname{vol}(\partial A)}{\operatorname{vol}(A)}$ the Cheeger's constant, and moreover, in some cases, for instance, for balls in the Euclidean space or Hadamard manifolds, $c(\Omega) = h(\Omega)$. Comparing with $h(\Omega)$, which is hard to get explicitly for general domains, $c(\Omega)$ can be computed for most of the time. This is the reason why we would like to give lower bounds, involving $c(\Omega)$, $\tilde{c}(\Omega)$, for eigenvalues.

Applying Lemma 2 directly, we can get the following result.

COROLLARY 1. *Let Ω be a normal domain with compact closure in a complete SMMS $(M, \langle \cdot, \cdot \rangle, e^{-f} dv)$. For the following BVP*

$$\begin{cases} \Delta_f v = 1, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

we have

$$\lambda_{1,p}^f(\Omega) \geq \left(\frac{1}{p|\nabla v|_\infty} \right)^p > 0.$$

Now, we would like to give another type lower bound for $\lambda_{1,p}^f(\Omega)$, which can be seen as an extension of Barta's lemma.

LEMMA 3. *Let $\mathcal{W}^{-1,1}(M)$ be the Sobolev space of all vector fields $X \in L_{loc}^1(M)$ possessing a weak weighted divergence¹ $\operatorname{div}_f X$ on a complete SMMS $(M, \langle, \rangle, e^{-f} dv)$. Then the first eigenvalue $\lambda_{1,p}^f(M)$ of the weighted p -Laplacian $\Delta_{p,f}$ satisfies*

$$\lambda_{1,p}^f(\Omega) \geq \sup_{\mathcal{W}^{-1,1}(M)} \left\{ \inf_M (\operatorname{div}_f X - (p-1)|X|^{p/(p-1)}) \right\}. \quad (39)$$

PROOF. Choose a function $\phi \in C_0^\infty(\Omega)$, with $C_0^\infty(\Omega)$ the set of compactly supported smooth functions defined on Ω , and then, clearly, for $X \in \mathcal{X}(\Omega)$, one knows that $|\phi|^p X$ has compact support on Ω . By direct computation and Young's inequality, we have

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}_f (|\phi|^p X) d\mu \\ &\geq \int_\Omega (\operatorname{div}_f X |\phi|^p - p|\phi|^{p-1} |\nabla \phi| |X|) d\mu \\ &\geq \int_\Omega \operatorname{div}_f X |\phi|^p d\mu - p \int_\Omega \left(\frac{1}{p} |\nabla \phi|^p + \frac{p-1}{p} (|\phi|^{p-1} |X|)^{p/(p-1)} \right) d\mu \\ &= \int_\Omega (\operatorname{div}_f X - |X|^{p/(p-1)}) |\phi|^p d\mu - \int_\Omega |\nabla \phi|^p d\mu \\ &\geq \inf_\Omega (\operatorname{div}_f X - |X|^{p/(p-1)}) \int_\Omega |\phi|^p d\mu - \int_\Omega |\nabla \phi|^p d\mu, \end{aligned}$$

which implies

$$\frac{\int_\Omega |\nabla \phi|^p d\mu}{\int_\Omega |\phi|^p d\mu} \geq \inf_\Omega (\operatorname{div}_f X - |X|^{p/(p-1)}). \quad (40)$$

¹ For a SMMS $(M, \langle, \rangle, e^{-f} dv)$, a function $g \in L_{loc}^1(M)$, in the sense of weighted density $d\mu$, is a weak weighted divergence of X if $\int_M g \psi d\mu = - \int_M \langle \operatorname{grad} \psi, X \rangle d\mu$, $\forall \psi \in C_0^\infty(M)$. There exists at most one $g \in L_{loc}^1(M)$ for a given vector field $X \in L_{loc}^1(M)$ and we can write $g = \operatorname{div}_f X$. Clearly, for a C^1 vector field X , its classical weighted divergence coincides with the weak weighted divergence $\operatorname{div}_f X$.

By the characterization (8), it is clear that (39) follows from (40) directly. This completes the proof of Lemma 3.

