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On cohomologically complete intersection modules

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Abstract. In this paper, several necessary and sufficient conditions are presented for a module M to be cohomologically complete intersection module with respect to I, *i.e.* $H_I^i(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$. This notion is a generalization of cohomologically complete intersection ideals.

1. Introduction

The local cohomology theory has become a technical tool in the fields of commutative algebra and algebraic geometry. It was introduced by Grothendieck (see [4] for details). Let I be an ideal of a commutative Noetherian local ring R. For an R-module M, $H_I^i(M)$, $i \in \mathbb{Z}$, is called the i-th local cohomology module of M with respect to I (see [2] and [4]). If $H_I^i(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$, then M is called a cohomologically complete intersection module with respect to I. The cohomologically complete intersection property of the R-module R with respect to I is necessary for the set-theoretic complete intersection property of I.

In recent years, many authors have studied cohomologically complete intersection ideals for $M = R$. The first attempt in this direction was made by M. Hellus and P. Schenzel in [8, Theorem 0.1]. Later on, M. Zargar provided with some conditions for a maximal Cohen-Macaulay module of finite injective dimension to be a cohomologically complete intersection (see [27, Theorem 1.1]). The similar results are obtained for the canonical modules in [15, Theorem 1.1]. Afterwards M. Hellus and P. Schenzel proved this property for an arbitrary module M in [10, Theorem 4.4].

Recently, the author has given a new characterization of an ideal in a local Gorenstein ring to be a cohomologically complete intersection (see [16, Theorem 1.1]). He proved that this property is equivalent to the following property of Betti numbers:

$$\dim_{k(\mathfrak{p})}(\operatorname{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}}))) = \delta_{c,i},$$

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for all $\mathfrak{p} \in V(I)$. Here, $k(\mathfrak{p})$ denotes the residue field of the local Gorenstein ring $R_{\mathfrak{p}}$. In this article, some new equivalent conditions will be proved for an arbitrary module M to be a cohomologically complete intersection. This provides some new necessary conditions for an ideal to be a set-theoretic complete intersection in an arbitrary local ring.

Theorem 1. *Let M be a non-zero finitely generated module a over local ring R. Suppose that I is an ideal with $c = \operatorname{grade}(I, M)$. Then the following conditions are equivalent:*

(1) *M is cohomologically complete intersection with respect to I.*

(2) *For all $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Ext}^{i+c}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, (M_{\mathfrak{p}}))$$

are isomorphisms for all $i \in \mathbb{Z}$.

(3) *For all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Tor}^{R_{\mathfrak{p}}}_{i+c}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Tor}^{R_{\mathfrak{p}}}_i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

(4) *For any finitely generated R-module N with $\operatorname{Supp}_R(N) \subseteq V(I)$ and for all $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Ext}^{i+c}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

(5) *For any finitely generated R-module N with $\operatorname{Supp}_R(N) \subseteq V(I)$ and for all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Tor}^{R_{\mathfrak{p}}}_{i+c}(N_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Tor}^{R_{\mathfrak{p}}}_i(N_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

In the second section, the natural homomorphisms of Theorem 1 are discussed while the proof of Theorem 1 is presented in section 3. Note that the equivalence of the conditions (1) and (2) is a slight variation of the argument of Hellus and Schenzel (see [10]).

2. Some vanishing results and natural homomorphisms

In this section, we will stick to the following notations: A commutative Noetherian ring will be denoted by R. In addition, if R is local, then the unique maximal ideal of this will be denoted by $\mathfrak{m}$. Let $E = E_R(k)$ be the injective hull of the residue field $k = R/\mathfrak{m}$. Then, $D(\cdot)$ will be denoting the

Matlis dual functor. For the basic results on commutative algebra and homological algebra, see [2], [5], [13], and [25].

The symbol $\cong$ is used to denote an isomorphism of R-modules.

