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On unicity of meromorphic functions when two differential polynomials share one value

Chao Meng

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Abstract. In this article, we deal with the uniqueness problems of meromorphic functions concerning differential polynomials and prove the following result: Let f and g be two nonconstant meromorphic functions and let $n(\geq 14)$ be an integer such that $n+1$ is not divisible by 3. If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share $(1,2)$ or "$(1,2)$", then $f \equiv g$. If $\bar{E}_{4)}(1, f^n(f^3-1)f') = \bar{E}_{4)}(1, g^n(g^3-1)g')$ and $E_{2)}(1, f^n(f^3-1)f') = E_{2)}(1, g^n(g^3-1)g')$, then $f \equiv g$.

1. Introduction, definitions and results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\bar{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\bar{E}(a,f) = \bar{E}(a,g)$, then we say that f and g share the value a IM. Let m be a positive integer or infinity and $a \in \mathbf{C} \cup \{\infty\}$. We denote by $E_{m)}(a,f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\bar{E}_{m)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than m. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

By the above definition, we have

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$$\overline{N}\left(r,\frac{1}{h}\right)+\overline{N}_{(2}\left(r,\frac{1}{h}\right)=N_2\left(r,\frac{1}{h}\right)\leq N\left(r,\frac{1}{h}\right).$$

Let $N_E(r,a;f,g)$ $(\overline{N}_E(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)$ $(\overline{N}_0(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM". It is assumed that the reader is familiar with the notations of Nevanlinna theory, that can be found, for instance, in [8] and [16].

In 1976, F. Gross proposed the following question.

Question A [7]. Whether there exists a finite set S such that $E(S,f)=E(S,g)$ can imply $f\equiv g$?

H. X. Yi gave a positive answer to Question A. He proved the following.

Theorem A [18]. *There exists a set S with* 7 *elements such that $E(S,f)=E(S,g)$ can imply $f\equiv g$ for any pair of nonconstant entire functions f and g.*

H. X. Yi [19], P. Li and C. C. Yang [12] and G. Frank and M. Reinders [6] studied the problem for meromorphic functions. G. Frank and M. Reinders proved the following.

Theorem B [6]. *There exists a set S with* 11 *elements such that $E(S,f)=E(S,g)$ can imply $f\equiv g$ for any pair of nonconstant meromorphic functions f and g.*

In fact, Question A can be stated as follows: whether there exists a polynomial P such that for any pair of nonconstant meromorphic functions f and g we can get $f\equiv g$ if $P(f)$ and $P(g)$ share one value CM? Naturally, we pose the following question:

Question B. Whether there exists a differential polynomial d such that for any pair of nonconstant meromorphic functions f and g we can get $f\equiv g$ if $d(f)$ and $d(g)$ share one value CM?

Some works have already been done in this direction ([5], [4], [14]). In 2006, I. Lahiri and R. Pal found a differential polynomial d for Question B and proved the following result.

Theorem C [11]. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 14)$ be an integer. If $E_{3)}(1, f^n(f^3-1)f') = E_{3)}(1, g^n(g^3-1)g')$, then $f \equiv g$.*

Naturally, we pose the following question:

Question C. Can the nature of sharing value in Theorem C be relaxed in another way?

In this paper, we give a positive answer to Question C. To state the main results of this paper, we require the following notion of weighted sharing which was introduced by I. Lahiri.

Definition 1 [9], [10]. For a complex number $a \in \mathbf{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in \mathbf{C} \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We prove the following results which give a positive answer to Question C.

Theorem 1. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 14)$ be an integer such that $n+1$ is not divisible by 3. If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share $(1, 2)$, then $f \equiv g$.*

Theorem 2. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 17)$ be an integer such that $n+1$ is not divisible by 3. If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share $(1, 1)$, then $f \equiv g$.*

Theorem 3. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 35)$ be an integer such that $n+1$ is not divisible by 3. If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share $(1, 0)$, then $f \equiv g$.*

Recently, Lin and Lin [13] introduced the following notion of weakly weighted sharing.

Definition 2 [13]. Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}_{k)}^{E}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k. $\overline{N}_{(k}^{O}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.

