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On locally pseudo-valuation domains

Mitsuo KANEMITSU

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Introduction

The purpose of this paper is to study locally pseudo-valuation domains which are quasinormal domains. In particular, we give some results on locally pseudo-valuation semigroup rings. Throughout this paper all rings are assumed to be commutative with identity.

Pseudo-valuation domains (shortly, PVD's) were introduced by J. R. Hedstrom and E. G. Houston in [9]. Also, locally pseudo-valuation domains (shortly, LPVD's) were introduced by D. E. Dobbs and M. Fontana in [4]. Examples of LPVD's are all Prüfer domains, some instances of the $D+M$ construction (cf. [5]) and certain subrings of a number field.

In the first section, we will consider the relation between the LPVD's and i-domains which were defined by I. J. Papick. In particular, we shall characterize an LPVD with the property that its integral closure is a Prüfer domain in terms of seminormality. We also note that a one dimensional Noetherian domain with finite integral closure is an LPVD if and only if it is quasinormal.

In the final section, we will give the main result. Let R be an integral domain, let S be a commutative monoid, with operation written additively, and let $R[S]$ be a monoid ring of S over R. We give a result on the problem of determining conditions under which $R[S]$ is an LPVD.

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Notation and terminology

Let R be a commutative ring with identity. We let $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ stand for the set of all prime ideals of R and that of all maximal ideals of R respectively. An overring of R is a subring between R and its total quotient ring $Q(R)$. $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{Z}_0$ and $\boldsymbol{Q}_0$ denote respectively the ring of rational integers, the field of rational numbers, the set of nonnegative rational integers and the set of nonnegative rational numbers. We denote by $\bar{R}$ the integral closure of R and denote by (R, M) the quasilocal ring R with the maximal ideal M.

§ 1. The relation between LPVD's and i-domains

First we shall give several definitions and known results which we shall need later. Let R be an integral domain and K be its quotient field. A prime ideal P of R is called *strongly prime* if P satisfies the following condition: For $x, y \in K$, $xy \in P$ implies $x \in P$ or $y \in P$ (cf. [9] and [4]). An integral domain R is called a *pseudo-valuation domain* (or, in short, a *PVD*) if every prime ideal of R is strongly prime. An integral domain R is called a *locally pseudo-valuation domain* (or, in short, an *LPVD*) in case R_P is a PVD for every prime ideal P of R. If (R, M) is a quasilocal domain, then the following statements are valid.

(i) R is a PVD if and only if it has a valuation overring V with maximal ideal M. In this case, V is unique and is called the *associated valuation domain* of R.

(ii) If R is a PVD and P is a nonmaximal prime ideal of R, then R_P is a valuation domain.

(iii) If R is a Noetherian PVD, then the Krull dimension of R is one at most ([9, Theorem 2.7, Proposition 2.6 and Proposition 3.2]).

We say that R is an *i-domain* if, for every prime ideal P of R and every overring T of R, at most one prime ideal of T lies over P. It is well known that R is an i-domain if and only if $\bar{R}$ is a Prüfer domain and $\mathrm{Spec}(\bar{R}) \to \mathrm{Spec}(R)$ is injective ([12, Theorem 3.4]).

An integral domain R is called *seminormal* (resp. *quasinormal*) if the canonical homomorphism $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X])$ (resp. $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X, X^{-1}])$) is an isomorphism, where X is an indeterminate. R is seminormal if and only if for $x \in Q(R)$, $x \in R$ whenever $x^2, x^3 \in R$ (cf. [8] or [14]). We say that ${}^{+}_{\bar{R}}R$ (or, in short, ${}^{+}R$) is the *seminormalization* of R if ${}^{+}R$ is the largest ring C between R and $\bar{R}$ such that the canonical mapping f: $\mathrm{Spec}(C) \to \mathrm{Spec}(R)$ is bijective and, for $P \in \mathrm{Spec}(C)$, the residue field of P coincides with that of $\mathfrak{p} = f(P)$. We see that R is seminormal if and only if ${}^{+}R = R$ (cf. [15]).