Lemma 1. *Let I be an ideal of R and X an arbitrary complex of R-modules. Then, for any integer $s \in \mathbb{N}$, the following conditions are equivalent:*

(1) $H_I^i(X) = 0$ *for all* $i < s$.

(2) $\operatorname{Ext}_R^i(R/I, X) = 0$ *for all* $i < s$.

If one of the above conditions holds, then there is an isomorphism:

$$\operatorname{Hom}_R(R/I, H_I^s(X)) \cong \operatorname{Ext}_R^s(R/I, X).$$

Proof. To prove this result, one can proceed along the similar lines to those of [8, Proposition 1.4].

Proposition 1. *Let I be an ideal of R. For any left bounded complex X of R-modules, the following conditions are equivalent:*

(1) $H_I^i(X) = 0$ *for all* $i \in \mathbb{Z}$.

(2) $\operatorname{Ext}_R^i(R/I, X) = 0$ *for all* $i \in \mathbb{Z}$.

(3) *For any R-module N such that* $\operatorname{Supp}_R(N) \subseteq V(I)$, *we have* $\operatorname{Ext}_R^i(N, X) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. Note that (2) is equivalent to (1) (see Lemma 1). Also, the proof that (3) implies (2) is obvious. Now, we prove that (1) implies (3). Suppose that (1) is true. Let $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x} = x_1, \ldots, x_s \in I$ such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{x})R$. Suppose that $E_R^{\cdot}$ denotes an injective resolution of X (see [1, p. 134] or [19, Theorem C]). The result will be proved in the following two steps:

Case-i: Assume that N is a finitely generated R-module with $\operatorname{Supp}_R(N) \subseteq V(I)$. Then the support of each module of the complex $\operatorname{Hom}_R(N, E_R^{\cdot})$ is contained in $V(I)$. It follows that there are the following isomorphisms of complexes:

$$\check{C}_{\underline{x}} \otimes_R^L \operatorname{Hom}_R(N, E_R^{\cdot}) \cong \mathrm{R}\Gamma_I(\operatorname{Hom}_R(N, E_R^{\cdot})) \cong \mathrm{R}\operatorname{Hom}_R(N, E_R^{\cdot}) \qquad (1)$$

(see [20, Theorem 3.2]). Also, $\check{C}_{\underline{x}}$ is a bounded complex of flat R-modules. So, by [5, Proposition 5.14] or [3, Proposition 1.1] or [24, Lemma 11.1.2], we have

$$\check{C}_{\underline{x}} \otimes_R^L \mathrm{R}\operatorname{Hom}_R(N, E_R^{\cdot}) \cong \mathrm{R}\operatorname{Hom}_R(N, \check{C}_{\underline{x}} \otimes_R^L E_R^{\cdot}). \qquad (2)$$

From assumption (1) and [20, Theorem 3.2], it follows that $\mathrm{R}\Gamma_I(E_R^{\cdot}) \cong \check{C}_{\underline{x}} \otimes_R^L E_R^{\cdot}$ is an exact complex. This proves the exactness of the complex $\mathrm{R}\operatorname{Hom}_R(N, \check{C}_{\underline{x}} \otimes_R^L E_R^{\cdot})$. Hence, the complex $\mathrm{R}\operatorname{Hom}_R(N, E_R^{\cdot})$ is exact (see the isomorphisms (1) and (2)).

Since $E_R^{\cdot}$ is a complex of injective R-modules, the complex $\mathrm{R}\operatorname{Hom}_R(N, E_R^{\cdot})$ can be represented by $\operatorname{Hom}_R(N, E_R^{\cdot})$. It follows that

$$\operatorname{Ext}_R^i(N, X) = 0 \qquad \text{for all } i \in \mathbb{Z}.$$

Case-ii: Assume that N is any R-module such that $\operatorname{Supp}_R(N) \subseteq V(I)$. Let $\{N_\alpha : \alpha \in \mathbb{N}\}$ be a family of all finitely generated R-submodules of N so that

$$N \cong \varinjlim N_\alpha.$$

Then the support of each module N_α is a subset of $V(I)$. By Case-i, we have

$$\operatorname{Ext}_R^i(N_\alpha, X) = 0 \qquad \text{for all } i \in \mathbb{Z} \text{ and } \alpha \in \mathbb{N}. \tag{3}$$