Definition 3 [13]. For $a \in \mathbf{C} \cup \{\infty\}$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}\left(r,\frac{1}{f-a}\right) - \overline{N}_{k)}^{E}(r,a;f,g) = S(r,f),$$

$$\overline{N}_{k)}\left(r,\frac{1}{g-a}\right) - \overline{N}_{k)}^{E}(r,a;f,g) = S(r,g),$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{f-a}\right) - \overline{N}_{(k+1}^{O}(r,a;f,g) = S(r,f),$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{g-a}\right) - \overline{N}_{(k+1}^{O}(r,a;f,g) = S(r,g),$$

or if $k = 0$ and

$$\overline{N}\left(r,\frac{1}{f-a}\right) - \overline{N}_{0}(r,a;f,g) = S(r,f), \qquad \overline{N}\left(r,\frac{1}{g-a}\right) - \overline{N}_{0}(r,a;f,g) = S(r,g),$$

then we say f and g weakly share a with weight k. Here we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Now it is clear that weighted sharing and weakly weighted sharing are respectively scalings between IM, CM and "IM", "CM". Also weakly weighted sharing includes the definition of weighted sharing.

With the notion of weakly weighted sharing, we prove the following result which also gives a positive answer to Question C.

Theorem 4. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 14)$ be an integer such that $n+1$ is not divisible by* 3. *If $f^n(f^3-1)f'$ and $g^n(g^3-1)g'$ share "$(1,2)$", then $f \equiv g$.*

Without the notions of weighted sharing and weakly weighted sharing, we can prove the following result.

Theorem 5. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 14)$ be an integer such that $n+1$ is not divisible by* 3. *If*

$\bar{E}_{4)}(1, f^n(f^3-1)f') = \bar{E}_{4)}(1, g^n(g^3-1)g')$ *and* $E_{2)}(1, f^n(f^3-1)f') = E_{2)}(1, g^n(g^3-1)g')$, *then* $f \equiv g$.

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1 [15]. *Let f be a nonconstant meromorphic function, and let $a_0, a_1, a_2, \ldots, a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. *Let f and g be two nonconstant meromorphic functions, $n(>6)$ a positive integer and let $F = f^n(f^3-1)f'$, $G = g^n(g^3-1)g'$. If F and G share $(1, 0)$, then $S(r, f) = S(r, g)$.*

Proof. By Lemma 1 we have

$$\begin{aligned}(n+3)T(r, f) &= T(r, f^n(f^3-1)) + S(r, f) \leq T(r, F) + T(r, f') + S(r, f)\\ &\leq T(r, F) + 2T(r, f) + S(r, f).\end{aligned}$$

Therefore

$$T(r, F) \geq (n+1)T(r, f) + S(r, f).$$

By the second fundamental theorem, we have

$$\begin{aligned}T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-1}\right) + S(r, f)\\ &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^3-1}\right)\\ &\quad + \bar{N}\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{G-1}\right) + S(r, f)\\ &\leq 7T(r, f) + T(r, G) + S(r, f).\end{aligned}$$

Note that

$$T(r, G) \leq T(r, g^n(g^3-1)) + T(r, g') \leq (n+5)T(r, g) + S(r, g).$$

We deduce that

$$(n-6)T(r,f) \le (n+5)T(r,g) + S(r,f) + S(r,g).$$

It follows that the conclusion of Lemma 2 holds.

Lemma 3 [10]. *Let F and G be two nonconstant meromorphic functions. If F and G share $(1,2)$. Then one of the following cases holds:*

$$(1)\quad T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$; (2) $F \equiv G$; (3) $FG \equiv 1$.

Lemma 4 [17]. *Let f be a nonconstant meromorphic function. Then*

$$N\left(r,\frac{1}{f^{(k)}}\right) \le N\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 5 [11]. *Let f and g be two nonconstant meromorphic functions. Then $f^n(f^3-1)f'g^n(g^3-1)g' \not\equiv 1$, where n is a positive integer.*

Lemma 6. *Let $F^* = f^{n+1}\left(\frac{f^3}{n+4} - \frac{1}{n+1}\right)$, $G^* = g^{n+1}\left(\frac{g^3}{n+4} - \frac{1}{n+1}\right)$, where $n(\ge 4)$ is an integer and $n+1$ is not divisible by 3. If $F^* \equiv G^*$, then $f \equiv g$.*

