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Page moves on arc presentations

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ABSTRACT. Cromwell introduced the notion of an arc presentation of a knot. An arc presentation is an embedding of a knot in an open-book. In this note, we define a transformation on an arc presentation called a page move, and prove that any knot is transformed into the trivial knot by a single page move. We also study a relationship with the unknotting number.

1. Introduction

An *open-book* in $\mathbf{R}^3$ is the union of a finite number of half planes $\bigcup_{i=1}^{n} H_i$ with $H_i = \{(r\cos\theta_i, r\sin\theta_i, z) \mid r \geq 0, z \in \mathbf{R}\}$ for some $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$. The z-axis is called the *binding* of the open-book and H_i is called a *page*. By adding the point ∞ to $\mathbf{R}^3$, the binding becomes the circle (z-axis) $\cup \{\infty\}$ and the pages become disks, so that we obtain an open-book in $S^3 = \mathbf{R}^3 \cup \{\infty\}$.

An *arc presentation* of a knot $K \subset S^3$ is an embedding of K in an open-book such that each of the pages meets K in a single simple arc. Every knot admits an arc presentation. Such presentations were originally described by Brunn [2], and have been revised by the work of Birman and Menasco [1], and have been studied further by Cromwell and Nutt [3, 4, 6].

In this paper, we introduce a move on an arc presentation.

DEFINITION 1. A *page move* on an arc presentation of a knot is a transformation which removes a page with the arc on it from the open-book and inserts the page with the arc into a different position. See Fig. 1.

We consider the problem whether the page move is an unknotting operation or not. The aim of this paper is to prove the following.

THEOREM 1. *Let K be a knot. Then there is an arc presentation P of K such that K is transformed into the trivial knot by a single page move on P.*

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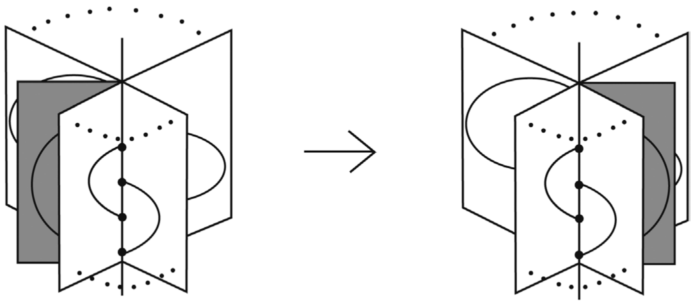

Fig. 1

This paper is organized as follows. We prove Theorem 1 in Section 2. In Section 3, we study a special kind of a page move, which we call a *page change*.

2. Proof of Theorem 1

LEMMA 1. *Let K be a knot and K' a knot obtained from K by n crossing changes. Then there are arc presentations P and P' of K and K', respectively, which satisfy the following.*

(i) *The intersection of the binding with K is identical to that with K'. Let $a_1, \dots, a_m$ be the points of the intersection appearing in this order with respect to the z-coordinate, where m is the number of the pages of P and P' with $m \geq 2n+2$.*

(ii) *The arc presentation P is identical to the arc presentation P' except in $n+1$ pages, $H_1, H_2, \dots, H_{n+1}$.*

(iii) *In P, $H_i \cap K$ is an arc connecting a_i and a_{2n+2-i} $(1 \leq i \leq n)$, and $H_{n+1} \cap K$ is an arc connecting a_{n+1} and a_{2n+2}.* Figure 2 *shows the case $n = 3$.*

(iv) *In P', $H_1 \cap K$ is an arc connecting a_{n+1} and a_{2n+2}, and $H_i \cap K$ is an arc connecting a_{i-1} and a_{2n+3-i} $(2 \leq i \leq n+1)$.*

PROOF. Since K' is obtained from K by n crossing changes, we can assume that the n crossings are consecutive on a certain diagram D of K (cf. [5]). Let D' be the diagram obtained from D by changing the n consecutive crossings as shown in Fig. 3. Then D' presents the knot K'.