The vanishing of $\operatorname{Ext}_R^i(N, X)$ will be proved by induction on i. This holds for any $i < 0$. Let us prove the case for $i = 0$. Since, Hom-functor transforms the direct limits into inverse limits in the first variable (see [25]), it follows that

$$\operatorname{Hom}_R(N, X) \cong \varprojlim \operatorname{Hom}_R(N_\alpha, X) = 0, \qquad \text{by Equation (3).}$$

Suppose that the assertion holds for $i = k - 1$. By definition of the direct limits, there is a short exact sequence

$$0 \to Y \to \bigoplus_{\alpha \in \mathbb{N}} N_\alpha \xrightarrow{f} N \to 0$$

where $Y = \ker(f)$. Apply the functor $\mathrm{R}\operatorname{Hom}_R(\cdot, X)$ to this sequence, we get the following exact sequence

$$0 \to \mathrm{R}\operatorname{Hom}_R(N, X) \to \mathrm{R}\operatorname{Hom}_R\left(\bigoplus_{\alpha \in \mathbb{N}} N_\alpha, X\right) \to \mathrm{R}\operatorname{Hom}_R(Y, X) \to 0. \tag{4}$$

Note that, in the derived category, the complexes $\mathrm{R}\operatorname{Hom}_R(N, X)$, $\mathrm{R}\operatorname{Hom}_R(Y, X)$ and $\mathrm{R}\operatorname{Hom}_R(\bigoplus_{\alpha \in \mathbb{N}} N_\alpha, X)$ are represented by $\operatorname{Hom}_R(N, E_R^{\cdot})$, $\operatorname{Hom}_R(Y, E_R^{\cdot})$ and $\operatorname{Hom}_R(\bigoplus_{\alpha \in \mathbb{N}} N_\alpha, E_R^{\cdot})$ respectively. Then the sequence (4) induces the following exact sequence of cohomologies:

$$\operatorname{Ext}_R^{k-1}(Y, X) \to \operatorname{Ext}_R^k(N, X) \to \operatorname{Ext}_R^k\left(\bigoplus_{\alpha \in \mathbb{N}} N_\alpha, X\right).$$

Since Ext-functor transforms the direct sums into direct products in the first variable (see [25]), the aforementioned sequence becomes the following exact sequence:

$$\operatorname{Ext}_R^{k-1}(Y, X) \to \operatorname{Ext}_R^k(N, X) \to \prod_{\alpha \in \mathbb{N}} \operatorname{Ext}_R^k(N_\alpha, X).$$

By Equation (3) and the inductive hypothesis applied to Y, it follows that $\mathrm{Ext}_R^k(N, X) = 0$. This completes the proof of Proposition.

Corollary 1. *With the same assumptions as in Proposition* 1, *suppose that R is local. Then the following conditions are equivalent:*

(1) $H_I^i(X) = 0$ *for all* $i \in \mathbb{Z}$.
(2) $H_I^i(D(X)) = 0$ *for all* $i \in \mathbb{Z}$.
(3) *For any R-module N such that* $\mathrm{Supp}_R(N) \subseteq V(I)$, *we have* $\mathrm{Ext}_R^i(N, X) = 0$ *for all* $i \in \mathbb{Z}$.
(4) $\mathrm{Ext}_R^i(R/I, X) = 0$ *for all* $i \in \mathbb{Z}$.
(5) *For any R-module N such that* $\mathrm{Supp}_R(N) \subseteq V(I)$, *we have* $\mathrm{Tor}_i^R(N, X) = 0$ *for all* $i \in \mathbb{Z}$.
(6) $\mathrm{Tor}_i^R(R/I, X) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. By Proposition 1, the statements (1)–(4) are equivalent. Also, the proof of (5) $\Rightarrow$ (6) $\Rightarrow$ (1) is obvious. Now we prove (3) implies (5). Suppose that N is an R-module with $\mathrm{Supp}_R(N) \subseteq V(I)$. The result is proved in the following two cases:

Case-i: Assume that N is a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(I)$. By [26, Corollary 1.2], it follows that $\mathrm{Tor}_i^R(N, X) = 0$ for all $i \in \mathbb{Z}$.