Proof. Let $h = g/f$. If possible, suppose that h is not a constant. Since $F^* \equiv G^*$, it follows that

$$f^3 \equiv \frac{n+4}{n+1}\cdot\frac{h^{n+1}-1}{h^{n+4}-1}.$$

The above equality holds under the assumption $h^{n+1} \not\equiv 1$. If $h^{n+1} = h^{n+4} \equiv 1$, then we have the trivial equation $0 = 0$. So we must assume that $n+1$ is not divisible by 3. Let $d = (n+1, n+4)$. We can write $n+1 = dm_1$, $n+4 = dm_2$, where m_1, m_2 are two positive integers and $m_1 < m_2$. So $(n+4) - (n+1) = d(m_2 - m_1)$, that is, $d(m_2 - m_1) = 3$. Since d is a positive integer, we deduce that $d = 1$ or $d = 3$. Thus the number of the common zeros of $h^{n+1} - 1$ and $h^{n+4} - 1$ is at most 3 and $h^{n+4} - 1$ has at least $n+1$ zeros which are not the zeros of $h^{n+1} - 1$. Denote these $n+1$ zeros by u_k, $k = 1, 2, \dots n+1$. Since f^3 has no simple pole, it follows that $h - u_k = 0$ has no simple root for $k = 1, 2, \dots, n+1$. Hence $\Theta(u_k; h) \ge 1/2$ for $k = 1, 2, \dots, n+1 (\ge 5)$, which is impossible. Therefore h is a constant. If $h \ne 1$, it follows that f is a constant, which is not the case. So $h = 1$ and hence $f \equiv g$. This proves the lemma.

Lemma 7 [1]. *Let H be defined as above. If F and G share $(1,1)$ and $H \not\equiv 0$, then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)$$
$$+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 8 [20]. *Let* H *be defined as above. If* F *and* G *share* $(1,0)$ *and* $H \not\equiv 0$, *then*

$$N_E^{1)}\left(r,\frac{1}{F-1}\right) \le N(r,H) + S(r,F) + S(r,G).$$

Lemma 9 [20]. *Let* F *and* G *be two nonconstant meromorphic functions such that* F *and* G *share 1 IM. If* $H \not\equiv 0$, *then*

$$T(r,F) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + N_E^{1)}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{F-1}\right) - N_0\left(r,\frac{1}{F'}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,F) + S(r,G),$$

where $N_0(r,1/F')$ *denotes the counting function corresponding to the zeros of* F' *that are not the zeros of* F *and* $F-1$, $N_0(r,1/G')$ *denotes the counting function corresponding to the zeros of* G' *that are not the zeros of* G *and* $G-1$.

Lemma 10 [21]. *Let* F *and* G *be two nonconstant meromorphic functions such that* F *and* G *share the value* 1 *IM. Then*

$$\overline{N}_L\left(r,\frac{1}{F-1}\right) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,F),$$
$$\overline{N}_L\left(r,\frac{1}{G-1}\right) \le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,G).$$

Lemma 11 [3]. *Let* H *be defined as above. If* F *and* G *share "*$(1,2)$*" and* $H \not\equiv 0$, *then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G)$$
$$-\sum_{p=3}^{\infty}\overline{N}_{(p}\left(r,\frac{G}{G'}\right) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$.

Lemma 12 [2]. *Let H be defined as above. If $\overline{E}_{4)}(1,F)=\overline{E}_{4)}(1,G)$ and $E_{2)}(1,F)=E_{2)}(1,G)$ and $H\not\equiv 0$, then*

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}$$
$$+S(r,F)+S(r,G).$$

3. Proof of Theorem 1

We can prove $S(r,f)=S(r,g)$ by the same method as Lemma 2. Let

$$F=f^n(f^3-1)f', \qquad G=g^n(g^3-1)g', \tag{1}$$

and

$$F^*=f^{n+1}\left(\frac{f^3}{n+4}-\frac{1}{n+1}\right), \qquad G^*=g^{n+1}\left(\frac{g^3}{n+4}-\frac{1}{n+1}\right).$$