REMARK 4. (1) Clearly, if $f = \text{const.}$ or $p = 2$, then the estimate (39) becomes the ones in [24, Lemma 4.2]. Besides, one can find that Lemma 3 can be seen as an extension of Barta's lemma. In fact, if $f = \text{const.}$ (or $p = 2$) and M is a bounded manifold with nonempty boundary ∂M , by (39) one has

$$\lambda_1(\Omega) \geq \sup_{\mathcal{H}^{1,1}(M)} \left\{ \inf_M (\operatorname{div} X - |X|^2) \right\},$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the Laplacian Δ . Choosing the vector field X as $X = -\operatorname{grad} \log h$, where $h \in C^2(M) \cap C^0(\bar{M})$ is a positive function, one has

$$\lambda_1(M) \geq \inf_M \left(\frac{-\Delta h}{h} \right),$$

which is exactly the conclusion of classical Barta's lemma. (2) Given a complete SMMS $(M, \langle, \rangle, e^{-f} dv)$, for a point $q \in M$, define the distance function $r(x) = \operatorname{dist}(x, q)$ for any $x \in M \setminus \{q\}$, and then we have $\nabla r = \partial_r$, $|\nabla r| = 1$, with ∂_r the radial unit vector field. Choosing $X = \nabla r$, then $\operatorname{div}_f X = \Delta_f r = m_f := m - \partial_r f$, $|X| = 1$, where m is the mean curvature of geodesic spheres with respect to the outward unit normal vector. Applying Lemma 3 directly, we have

$$\lambda_{1,p}^f(M) \geq \inf_M (m_f - (p-1)).$$

We want to give lower bound estimates for the first eigenvalue of the weighted p -Laplacian on submanifolds with locally bounded weighted mean curvature.

Let $\varphi: M \hookrightarrow N$ be an isometric immersion, where $(M, \langle, \rangle, e^{-f} dv)$ is an n -dimensional ($n \geq 2$) complete SMMS, and N is a complete Riemannian manifold. In order to state our estimates, first, we need the following notion.

DEFINITION 4. An isometric immersion $\varphi: M \hookrightarrow N$ has locally bounded weighted mean curvature vector $\mathbf{H}_f := n\mathbf{H} + (\bar{\nabla} f)^\perp$ if for any $q \in N$ and $r > 0$, the number $h_f(q, r) := \sup\{|(n\mathbf{H} + (\bar{\nabla} f)^\perp)(x)| \mid x \in \varphi(M) \cap B_N(q, r)\}$ is finite, where, as before, $\mathbf{H}$ is the mean curvature vector of the immersed submanifold $\varphi(M)$, and $(\bar{\nabla} f)^\perp$ denotes the projection of $\bar{\nabla} f$ on the normal bundle of $\varphi(M)$, with $\bar{\nabla}$ the gradient operator on N .

REMARK 5. The concept of the weighted mean curvature in Definition 4 seems like different from the one used in [24, Section 5]. However, essentially,

they are the same, since in different literatures the norm of the mean curvature vector is defined to be $|\mathbf{H}| = \sum_{i=1}^n \lambda_i$ or $|\mathbf{H}| = \sum_{i=1}^n \lambda_i/n$, where λ_i , $i = 1, 2, \dots, n$, are eigenvalues of the second fundamental form of the n -dimensional submanifolds of M .

Applying Lemma 3, the following conclusion can be proven.

THEOREM 5. *Let $\varphi: M \hookrightarrow N$ be an isometric immersion with locally bounded mean curvature and let Ω , with piecewise smooth boundary (if exists), be any connected component of $\varphi^{-1}(B_N(q, r))$, where $q \in N \setminus \varphi(M)$, $r > 0$ and $\dim(M) = n$, $n \geq 2$. Let $\kappa(q, r) = \sup\{K_N(x) \mid x \in B_N(q, r)\}$, where $K_N(x)$ is the sectional curvature at x , and $B_N(q, r)$ is the geodesic ball, with center q and radius r , on N . Denote by $\text{inj}(q)$ the injectivity radius of N at the point q . Choosing r properly, we have the following estimates:*

(1) *If $\kappa(q, \text{inj}(q)) = k^2 < \infty$, $k > 0$, choose*

$$r < \min\left\{\text{inj}(q), \frac{\pi}{2k}, \cot^{-1}\left[\frac{h_f(q, \text{inj}(q))}{(n-1)k}\right] \middle/ k\right\}.$$

Then we have

$$\lambda_{1,p}^f(\Omega) \geq \left[\frac{(n-1)k \cot(kr) - h_f(q, r)}{p}\right]^p.$$