Case-ii: Assume that N is any R-module such that $\mathrm{Supp}_R(N) \subseteq V(I)$. Along the same steps as followed in Proposition 1, one can obtain the vanishing of $\mathrm{Tor}_i^R(N, X)$. Recall that Tor commutes with direct sums. This proves the result.

In order to derive the natural homomorphisms of Theorem 2, we need the definition of the truncation complex. The truncation complex is introduced in [21, Definition 4.1]. Suppose that I is an ideal of R. Let $E_R^{\cdot}(M)$ be a minimal injective resolution of an R-module M with $\mathrm{grade}(I, M) = c$. Then there is an exact sequence of R-modules

$$0 \to H_I^c(M) \to \Gamma_I(E_R^{\cdot}(M))^c \to \Gamma_I(E_R^{\cdot}(M))^{c+1}.$$

Hence, this induces an embedding of complexes of R-modules $H_I^c(M)[-c] \to \Gamma_I(E_R^{\cdot}(M))$.

Definition 1. Let $C_M^{\cdot}(I)$ be the cokernel of the embedding $H_I^c(M)[-c] \to \Gamma_I(E_R^{\cdot}(M))$. Then $C_M^{\cdot}(I)$ is called the truncation complex of M with respect to I. Note that there exists the following short exact sequence of complexes

$$0 \to H_I^c(M)[-c] \to \Gamma_I(E_R^{\cdot}(M)) \to C_M^{\cdot}(I) \to 0. \tag{5}$$

In particular, the modules $H^i(C^{\cdot}_M(I))$ are zero, for all $i \le c$. Also, there is an isomorphism

$$H^i(C^{\cdot}_M(I)) \cong H^i_I(M) \qquad \text{for all } i > c.$$

Definition 2. Let M be an R-module and I an ideal of R with $\operatorname{grade}(I, M) = c$. Then M is called a cohomologically complete intersection module with respect to I, if $H^i_I(M) = 0$ for all $i \ne c$.

In the following result, some natural homomorphisms will be obtained with the help of truncation complex. These maps will be used to investigate the property of cohomologically complete intersection modules.

Theorem 2. *Let M be a non-zero finitely generated R-module. Suppose that* $\operatorname{grade}(I, M) = c$, *where I is an ideal. For any finitely generated R-module N with* $\operatorname{Supp}_R(N) \subseteq V(I)$, *we have:*

(1) *For all $i \in \mathbb{Z}$, there are natural homomorphisms*

$$\operatorname{Tor}^R_{c+i}(N, H^c_I(M)) \to \operatorname{Tor}^R_i(N, M).$$

These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $\operatorname{Tor}^R_i(N, C^{\cdot}_M(I)) = 0$ *for all $i \in \mathbb{Z}$.*

(2) *For all $i \in \mathbb{Z}$, there are natural homomorphisms*

$$\operatorname{Ext}^{i-c}_R(N, H^c_I(M)) \to \operatorname{Ext}^i_R(N, M).$$

These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $\operatorname{Ext}^i_R(N, C^{\cdot}_M(I)) = 0$ *for all $i \in \mathbb{Z}$.*

Proof. Since $\operatorname{Supp}_R(N) \subseteq V(I)$, there are the following isomorphisms in the derived category

$$N \otimes^L_R \mathrm{R}\Gamma_I(M) \cong \mathrm{R}\Gamma_I(N \otimes^L_R M) \cong N \otimes^L_R M \qquad \text{and}$$

$$\mathrm{R}\operatorname{Hom}_R(N, \mathrm{R}\Gamma_I(M)) \cong \mathrm{R}\Gamma_I(\mathrm{R}\operatorname{Hom}_R(N, M)) \cong \mathrm{R}\operatorname{Hom}_R(N, M).$$