Thus we obtain that F and G share $(1,2)$. If possible, let the case (1) of Lemma 3 occur, that is

$$T(r,F)\le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G). \tag{2}$$

Moreover, by Lemma 1, we have

$$T(r,F^*)=(n+4)T(r,f)+S(r,f), \qquad T(r,G^*)=(n+4)T(r,g)+S(r,g). \tag{3}$$

Since $(F^*)'=F$, we deduce

$$m\left(r,\frac{1}{F^*}\right)\le m\left(r,\frac{1}{F}\right)+S(r,f), \tag{4}$$

and by the first fundamental theorem

$$T(r,F^*)\le T(r,F)+N\left(r,\frac{1}{F^*}\right)-N\left(r,\frac{1}{F}\right)+S(r,f). \tag{5}$$

Note that

$$N\left(r,\frac{1}{F^*}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right), \tag{6}$$

$$N\left(r,\frac{1}{F}\right)=nN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f^3-1}\right). \tag{7}$$

It follows from (5)–(7) that

$$T(r,F^*) \leq T(r,F) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right) - N\left(r,\frac{1}{f'}\right) - N\left(r,\frac{1}{f^3-1}\right) + S(r,f). \tag{8}$$

It follows from (1) that

$$N_2\left(r,\frac{1}{F}\right) + N_2(r,F) \leq 2\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f'}\right) + N_2\left(r,\frac{1}{f^3-1}\right) + 2\overline{N}(r,f), \tag{9}$$

$$N_2\left(r,\frac{1}{G}\right) + N_2(r,G) \leq 2\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g'}\right) + N_2\left(r,\frac{1}{g^3-1}\right) + 2\overline{N}(r,g). \tag{10}$$

From (2), (8), (9) and (10) we obtain

$$T(r,F^*) \leq 3N\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right) + 2N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g'}\right) + N\left(r,\frac{1}{g^3-1}\right) + 2\overline{N}(r,g) + S(r,f). \tag{11}$$

By Lemma 4 we have

$$N\left(r,\frac{1}{g'}\right) \leq N(r,g) + N\left(r,\frac{1}{g}\right) \leq 2T(r,g) + S(r,g). \tag{12}$$

We have from (3), (11) and (12) that

$$(n-4)T(r,f) \leq 9T(r,g) + S(r,g). \tag{13}$$

In the same manner as above, we have

$$(n-4)T(r,g) \leq 9T(r,f) + S(r,g). \tag{14}$$

Therefore by (13) and (14), we obtain that $n \leq 13$, which contradicts $n \geq 14$. Thus by Lemma 3, we get $F \equiv G$ or $FG \equiv 1$. If $FG \equiv 1$, that is

$$f^n(f^3-1)f'g^n(g^3-1)g' \equiv 1.$$

By Lemma 5, we get a contradiction. If $F \equiv G$, that is

$$F^* = G^* + c, \tag{15}$$

where c is a constant. It follows that $T(r,f) = T(r,g) + S(r,f)$. Suppose that $c \neq 0$, by the second fundamental theorem, we have

$$(n+4)T(r,g) = T(r,G^*) < \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{G^*+c}\right) + \overline{N}(r,G^*) + S(r,g)$$
$$\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^3-\frac{n+4}{n+1}}\right) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right)$$
$$+ \overline{N}\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right) + S(r,f) \leq 9T(r,f) + S(r,f), \tag{16}$$

which contradicts the assumption. Therefore $F^* \equiv G^*$. Thus by Lemma 6, we have $f \equiv g$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

We can prove $S(r,f) = S(r,g)$ by the same method as Lemma 2. Let

$$F = f^n(f^3-1)f', \qquad G = g^n(g^3-1)g', \tag{17}$$

and

$$F^* = f^{n+1}\left(\frac{f^3}{n+4} - \frac{1}{n+1}\right), \qquad G^* = g^{n+1}\left(\frac{g^3}{n+4} - \frac{1}{n+1}\right).$$

Thus we obtain that F and G share $(1,1)$. If possible, we suppose that $H \not\equiv 0$. Thus, by Lemma 7, we have

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)$$
$$+ \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G). \tag{18}$$

It follows from (17) that

$$N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F)$$
$$\leq 2\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f'}\right) + N_2\left(r,\frac{1}{f^3-1}\right) + 2\overline{N}(r,f)$$
$$+ \frac{1}{2}\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{2}\overline{N}\left(r,\frac{1}{f^3-1}\right) + \frac{1}{2}\overline{N}\left(r,\frac{1}{f'}\right) + \frac{1}{2}\overline{N}(r,f), \tag{19}$$

$$N_2\left(r,\frac{1}{G}\right) + N_2(r,G) \leq 2\overline{N}\left(r,\frac{1}{g}\right) + N_2\left(r,\frac{1}{g'}\right) + N_2\left(r,\frac{1}{g^3-1}\right) + 2\overline{N}(r,g). \tag{20}$$