LEMMA 1.1. *Let R be an integral domain with the quotient field K. Then the following statements hold.*

(1) *If R is a PVD and $\bar{R}$ is a Prüfer domain, then $\bar{R}$ is the associated valuation domain of R.*

(2) *R is a PVD and $\bar{R}$ is the associated valuation domain of R if and only if $x^{-1} \in \bar{R}$ whenever $x \in K \setminus R$.*

PROOF. The proof of (1) is easy. We shall give the proof of (2). R is a PVD if and only if R is a quasilocal domain and there is a valuation overring V, which dominates R and for which $x^{-1} \in V$ whenever $x \in K \setminus R$. Hence, the "only if" part is clear. Conversely, assume that $x^{-1} \in \bar{R}$ whenever $x \in K \setminus R$. This implies that $\bar{R}$ is a valuation domain; therefore R is a quasilocal domain. The condition also shows that the maximal ideal of R coincides with that of $\bar{R}$, and the assertion follows from (i). Q. E. D.

We can now give the following theorem.

THEOREM 1.2. *Let R be an integral domain. Then R is an LPVD and $\bar{R}$ is a Prüfer domain if and only if R is a seminormal i-domain and every prime ideal in the support of the R-module $\bar{R}/R$ is maximal.*

PROOF. Suppose first that R is an LPVD and $\bar{R}$ is a Prüfer domain. Then R is seminormal by [4, Remark 2.4 (a)]. It is easy to see that $\bar{R}_M$ is a Prüfer domain for every maximal ideal M of R. It follows from Lemma 1.1 (1) that $\bar{R}_M$ is the associated valuation domain of R_M. Hence R is an i-domain. Also, if P is a nonmaximal prime ideal of R, R_P is a valuation domain by (ii). Hence every prime ideal in the support of $\bar{R}/R$ is maximal.

Conversely, assume that R is a seminormal i-domain and every prime ideal in the support of $\bar{R}/R$ is maximal. Since R is an i-domain, $\bar{R}$ is a Prüfer domain. We assert that R is an LPVD. To prove this, it is enough to assume that (R, M) is a quasilocal domain such that $\bar{R}$ is a valuation domain with the maximal ideal $\bar{M}$. To show that R is a PVD, it sufficies to show $M = \bar{M}$ by (i). $R + \bar{M}$ is an intermediate ring between R and $\bar{R}$. Since $R + \bar{M}/\bar{M} \cong R/M$ and R is an i-domain, $\bar{M}$ is the maximal ideal of $R + \bar{M}$. Since every prime ideal in the support of $\bar{R}/R$ is maximal and R is an i-domain, the canonical mapping f: Spec $(R + \bar{M}) \to$ Spec(R) is bijective and, for $\mathfrak{p} \in \mathrm{Spec}(R)$ and $P \in \mathrm{Spec}(R + \bar{M})$ with $f(P) = \mathfrak{p}$, the residue field of P coincides with that of $\mathfrak{p}$. Therefore we see that $R + \bar{M}$ is an intermediate ring between ${}^{+}R$ and R. Since R is a seminormal ring, we have $R = R + \bar{M}$. Hence $M = \bar{M}$ and so R is a PVD. Q. E. D.

We say that an integral domain R is a *finite-conductor domain* if $Rx \cap Ry$ is always a finitely generated ideal for $x, y \in R$. We also say that an integral domain R is a *locally finite-conductor domain* if R_M is a finite-conductor domain for every maximal ideal M of R.

COROLLARY 1.3. *Let R be a locally finite-conductor domain. Then R is an LPVD if and only if R is a seminormal i-domain and every prime ideal in the support of the R-module $\bar{R}/R$ is maximal.*

PROOF. Since "if" part is clear by Theorem 1.2, it is enough to prove the "only if" part. Since R_M is a finite-conductor PVD for a maximal ideal M of R, it follows from [3, Proposition 4.2] that $\bar{R}_M$ is a valuation domain. Hence $\bar{R}$ is a Prüfer domain. The assertion now follows immediately from Theorem 1.2.
Q. E. D.