Then the exact sequence (5) induces the following morphisms in the derived category

$$N \otimes^L_R H^c_I(M)[-c] \to N \otimes^L_R \mathrm{R}\Gamma_I(M) \cong N \otimes^L_R M \qquad \text{and}$$

$$\mathrm{R}\operatorname{Hom}_R(N, H^c_I(M))[-c] \to \mathrm{R}\operatorname{Hom}_R(N, \mathrm{R}\Gamma_I(M)) \cong \mathrm{R}\operatorname{Hom}_R(N, M).$$

It is an isomorphism if and only if the complex $N \otimes^L_R C^{\cdot}_M(I)$ resp. $\mathrm{R}\operatorname{Hom}_R(N, C^{\cdot}_M(I))$ is exact. This completes the proof of Theorem.

Proposition 2. *With the above notation, let I and J be two ideals of R such that $I \subseteq J$. Then, there exist natural isomorphisms*

$$H_J^{i+c}(D(H_I^c(M))) \to H_J^{i+1}(D(C_M^{\cdot}(I))), \qquad \textit{for all } i \in \mathbb{Z}.$$

Proof. Apply the Matlis dual functor to the sequence (5). The following exact sequence will be obtained

$$0 \to D(C_M^{\cdot}(I)) \to D(\mathrm{R}\Gamma_I(M)) \to D(H_I^c(M))[c] \to 0.$$

Let $s \in \mathbb{N}$ be fixed. Apply the functor $\mathrm{R}\operatorname{Hom}_R(R/J^s, .)$ to the above sequence, in order to obtain the following short exact sequence

$$\begin{aligned} 0 &\to \mathrm{R}\operatorname{Hom}_R(R/J^s, D(C_M^{\cdot}(I))) \to \mathrm{R}\operatorname{Hom}_R(R/J^s, D(\mathrm{R}\Gamma_I(M))) \\ &\to \mathrm{R}\operatorname{Hom}_R(R/J^s, D(H_I^c(M)))[c] \to 0. \end{aligned}$$

We are interested in the cohomologies of the complex $\mathrm{R}\operatorname{Hom}_R(R/J^s, D(\mathrm{R}\Gamma_I(M)))$, denoted by X. Note that there are the following isomorphisms of complexes

$$\begin{aligned} X &\cong \mathrm{R}\operatorname{Hom}_R(R/J^s \otimes_R^L \mathrm{R}\Gamma_I(M), E) \cong \mathrm{R}\operatorname{Hom}_R(R/J^s \otimes_R^L M, E) \\ &\cong \mathrm{R}\operatorname{Hom}_R(R/J^s, D(M)) \end{aligned}$$

(see [5, Proposition 5.15]). Hence, $H^i(X) \cong \operatorname{Ext}_R^i(R/J^s, D(M))$ for all $i \in \mathbb{Z}$. Then the aforementioned sequence induces the following exact sequence of cohomologies

$$\operatorname{Ext}_R^i(R/J^s, D(M)) \to \operatorname{Ext}_R^{i+c}(R/J^s, D(H_I^c(M))) \to \operatorname{Ext}_R^{i+1}(R/J^s, D(C_M^{\cdot}(I))),$$

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. On passing to the direct limit, the following natural homomorphisms are obtained

$$H_J^i(D(M)) \to H_J^{i+c}(D(H_I^c(M))) \to H_J^{i+1}(D(C_M^{\cdot}(I))), \qquad \text{for all } i \in \mathbb{Z}.$$

These are isomorphisms for all $i \in \mathbb{Z}$ because of $H_J^i(D(M)) = 0$ for all $i \in \mathbb{Z}$. Here, we used that $D(M)$ is an Artinian R-module.