Proceeding as in the proof of Theorem 1, we also have (8) and (12). We have from (18)–(20) and (8) that

$$\begin{aligned} T(r,F^*) &= (n+4)T(r,f)+S(r,f) \\ &\le 4N\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+\frac{1}{2}N\left(r,\frac{1}{f^3-1}\right) \\ &\quad + N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right)+3N\left(r,\frac{1}{g}\right)+3\overline{N}(r,g) \\ &\quad + N\left(r,\frac{1}{g^3-1}\right)+S(r,f). \end{aligned} \tag{21}$$

Thus we have

$$\left(n-\frac{15}{2}\right)T(r,f)\le 9T(r,g)+S(r,f). \tag{22}$$

In the same manner as above, we have

$$\left(n-\frac{15}{2}\right)T(r,g)\le 9T(r,f)+S(r,f). \tag{23}$$

From (22) and (23) we obtain that $n\le 16$, which contradicts $n\ge 17$. Therefore, $H\equiv 0$. That is

$$\frac{F''}{F'}-2\frac{F'}{F-1}\equiv\frac{G''}{G'}-2\frac{G'}{G-1}. \tag{24}$$

By integration, we have from (24)

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \tag{25}$$

where $A(\ne 0)$ and B are constants. It follows from (25) that F and G share 1 CM. Thus by Theorem 1, we get $f\equiv g$. This completes the proof of Theorem 2.

5. Proof of Theorem 3

We can prove $S(r,f)=S(r,g)$ by the same method as Lemma 2. Let

$$F=f^n(f^3-1)f', \qquad G=g^n(g^3-1)g', \tag{26}$$

and

$$F^* = f^{n+1}\left(\frac{f^3}{n+4} - \frac{1}{n+1}\right), \qquad G^* = g^{n+1}\left(\frac{g^3}{n+4} - \frac{1}{n+1}\right).$$

Thus we obtain that F and G share $(1,0)$, that is F and G share 1 IM. If possible, we suppose that $H \not\equiv 0$. From Lemma 8, we have

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \leq N(r,H) + S(r,F) + S(r,G). \tag{27}$$

Also by the definition of H, we have

$$N(r,H) \leq \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right). \tag{28}$$

By (27), (28) and Lemma 9, we have

$$T(r,F) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \tag{29}$$

By Lemma 10 and (29), we have

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,F) + S(r,G). \tag{30}$$

It follows from (26) that

$$N_2\left(r,\frac{1}{F}\right) + 4\overline{N}(r,F) + 2\overline{N}\left(r,\frac{1}{F}\right) \leq 4\overline{N}\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{f'}\right) + 2\overline{N}\left(r,\frac{1}{f'}\right) + N_2\left(r,\frac{1}{f^3-1}\right) + 2\overline{N}\left(r,\frac{1}{f^3-1}\right) + 4\overline{N}(r,f), \tag{31}$$

$$N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\leq 3\overline{N}\left(r,\frac{1}{g}\right)+2N_2\left(r,\frac{1}{g'}\right)$$
$$+2N_2\left(r,\frac{1}{g^3-1}\right)+3\overline{N}(r,g). \qquad (32)$$

Proceeding as in the proof of Theorem 1, we also have (8). We have from (30), (31), (32) and (8) that

$$T(r,F^*)\leq 7N\left(r,\frac{1}{f}\right)+6\overline{N}(r,f)+2N\left(r,\frac{1}{f^3-1}\right)+N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right)$$
$$+5N\left(r,\frac{1}{g}\right)+5\overline{N}(r,g)+2N\left(r,\frac{1}{g^3-1}\right)+S(r,f). \qquad (33)$$

Thus we have

$$(n-18)T(r,f)\leq 16T(r,g)+S(r,f). \qquad (34)$$

In the same manner as above, we have

$$(n-18)T(r,g)\leq 16T(r,f)+S(r,f). \qquad (35)$$

From (34) and (35) we obtain that $n\leq 34$, which contradicts $n\geq 35$. Therefore $H\equiv 0$. Proceeding as in the proof of Theorem 2, we get the conclusion. This completes the proof of Theorem 3.