COROLLARY 1.4. *Let R be a one dimensional integral domain. Then R is an LPVD and $\bar{R}$ is a Prüfer domain if and only if R is a seminormal i-domain.*

PROOF. Since R is a one dimensional integral domain, the condition that every prime ideal in the support of $\bar{R}/R$ is maximal is automatically satisfied and the assertion follows immediately from Theorem 1.2. Q. E. D.

COROLLARY 1.5. *Let R be a Noetherian domain. Then R is an LPVD if and only if it is a seminormal i-domain.*

PROOF. If R is a Noetherian i-domain, then $\dim R \leqq 1$ by [13, Corollary 2.16]. Therefore we have only to prove the "only if" part. By (iii) and the Krull-Akizuki Theorem, $\bar{R}$ is a Dedekind domain. Thus the assertion follows immediately from Corollary 1.4. Q. E. D.

REMARK 1.6. If we drop the assumption that every prime ideal in the support of $\bar{R}/R$ is maximal, then the "if" part of Theorem 1.2 fails to hold. For example, let k be a field, t be an indeterminate and $k((t))$ be the quotient field of the formal power series ring $k[[t]]$. We consider the polynomial ring $k((t))[X]$ over $k((t))$. If $\psi: k((t))[X]_{(X)} \rightarrow k((t))$ is the natural homomorphism, then $\psi^{-1}(k[[t^2]])$ is a seminormal i-domain. However it is not an LPVD.

REMARK 1.7. Let R be a Noetherian domain. Then R is an i-domain if and only if ${}^{+}R$ is an LPVD.

REMARK 1.8. Let R be a one dimensional Noetherian domain such that $\bar{R}$ is a finite R-module. Then R is an LPVD if and only if it is quasinormal. In fact, it is an immediate consequence of [7, Theorem 4.5].

REMARK 1.9. We give an example of an integral domain which is not quasinormal but seminormal. Let R be the ring $k[X, Y]/(Y^2 - X^2 - X^3) = k[x, y]$ where k is a field. Since $\bar{R} = k[y/x]$ and $\mathrm{Spec}(\bar{R}) \rightarrow \mathrm{Spec}(R)$ is not injective, R is not an i-domain but it is seminormal. By Corollary 1.5, R is not an LPVD. Also, it is not quasinormal by Remark 1.8.

§ 2. Group rings, monoid rings and LPVD's

Throughout this section, S will stand for a monoid, namely a semi-group with identity. Moreover we assume that S is commutative; the operation is written additively and 0 is the identity of S. Let R be an integral domain and $R[S]$ be the monoid ring of S over R. We follow the notation of Northcott [11, p. 128] in writing elements of $R[S]$ as "polynomials" $r_1 X^{s_1} + \cdots + r_n X^{s_n}$ with coefficients in R and exponents in S.

We summarize here some terminologies and known facts which will be used later. A group G is *locally cyclic* if every finitely generated subgroup of G is cyclic. For example, $\boldsymbol{Q}$ is such a group. A monoid ring $R[S]$ is an integral domain if and only if R is an integral domain and S is a torsion-free cancellative monoid ([6, Theorem 8.1]). Also, a monoid ring $R[S]$ is Noetherian if and only if R is a Noetherian ring and S is finitely generated ([6, Theorem 7.7]). We understand that a submonoid of a monoid always contains the identity of the monoid. Let S be a submonoid of a monoid T. We say that an element t of T is *integral* over S if $nt \in S$ for some positive integer n. The set $\bar{S}$ of elements t of T which are integral over S is a submonoid of T containing S. This submonoid is called the *integral closure of S in T*. In case S is cancellative and T is the quotient group of S, the submonoid $\bar{S}$ is called simply the *integral closure* of S and we say that S is *normal* if $S=\bar{S}$.

First we give the following lemma.