Note that the following Lemma is already proved in [10, Lemma 4.2] for $J = \mathfrak{m}$.

Lemma 2. *Under the assumption of Proposition 2, there are natural homomorphisms*

$$H_J^i(H_I^c(M)) \to H_J^{i+c}(M)$$

for all $i \in \mathbb{Z}$. These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $H_J^i(C_M^{\cdot}(I)) = 0$ for all $i \in \mathbb{Z}$.

Proof. To prove the claim, follow the same steps as in proof of [10, Lemma 4.2].

3. Cohomologically complete intersection modules

In this section, the natural homomorphisms of Theorem 2 and Proposition 2 will be investigated for isomorphisms. In addition, the cohomologically complete intersection modules will be studied from various homological aspects.

Corollary 2. *Let I and J be two ideals of R such that $I \subseteq J$. Let M be a non-zero finitely generated R-module with $c = \operatorname{grade}(I, M)$. Assume that M is a cohomologically complete intersection module with respect to I. Then, for any finitely generated R-module N with $\operatorname{Supp}_R(N) \subseteq V(I)$, the following statements are true:*

(1) *The natural homomorphisms*

$$\operatorname{Tor}_{i+c}^R(N, H_I^c(M)) \to \operatorname{Tor}_i^R(N, M)$$

are isomorphisms for all $i \in \mathbb{Z}$.

(2) *The natural homomorphisms*

$$\operatorname{Ext}_R^i(N, H_I^c(M)) \to \operatorname{Ext}_R^{i+c}(N, M)$$

are isomorphisms for all $i \in \mathbb{Z}$.

(3) *The natural homomorphisms*

$$H_J^i(H_I^c(M)) \to H_J^{i+c}(M)$$

are isomorphisms for all $i \in \mathbb{Z}$.

Proof. To prove the statement in (1), consider a minimal free resolution $F_{\cdot}^R$ of the truncation complex $C_M^{\cdot}(I)$ (see [1, p. 134]). We claim that $\operatorname{Tor}_i^R(N, C_M^{\cdot}(I)) = 0$ for all $i \in \mathbb{Z}$. To prove this claim, take the following spectral sequence

$$E_2^{p,q} := \operatorname{Tor}_{-p}^R(N, H^q(F_{\cdot}^R)) \Rightarrow E_{\infty}^{p+q} = H^{-p-q}(N \otimes_R F_{\cdot}^R)$$

(see [25, Theorem 11.39]). Since the complex $C_M^{\cdot}(I)$ is exact, it follows that $H^i(F_{\cdot}^R) \cong H^i(C_M^{\cdot}(I)) = 0$ for all $i \in \mathbb{Z}$. This proves the vanishing of all the initial terms in the aforementioned sequence. Therefore, $\operatorname{Tor}_{-i}^R(N, C_M^{\cdot}(I)) \cong H^i(N \otimes_R F_{\cdot}^R) = 0$ for all $i \in \mathbb{Z}$. The statement in (1) is obvious in light of Theorem 2(1).

Note that $\operatorname{Ext}_R^i(N, C_M^{\cdot}(I)) = 0 = H_J^i(C_M^{\cdot}(I))$ for all $i \in \mathbb{Z}$, see Corollary 1. This completes the proof in view of Theorem 2(2) and Lemma 2.

3.1. Characterization with Tor modules.

Lemma 3 ([14, Proposition 2.7]). *Let $(R, \mathfrak{m})$ be a local ring and $0 \neq M$ be a finitely generated R-module. Let $I \subseteq R$ be an ideal of R with $H_I^i(M) = 0$ for all $i \neq c$. Then*

(1) $c = \operatorname{grade}(IR_{\mathfrak{p}}, M_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$.

(2) $H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ *for all* $i \neq c$.

Before proving the next result, we will fix some notations. Let I be an ideal of a ring R. Let M be a non-zero finitely generated R-module such that $c = \operatorname{grade}(I, M)$. Let $E_{R_{\mathfrak{p}}}(k(\mathfrak{p}))$ be the injective hull of the residue field $k(\mathfrak{p})$ of $R_{\mathfrak{p}}$, where $\mathfrak{p} \in V(I)$.