We can take a circle C embedded in the plane where D lies such that

- C intersects transversely with K in a finite number of points which are different from the crossings of K,
- C runs around the n consecutive crossings as shown in the left of Fig. 4, and

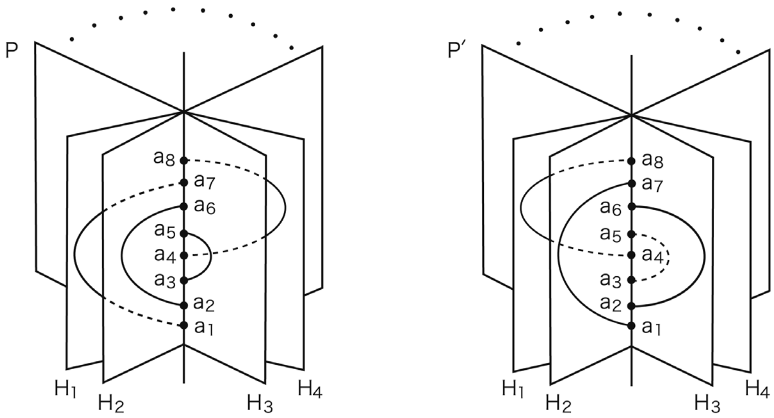

Fig. 2

Fig. 3

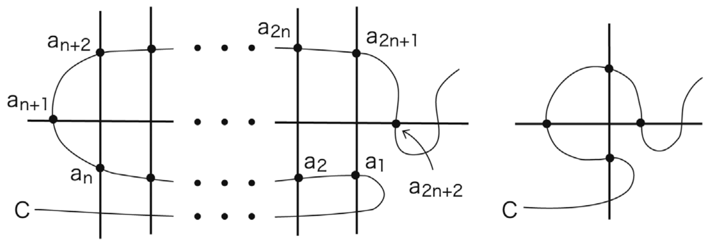

Fig. 4

- C runs around each crossing other than the n crossings as shown in the right.

Let $a_1, \dots, a_m$ be the points of $D \cap C$ appearing in this order such that the first $2n+2$ consecutive points $a_1, \dots, a_{2n+2}$ are near the n crossings as in the figure. By the argument in [4], K and K' admit arc presentations, P and P', respectively, which have C as the binding. Then we can see that, after a slight isotopy, P and P' satisfy the conditions (i)–(iv). □

Proof (Proof of Theorem 1). For the arc presentations P and P' in Lemma 1, P is transformed into P' by the page move such that we remove H_{n+1} from P and insert the page between H_m and H_1. □

3. Page change

A *page change* is a transformation on an arc presentation such that we exchange the positions of adjacent pages. The page change is a special kind of a page move.

PROPOSITION 1. *For any arc presentation P of a knot K, there exist a finite sequence of page changes on P to transform K into the trivial knot.*

PROOF. Any permutation of the pages of $P = \bigcup_{i=1}^{m} H_i$ is realized by a finite sequence of page changes. We consider the permutation as follows.
- We take an arbitrary page H_{i_1} as the first page.
- We take a page H_{i_2} as the second page so that the arcs in H_{i_1} and H_{i_2} have a common endpoint.
- For $k = 2, 3, \dots, m-1$, we take a page $H_{i_{k+1}} \neq H_{i_{k-1}}$ as the $(k+1)$-st page so that the arcs in H_{i_k} and $H_{i_{k+1}}$ have the common endpoint.

Then the resultant knot is trivial.

We denote by $\mathrm{pc}(P)$ the minimal number of page changes in Proposition 1, and by $\mathrm{u}(K)$ the unknotting number of K.

THEOREM 2. *Let K be a knot.*
(i) *For any arc presentation P of K, it holds that* $\mathrm{pc}(P) \geq \mathrm{u}(K)$.
(ii) *There exists an arc presentation P of K such that* $\mathrm{pc}(P) = \mathrm{u}(K)$.

PROOF. (i) A single page change on P induces a single crossing change on K or an isotopy.

(ii) We apply Lemma 1 to K and the trivial knot to obtain arc presentations P and P'. Then the page move in the proof of Theorem 1 is realized by the sequence of n page changes.

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