LEMMA 2.1. *Let k be a field, S be a non-zero submonoid of $\boldsymbol{Q}$ and G be the quotient group of S. Put $G_0 = G \cap \boldsymbol{Q}_0$. Then the following statements hold.*

(1) *S is a subgroup of $\boldsymbol{Q}$ if and only if $\bar{S}$ is a subgroup of $\boldsymbol{Q}$.*

(2) *If S is contained in $\boldsymbol{Q}_0$, then G_0 is integral over S and hence coincides with the integral closure of S.*

(3) *$k[G]$ and $k[G_0]$ are one dimensional Bézout domains* ([6, Theorem 13.5]).

PROOF. (1) The proof follows immediately from the fact that if $\bar{S}$ is a submonoid of $\boldsymbol{Q}$ containing both positive and negative rationals, then $\bar{S}$ is a subgroup of $\boldsymbol{Q}$ ([6, Theorem 2.9]).

(2) Let g be any element of G_0. Since $g \in \boldsymbol{Q}_0$, there exists some positive integer n such that $ng \in \boldsymbol{Z}_0$. On the other hand, since $S \subset \boldsymbol{Q}_0$, there exists some positive integer m contained in S. Hence $mng \in S$, and so $g \in \bar{S}$. Q. E. D.

We give here the main result.

THEOREM 2.2. *Let G be a non-zero abelian group. Then the group ring $R[G]$ is an LPVD if and only if R is a field and G is isomorphic to a subgroup of $\boldsymbol{Q}$. Moreover, in this case, $R[G]$ is a one dimensional Bézout domain.*

PROOF. Suppose that $R[G]$ is an LPVD. Let $\mathfrak{m}$ be a maximal ideal of R, $\tilde{\phi}\colon R[G] \to R$ be the augmentation mapping defined by $\tilde{\phi}(\sum r_g X^g) = \sum r_g$, $g \in G$, and $\psi\colon R \to R/\mathfrak{m}$ be the natural homomorphism. Put $\phi = \psi \cdot \tilde{\phi}$. Then $\operatorname{Ker}\phi = \mathfrak{m}R[G] + \langle X^g - 1;\, g \in G\rangle$, where $\langle X^g - 1;\, g \in G\rangle$ is the ideal generated by $X^g - 1$, $g \in G$. We now put $M = \operatorname{Ker}\phi$. Then M is a maximal ideal of $R[G]$. Since $R[G]_M$ is a PVD, there exists the associated valuation overring (V, N) where $N = MR[G]_M$. Assume that R is not a field. Then there exists some non-zero element a of $\mathfrak{m}$. Let g be a fixed non-zero element of G. Since V is a valuation domain, one of the following three cases occurs:

(i) $aV \subsetneqq (X^g-1)V$ (ii) $aV \supsetneqq (X^g-1)V$ (iii) $a^2V \subsetneqq aV=(X^g-1)V$.

For the case (i), since $a/(X^g-1)\in N$, it follows that $af=(X^g-1)h$ for some $f=\sum a_g X^g \notin M$ and some $h\in M$. Since $\tilde{\phi}(af)=\tilde{\phi}((X^g-1)h)=0$, we have $a(\sum a_g)=0$. However, since both a and $\sum a_g$ are not zero, this is a contradiction. Hence the case (i) does not occur. The case (iii) also does not occur similarly. For the case (ii), $(X^g-1)f=ah$ for some $f=\sum a_g X^g \notin M$ and $h\in M$. Since $\sum a_g \notin \mathfrak{m}$ and $a\in \mathfrak{m}$, $(X^g-1)\bar{f}=0$ in $(R/\mathfrak{m})[G]$, where $\bar{f}(X)$ is the polynomial in $(R/\mathfrak{m})[G]$ obtained from $f(X)$ by reduction modulo $\mathfrak{m}$. Since $R/\mathfrak{m}$ is a field and G is a torsion-free group, $(R/\mathfrak{m})[G]$ is an integral domain. Hence $\bar{f}(X)=0$, which contradicts the fact $f\notin M$. Thus we have proved that R is a field. Let F be a free subgroup of G such that G/F is a torsion group. Then $R[G]$ is an integral extension of $R[F]$ and since F is a free group, $R[F]$ is a normal domain. The fact that $R[G]$ is an LPVD implies that $R[F]$ is also an LPVD by [4, Proposition 2.7 (2)]. Suppose that $F=F_1\oplus F_2$ for two subgroups F_1, F_2 of F. Since $R[F]=R[F_1][F_2]$ is an LPVD, this implies that $R[F_1]$ is a field, and so $F_1=(0)$. Hence F is indecomposable. Therefore $F\cong \boldsymbol{Z}\subset \boldsymbol{Q}$. Thus G is a subgroup of $\boldsymbol{Q}$. By Lemma 2.1 (3), $R[G]$ is a one dimensional Bézout domain.