We are now in a position to prove the major result.

Theorem 3. *Let the notation be as above. Then the following conditions are equivalent:*

(1) *M is cohomologically complete intersection with respect to I.*

(2) *For all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Tor}_{i+c}^{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Tor}_i^{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

(3) *For any finitely generated R-module N such that $\operatorname{Supp}_R(N) \subseteq V(I)$ and for all $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Tor}_{i+c}^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Tor}_i^{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

Proof. Note that for $N = R/\mathfrak{p}$, the proof that (3) implies (2) is obvious. To prove that (1) implies (3), assume that $H_I^i(M) = 0$ for all $i \neq c$. Note that the result can only be proved for $\mathfrak{p} = \mathfrak{m}$, the maximal ideal (see Lemma 3). The proof of this case is given in Corollary 2(1).

It only remains to prove that (2) implies (1). We will apply induction on $d := \dim_R(M/IM)$. In the case of $d = 0$, we have $V(I) \cap \operatorname{Supp}_R M \subseteq \{\mathfrak{m}\}$. Using it with the definition of the truncation complex, we obtain

$$\operatorname{Supp}_R(H^i(C^{\cdot}_M(I))) = \operatorname{Supp}_R(H^i_I(M)) \subseteq \{\mathfrak{m}\} \subseteq V(I), \qquad \text{for all } i > c.$$

By [14, Lemma 2.5], this induces the following isomorphisms

$$H^i_I(C^{\cdot}_M(I)) \cong H^i(C^{\cdot}_M(I)) \cong H^i_I(M), \qquad \text{for all } i > c. \tag{6}$$

Due to the assumption in (2), for $\mathfrak{p} = \mathfrak{m}$, it implies that $\operatorname{Tor}_i^R(R/\mathfrak{m}, C^{\cdot}_M(I)) = 0$ for all $i \in \mathbb{Z}$, see Theorem 2. By Corollary 1, it follows that $H^i_I(C^{\cdot}_M(I)) = 0$ for all $i \in \mathbb{Z}$. In view of the isomorphism (6), this proves the result for $d = 0$.

Now suppose that $d > 0$ and the statement is true for all smaller dimensions. Since $\dim(M_{\mathfrak{p}}/IM_{\mathfrak{p}}) < d$ for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M \setminus \{\mathfrak{m}\}$, by the induction hypothesis

$$H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0, \qquad \text{for all } i \neq c \text{ and } \mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M \setminus \{\mathfrak{m}\}.$$

Hence, $\operatorname{Supp}(H^i_I(M)) \subseteq V(\mathfrak{m}) \subseteq V(I)$ for all $i \neq c$. With similar arguments to those for the case $d = 0$, one can deduce that

$$H^i_I(M) = 0, \qquad \text{for all } i \neq c.$$

3.2. Characterization with Ext modules.

THEOREM 4. *Let $I \subseteq R$ be an ideal and M a non-zero finitely generated R-module with $c = \operatorname{grade}(I, M)$. Then the following conditions are equivalent:*

(1) *M is cohomologically complete intersection with respect to I.*

(2) *For all $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Ext}^{i+c}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

(3) *For any finitely generated R-module N such that $\operatorname{Supp}_R(N) \subseteq V(I)$ and for all $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(I)$, the natural homomorphisms*

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to \operatorname{Ext}^{i+c}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$$

are isomorphisms for all $i \in \mathbb{Z}$.

PROOF. By the similar arguments to those employed to prove Theorem 3, the above statements are equivalent.

REMARK 1. *The local conditions are necessary in Theorems* 3 *and* 4. *By Hellus and Schenzel, see* [8, Example 4.1], *it is not enough to assume the conditions for $R_{\mathfrak{m}}$ for any maximal ideal instead of $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$.*

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