(2) *If $\lim_{\ell \rightarrow \infty} \kappa(q, \ell) = \infty$, let*

$$r(s) := \min\left\{\frac{\pi}{2\sqrt{\kappa(q, s)}}, \cot^{-1}\left[\frac{h_f(q, s)}{(n-1)\sqrt{\kappa(q, s)}}\right] \middle/ \sqrt{\kappa(q, s)}\right\}, \quad s > 0.$$

Choose $r = \max_{s>0} r(s)$. Then we have

$$\lambda_{1,p}^f(\Omega) \geq \left[\frac{(n-1)\sqrt{\kappa(q, s)} \cot(\sqrt{\kappa(q, s)}r) - h_f(q, r)}{p}\right]^p.$$

(3) *If $\kappa(q, \text{inj}(q)) = 0$, choose $r < \min\left\{\text{inj}(q), \frac{n}{h_f(q, \text{inj}(q))}\right\}$. Assume that $\frac{n}{h_f(q, \text{inj}(q))} = \infty$ if $h_f(q, \text{inj}(q)) = 0$. Then we have*

$$\lambda_{1,p}^f(\Omega) \geq \left[\frac{\frac{n}{r} - h_f(q, r)}{p}\right]^p.$$

(4) *If $\kappa(q, \text{inj}(q)) = -k^2 < \infty$, $k > 0$, and $h_f(q, \text{inj}(q)) < (n-1)k$, choose $r < \text{inj}(q)$. Then we have*

$$\lambda_{1,p}^f(\Omega) \geq \left[\frac{(n-1)k - h_f(q, r)}{p}\right]^p.$$

(5) If $\kappa(q, \text{inj}(q)) = -k^2 < \infty$, $k > 0$, and $h_f(q, \text{inj}(q)) \geq (n-1)k$, choose

$$r < \min \left\{ \text{inj}(q), \coth^{-1} \left[\frac{h_f(q, \text{inj}(q))}{(n-1)k} \right] / k \right\}.$$

Then we have

$$\lambda_{1,p}^f(\Omega) \geq \left[\frac{(n-1)k \coth(kr) - h_f(q, r)}{p} \right]^p.$$

In (2), since $r(s) > 0$ for small s , $r > 0$. In (3)–(5), because of the non-positivity assumption on $\kappa(q, \text{inj}(q))$, the radius r is not necessary to be finite, which implies that the connected component Ω of $\varphi^{-1}(\overline{B_N(q, r)})$ may be unbounded as $r \rightarrow \infty$.

PROOF. The proof is almost the same as that of [24, Theorem 3.2], and one just needs to replace the number $h(q, r)$ by its weighted version $h_f(q, r)$.

Using Theorem 5 directly, we can get the following results.

COROLLARY 2. Let $\varphi : M^n \hookrightarrow \mathbb{R}^m$ be an isometric f -minimal immersion (i.e., $\mathbf{H}_f = 0$), where M is an n -dimensional ($n \geq 2$) complete SMMS, $\varphi(M^n) \subset B_{\mathbb{R}^m}(o, r)$, and $B_{\mathbb{R}^m}(o, r) \subset \mathbb{R}^m$, is the ball, with radius r and the origin o of $\mathbb{R}^m$ as its center. Then

$$\lambda_{1,p}^f(M) \geq \left(\frac{n}{pr} \right)^p.$$

COROLLARY 3. Let $\varphi : M \hookrightarrow N$ be an isometric immersion with locally bounded weighted mean curvature $|\mathbf{H}_f|$ satisfying $|\mathbf{H}_f| \leq \alpha < (n-1)a$ for some positive constant α , where M is an n -dimensional ($n \geq 2$) complete non-compact SMMS and N is an m -dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $K_N \leq -a^2 < 0$ for some constant $a > 0$. Then we have

$$\lambda_{1,p}^f(M) \geq \left(\frac{(n-1)a - \alpha}{p} \right)^p > 0,$$

where $\lambda_{1,p}^f(M)$ is defined as (9).

REMARK 6. Clearly, if $N = \mathbb{H}^m(-1)$ is the hyperbolic m -space with sectional curvature -1 , then we have $\lambda_{1,p}^f(M) \geq \left(\frac{n-1-a}{p} \right)^p > 0$, which implies that a strictly positive lower bound can be attained for the first eigenvalue of the weighted p -Laplacian on non-compact submanifolds of $\mathbb{H}^m(-1)$ with locally bounded weighted mean curvature.

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