Conversely, suppose that R is a field and G is a subgroup of $\boldsymbol{Q}$. Then $R[G]$ is a Bézout domain by Lemma 2.1 (3), and hence it is a Prüfer domain. Hence $R[G]$ is an LPVD. Q.E.D.

Remark 2.3. (1) Let G be a non-zero abelian group. The fact that the group ring $R[G]$ is an LPVD does not necessarily imply that $R[G]$ is a Dedekind domain. For example, $\boldsymbol{Q}[\boldsymbol{Q}]$ is not Noetherian, although it is an LPVD.

(2) For G as above, if the group ring $R[G]$ is an LPVD, then G is indecomposable and the Picard group $\operatorname{Pic}(R[G])$ is (0) by Theorem 2.2. In particular, $\boldsymbol{Q}[\boldsymbol{Q}\oplus\boldsymbol{Q}]$ is not an LPVD.

Corollary 2.4. *Let R be an integral domain of characteristic p, G be a non-zero abelian group and F be a free subgroup of G such that G/F is a torsion group. Assume that either $p=0$ or $p\neq 0$ and $(G/F)_p$ is finite, where $(G/F)_p$ denotes the p-primary component of G/F. We also assume that* $\operatorname{Max}(R[G])$ *is a Noetherian space. Then $R[G]$ is an LPVD if and only if R is a field and $R[G]$ is isomorphic to $R[X, X^{-1}]$.*

Proof. We have only to prove the "only if" part of the corollary. We see that R is a field and $R[G]$ is a one dimensional Bézout domain by Theorem 2.2. Since $\operatorname{Spec}(R[G])$ is a Noetherian space, it is well known that, P being a non-zero prime ideal, $P=\sqrt{Q}$ for some principal ideal Q of $R[G]$. Since P is a maximal ideal, it follows that Q is a primary ideal. Since $R[G]$ is a locally Noetherian domain by [1, Theorem A], we see that $R[G]_P$ is a discrete valuation domain.

Since $QR[G]_P = P^m R[G]_P$ for some positive integer m, we have $Q = P^m R[G]_P \cap R[G] = P^m$, and so P is an invertible ideal. Hence $R[G]$ is a Dedekind domain; this implies that G is a finitely generated indecomposable group. Thus we can see that $\boldsymbol{Z} \cong G$. Hence $R[G] \cong R[X, X^{-1}]$. Q.E.D.

COROLLARY 2.5. *Suppose that G is a non-zero abelian group. Then the following statements hold.*

(1) *$R[G]$ is an LPVD if and only if $R[G]$ is an i-domain.*

(2) *When G is finitely generated, $R[G]$ is an LPVD if and only if R is a field and $R[G]$ is isomorphic to $R[X, X^{-1}]$.*

PROOF. (1) Suppose that $R[G]$ is an LPVD. Then $R[G]$ is an i-domain by Theorem 2.2. Conversely, suppose that $R[G]$ is an i-domain. Then, since the integral closure of $R[G]$ is $\bar{R}[G]$, $\bar{R}[G]$ is a Prüfer domain. Hence $\bar{R}$ is a field and G is a subgroup of $\boldsymbol{Q}$ by Theorem 2.2. Thus R is a field. Again, by Theorem 2.2, we see that $R[G]$ is an LPVD.