6. Proof of Theorem 4

We can prove $S(r,f)=S(r,g)$ by the same method as Lemma 2. Let

$$F=f^n(f^3-1)f', \qquad G=g^n(g^3-1)g', \qquad (36)$$

and

$$F^*=f^{n+1}\left(\frac{f^3}{n+4}-\frac{1}{n+1}\right), \qquad G^*=g^{n+1}\left(\frac{g^3}{n+4}-\frac{1}{n+1}\right).$$

Thus we obtain that F and G share "$(1,2)$". If possible, we suppose that $H\not\equiv 0$. Then by Lemma 11 we have

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G). \qquad (37)$$

It follows from (36) that

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\le 2\overline{N}\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f'}\right)+N_2\left(r,\frac{1}{f^3-1}\right)+2\overline{N}(r,f), \quad (38)$$

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le 2\overline{N}\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{g'}\right)+N_2\left(r,\frac{1}{g^3-1}\right)+2\overline{N}(r,g). \quad (39)$$

Proceeding as in the proof of Theorem 1, we also have (8). From (37), (38), (39) and (8) we obtain

$$\begin{aligned} T(r,F^*) &= (n+4)T(r,f)+S(r,f) \\ &\le 3N\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right)+2N\left(r,\frac{1}{g}\right) \\ &\quad +N\left(r,\frac{1}{g'}\right)+N\left(r,\frac{1}{g^3-1}\right)+2\overline{N}(r,g)+S(r,f). \end{aligned} \quad (40)$$

By Lemma 4 we have

$$N\left(r,\frac{1}{g'}\right)\le N(r,g)+N\left(r,\frac{1}{g}\right)\le 2T(r,g)+S(r,g). \quad (41)$$

We have from (40) and (41) that

$$(n-4)T(r,f)\le 9T(r,g)+S(r,g). \quad (42)$$

In the same manner as above, we have

$$(n-4)T(r,g)\le 9T(r,f)+S(r,g). \quad (43)$$

Therefore by (42) and (43), we obtain that $n\le 13$, which contradicts $n\ge 14$. Therefore $H\equiv 0$. Proceeding as in the proof of Theorem 2, we get the conclusion. This completes the proof of Theorem 4.

7. Proof of Theorem 5

We can prove $S(r,f)=S(r,g)$ by the same method as Lemma 2. Let

$$F=f^n(f^3-1)f', \qquad G=g^n(g^3-1)g', \quad (44)$$

and

$$F^*=f^{n+1}\left(\frac{f^3}{n+4}-\frac{1}{n+1}\right), \qquad G^*=g^{n+1}\left(\frac{g^3}{n+4}-\frac{1}{n+1}\right).$$

Thus we obtain that $\overline{E}_{4)}(1,f^n(f^3-1)f')=\overline{E}_{4)}(1,g^n(g^3-1)g')$ and $E_{2)}(1,f^n(f^3-1)f')=E_{2)}(1,g^n(g^3-1)g')$. If possible, we suppose that $H\not\equiv 0$. Then by Lemma 12 we have

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}$$
$$+S(r,F)+S(r,G). \tag{45}$$

It follows from (44) that

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\le 2\overline{N}\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f'}\right)+N_2\left(r,\frac{1}{f^3-1}\right)+2\overline{N}(r,f), \tag{46}$$

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le 2\overline{N}\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{g'}\right)+N_2\left(r,\frac{1}{g^3-1}\right)+2\overline{N}(r,g). \tag{47}$$

Proceeding as in the proof of Theorem 1, we have

$$T(r,F^*)\le T(r,F)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right)-N\left(r,\frac{1}{f'}\right)$$
$$-N\left(r,\frac{1}{f^3-1}\right)+S(r,f), \tag{48}$$

and

$$T(r,G^*)\le T(r,G)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g^3-\frac{n+4}{n+1}}\right)-N\left(r,\frac{1}{g'}\right)$$
$$-N\left(r,\frac{1}{g^3-1}\right)+S(r,g). \tag{49}$$