(2) Suppose that $R[G]$ is an LPVD. Since G is finitely generated and a subgroup of $\boldsymbol{Q}$ by Theorem 2.2, $G \cong \boldsymbol{Z}$. The converse is clear. Q.E.D.

We now proceed to the case of monoid domains; we determine the structure of monoid domains which are LPVD's.

THEOREM 2.6. *Let S be a non-zero monoid. Then $R[S]$ is an LPVD if and only if R is a field and to within isomorphism, S is either a subgroup of $\boldsymbol{Q}$ or a normal submonoid of $\boldsymbol{Q}_0$. Moreover, in this case, $R[S]$ is a one dimensional Bézout domain.*

PROOF. First, we shall prove the "only if" part. Let $R[S]$ be an LPVD and G be the quotient group of S. Then we have $R[G] = R[S]_T$, where $T = \{X^s; s \in S\}$. This implies that $R[G]$ is an LPVD and therefore, we can see that R is a field and G is a subgroup of $\boldsymbol{Q}$; in particular S is a submonoid of $\boldsymbol{Q}$. If S contains both positive and negative rationals, then S is a subgroup of $\boldsymbol{Q}$ and $S = G$ by [6, Theorem 2.9]. Hence, if S is not a subgroup of $\boldsymbol{Q}$, then either $S \subset \boldsymbol{Q}_0$ or $-S \subset \boldsymbol{Q}_0$. Thus S is isomorphic to a submonoid of $\boldsymbol{Q}_0$, and so we may assume that $S \subset \boldsymbol{Q}_0$. Put $G_0 = G \cap \boldsymbol{Q}_0$. We claim that $S = G_0$. By Lemma 2.1 (3), $R[G_0]$ is a one dimensional Bézout domain. Let M be the ideal generated by the set $\{X^g; g \in G_0, g > 0\}$ in $R[G_0]$ and N be the ideal generated by the set $\{X^s; s \in S, s > 0\}$ in $R[S]$. Then M is a maximal ideal of $R[G_0]$ and N is a maximal ideal in $R[S]$. Put $V = R[G_0]_M$ and $T = R[S]_N$. Then, since $R[G_0]$ is a Bézout domain, V is a valuation domain and, by assumption, T is a PVD. Also T is dominated by V. Therefore $MV = NT$ and V is the associated valuation domain of T. To show that $S = G_0$, it is enough to prove $S \supset G_0$. Take any non-zero element $g \in G_0$. Since $X^g \in MV = NT$, we can write $X^g = f/k$ where

$f=\sum a_h X^h$ ($a_h \in R$, $h \in S$ and $h>0$) and $k=\sum b_t X^t$ ($b_t \in R$, $b_0=1$ and $t \in S$). Since $k=1+\sum_{t\neq 0} b_t X^t$, we have $f=X^g+\sum_{t\neq 0} b_t X^{g+t}$. Since $R[G_0]$ is a free R-module, we have $g \in S$. Hence $S=G_0$. Thus S is a normal submonoid of $\boldsymbol{Q}_0$. By Lemma 2.1 (3), $R[S]$ is a one dimensional Bézout domain.

Conversely suppose that S is either a subgroup of $\boldsymbol{Q}$ or a normal submonoid of $\boldsymbol{Q}_0$. If S is a subgroup, the assertion is clear by Theorem 2.2. If S is a normal submonoid of $\boldsymbol{Q}_0$, then $S=G \cap \boldsymbol{Q}_0$, where G is the quotient group of S in $\boldsymbol{Q}$; and the assertion follows from Lemma 2.1 (3). Q. E. D.

REMARK 2.7. Let S be a non-zero monoid. If the monoid ring $R[S]$ is an LPVD, then S is an indecomposable monoid and $\mathrm{Pic}(R[S])=(0)$. For, S is a normal submonoid of $\boldsymbol{Q}$ and $R[S]$ is a one dimensional Bézout domain by Theorem 2.6.

COROLLARY 2.8. *Let S be a non-zero monoid. Then $R[S]$ is an i-domain if and only if R is a field and S is a submonoid of $\boldsymbol{Q}$. Moreover, in this case, $R[S]$ is a one dimensional monoid ring.*

PROOF. Suppose that $R[S]$ is an i-domain. Then $\bar{R}[\bar{S}]$ is the integral closure of $R[S]$ and hence $\bar{R}[\bar{S}]$ is a Prüfer domain. By Theorem 2.6, we see that $\bar{R}$ is a field and so R is a field; also $\bar{S}$ is a submonoid of $\boldsymbol{Q}$ and so S is a submonoid of $\boldsymbol{Q}$.

Conversely, suppose that R is a field and S is a submonoid of $\boldsymbol{Q}$. By Theorem 2.6, $R[\bar{S}]$ is a one dimensional Bézout domain. We claim that the canonical mapping: $\mathrm{Spec}(R[\bar{S}]) \to \mathrm{Spec}(R[S])$ is injective. We may assume that S is a submonoid of $\boldsymbol{Q}_0$. Let P be any prime ideal of $R[S]$. If $X^s \notin P$ for some $s \in S$, $s>0$, then we have $R[S]_{X^s}=R[\bar{S}]_{X^s}$; in fact $\langle S, -s\rangle$ is a subgroup and hence coincides with $\langle \bar{S}, -s\rangle$, where $\langle S, -s\rangle$ means the monoid generated by S and $-s$. Let M and M' be prime ideals of $R[\bar{S}]$ lying over P. Then $M_{X^s}=P_{X^s}=M'_{X^s}$ and hence $M=M'$. We assume that $X^s \in P$ for every $s \in S$, $s>0$. Let M be a prime ideal of $R[\bar{S}]$ lying over P. Then it is easy to see that $X^t \in M$ for every $t \in \bar{S}$, $t>0$; this implies the uniqueness of such a prime ideal M. Thus $R[S]$ is an i-domain. Q. E. D.

REMARK 2.9. Let S be a non-zero monoid. Then $R[S]$ is an LPVD if and only if it is a seminormal i-domain. In fact, the assertion follows from Theorem 2.6, Corollary 2.8 and Corollary 1.4.

REMARK 2.10. Let S be a non-zero monoid. We assume that the monoid ring $R[S]$ is a locally Noetherian domain and $\mathrm{Max}(R[S])$ is a Noetherian space. Then $R[S]$ is an LPVD if and only if R is a field and $R[S]$ is isomorphic to $R[X]$ or $R[X, X^{-1}]$.

We use the notation gl. dim and f.p. dim to denote the global dimension and the finitely presented dimension respectively in the sense of H. K. Ng [10].

COROLLARY 2.11. *Let S be a non-zero monoid. Then* $R[S]$ *is an LPVD if and only if R is a field and* $R[S]$ *is one of the following types.*

(1) $R[S]$ *is isomorphic to* $R[X]$.

(2) $R[S]$ *is isomorphic to* $R[X, X^{-1}]$.

(3) $R[S]$ *is a one dimensional Bézout domain such that* $\mathrm{gl.dim}\, R[S]=2$ *and* $\mathrm{f.p.dim}\, R[S]=3$.

PROOF. It is enough to prove the "only if" part. By Theorem 2.6, $R[S]$ is a one dimensional Bézout domain in which every ideal is countably generated. Then we have $\mathrm{gl.dim}\, R[S] \leqq 1$ or $\mathrm{gl.dim}\, R[S]=2$, according as $R[S]$ is Noetherian or not by [2, vII.5]. Also, by [10, Corollary 3.5], we have $\mathrm{f.p.dim}\, R[S]=3$ in the case $\mathrm{gl.dim}\, R[S]=2$.

EXAMPLE. $\mathrm{gl.dim}\, \boldsymbol{Q}[\boldsymbol{Q}]=2$ and $\mathrm{f.p.dim}\, \boldsymbol{Q}[\boldsymbol{Q}]=3$.

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Department of Mathematics,
Aichi University of Education