From (45), (46), (47), (48) and (49) we obtain

$$T(r,F^*)+T(r,G^*)\le 5N\left(r,\frac{1}{f}\right)+4\overline{N}(r,f)+N\left(r,\frac{1}{f^3-\frac{n+4}{n+1}}\right)+N\left(r,\frac{1}{f'}\right)$$
$$+N\left(r,\frac{1}{f^3-1}\right)+5N\left(r,\frac{1}{g}\right)+4\overline{N}(r,g)+N\left(r,\frac{1}{g^3-\frac{n+4}{n+1}}\right)$$
$$+N\left(r,\frac{1}{g'}\right)+N\left(r,\frac{1}{g^3-1}\right)+S(r,f). \tag{50}$$

By Lemma 4 we have

$$N\left(r,\frac{1}{f'}\right)\le N(r,f)+N\left(r,\frac{1}{f}\right)\le 2T(r,f)+S(r,f), \tag{51}$$

$$N\left(r,\frac{1}{g'}\right)\le N(r,g)+N\left(r,\frac{1}{g}\right)\le 2T(r,g)+S(r,g). \tag{52}$$

We have from (50), (51) and (52) that

$$(n-13)T(r,f)+(n-13)T(r,g)\le S(r,f)+S(r,g). \tag{53}$$

We obtain that $n \le 13$, which contradicts $n \ge 14$. Therefore $H \equiv 0$. Proceeding as in the proof of Theorem 2, we get the conclusion. This completes the proof of Theorem 5.

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References

[1] A. Banerjee, Meromorphic functions sharing one value, Int. J. Math. Math. Sci. 22 (2005), 3587–3598.

[2] A. Banerjee, On uniqueness of meromorphic functions when two differential monomials share one value, Bull. Korean Math. Soc. 44(4) (2007), 607–622.

[3] A. Banerjee and S. Mukherjee, Uniqueness of meromorphic functions concerning differential monomial sharing the same value, Bull. Math. Soc. Sci. Math. Roumanie. 50 (2007), 191–206.

[4] C. Y. Fang and M. L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl. 44 (2002), 607–617.

[5] M. L. Fang and W. Hong, A unicity theorem for entire functions concerning differential polynomials, Indian J. Pure Appl. Math. 32 (2001), 1343–1348.

[6] G. Frank and M. Reinders, A unique range set for meromorphic functions with 11 elements, Complex Variables Theory Appl. 37 (1998), 185–193.

[7] F. Gross, Factorization of meromorphic functions and some open problems, Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977, pp. 51–67.

[8] W. K. Hayman, Meromorphic Functions, Clarendon, Oxford, 1964.

[9] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J. 161 (2001), 193–206.

[10] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Variables Theory Appl. 46 (2001), 241–253.

[11] I. Lahiri and R. Pal, Nonlinear differential polynomials sharing 1-points, Bull. Korean Math. Soc. 43(1) (2006), 161–168.

[12] P. Li and C. C. Yang, On the unique range set of meromorphic functions, Proc. Amer. Math. Soc. 124(1) (1996), 177–185.

[13] S. H. Lin and W. C. Lin, Uniqueness of meromorphic functions concerning weakly weighted sharing, Kodai Math. J. 29 (2006), 269–280.

[14] W. C. Lin and H. X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math. 35 (2004), 121–132.

[15] C. C. Yang, On deficiencies of differential polynomials, Math. Z. 125 (1972), 107–112.

[16] L. Yang, Value distribution theory, Springer-Verlag, Berlin, 1993.

[17] H. X. Yi, Uniqueness of meromorphic functions and a question of C. C. Yang, Complex Variables Theory Appl. 14 (1990), 169–176.
[18] H. X. Yi, A question of Gross and the uniqueness of entire functions, Nagoya Math. J. 138 (1995), 169–177.
[19] H. X. Yi, On uniqueness theorems for meromorphic functions, Chinese Math. Ann. 17A(4) (1996), 397–404.
[20] H. X. Yi, Uniqueness theorems for meromorphic functions whose n-th derivatives share the same 1-points, Complex Variables Theory Appl. 34 (1997), 421–436.
[21] H. X. Yi, Meromorphic functions that share one or two values II, Kodai Math. J. 22 (1999), 264–272.

Chao Meng
Department of Mathematics
Shandong University
Jinan 250100, *P.R. China*
E-mail: mengchao@mail.sdu.edu.cn