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Riesz's lemma and orthogonality in normed spaces

Dedicated to Professor Isao Miyadera on his 60th birthday

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Introduction

This paper is concerned with orthogonality in normed spaces and geometric structure of normed spaces as well as their dual spaces. The orthogonality problem is to discuss the existence and properties of elements that are orthogonal in an appropriate sense to a given closed subspace of a normed or Banach space, and problems of this kind are important in connection with the geometry of normed spaces. Our work is mainly devoted to two problems: The first aim is to seek natural notions of orhtogonality in general normed spaces; and the second purpose is to investigate various geometric properties of Banach spaces as well as those of incomplete normed spaces via the above notions of orthogonality.

Here we give a geometric interpretation of Riesz's Lemma in terms of duality theory of normed spaces and make an attempt to generalize the notion of Birkhoff orthogonality (see [2]) which is known as the most natural notion of orthogonality in general normed spaces. So-called Riesz's Lemma states that given a proper closed subspace M of a normed space X and a number $\varepsilon \in (0, 1)$ there is an element x_ε of X satisfying $\|x_\varepsilon\| = 1$ and $\mathrm{dist}(x_\varepsilon, M) \geqq 1-\varepsilon$. The standard use of Riesz's Lemma indicates that the Lemma is solely employed to find an element of norm 1 at a positive distance from a given proper closed subspace of a normed space, although the Lemma is directly related to the orthogonality problem in the following sense: If $\varepsilon = 0$ can be taken in the Lemma, then the associated unit vector x_0 turns out to be orthogonal to M in the sense of Birkhoff.

On the other hand, the James theorem and the Bishop-Phelps theorem, both of which are fundamental in Banach space theory, can be formulated as geometrical results concerning the orthogonality problem. First the former theorem asserts that a Banach space X is nonreflexive iff there is a hyperplane H in X such that none of the elements of X is orthogonal to H. This means that it is impossible to take $\varepsilon = 0$ in Riesz's Lemma if X is a nonreflexive Banach space. Now the latter theorem ensures a geometric property which compensates for this situation. Namely, the Bishop-Phelps theorem states that given a proper closed subspace M of a nonreflexive Banach space we can find a hyperplane H which is as close as we please to M and admits an element orthogonal to H. In other words, there is a sequence of unit vectors x_n, $n = 1, 2, 3, \ldots$, orthogonal respectively to hyperplanes

H_n, $n=1, 2, 3,\dots$, which come close to M as $n\to\infty$ in a certain sense. In this paper we say that such a sequence is asymptotically orthogonal to M. This observation leads us to a generalization of the Birkhoff orthogonality to closed subspaces of Banach spaces. In terms of this new notion we establish an orthogonality theorem for proper closed subspaces of Banach spaces and give a characterization of nonreflexivity of Banach spaces.

For incomplete normed spaces the Bishop-Phelps theorem is no longer valid in general. In fact, there is an incomplete normed space X which contains a proper closed subspace M with the property that none of the sequences of unit vectors is asymptotically orthogonal to M in the above sense; and such situation does happen even though the completion of X is reflexive. Hence it is an interesting problem to explore other notions of asymptotic orthogonality which would be adequate for general incomplete normed spaces. We here employ the geometric aspect of Riesz's Lemma. Namely, given a proper closed subspace M of a normed space X we choose a null sequence (ε_n) of positive numbers and a sequence (x_n) of unit vectors with $\operatorname{dist}(x_n, M)\geqq 1-\varepsilon_n$ for $n=1, 2, 3,\dots$, and we say that the sequence (x_n) is asymptotically orthogonal to M. We shall see that this asymptotic orthogonality is adequate to treat orthogonality problems in incomplete normed spaces. An orthogonality theorem for proper closed subspaces of general normed spaces is established by use of the generalized notion and reflexivity as well as nonreflexivity of the completions of incomplete normed spaces are discussed. Also, the structural difference between the unit ball of an incomplete normed space and that of its completion is investigated in some detail. Moreover, it should be mentioned that if a sequence (x_n) of unit vectors is asymptotically orthogonal to a proper closed subspace of a normed space X, then any weakstar cluster point of (x_n), viewed as a net in the second dual X^{**}, has norm 1 and is orthogonal to M in X^{**} in the sense of Birkhoff.

In connection with the orthogonality problems as mentioned above, it is interesting to investigate as to whether or not an incomplete normed space has bounded linear functionals which do not achieve their norms. We here treat important and typical classes of those incomplete normed spaces X such that there exist functionals in X^* which do not attain their norms.

If a Banach space X is nonreflexive, then X is regarded as a proper closed subspace of its second dual X^{**} and it becomes another significant problem to discuss the existence and properties of elements of X^{**} orthogonal to X in the sense of Birkhoff. If the orthogonality theorems as mentioned above would be applied to this case, the fourth dual of X must be taken into account in order to find elements orthogonal to X. These problems are therefore important in connection with the structure of second dual spaces. We here introduce for a Banach space X the set $L(X)$ of all "left-orthogonal" elements to X and the set $R(X)$ of all "right-orthogonal" elements to X. Both of $L(X)$ and $R(X)$ are extremely complicated subsets of X^{**} and their structures depend strongly upon the geometric properties of X. The set $L(X)$ is characterized by means of the duality mapping of X^{**}, while $R(X)$ is related to the existence of the projection with norm 1 of X^{**} onto X. Detailed properties of these sets are investigated and orthogonal decompositions in a generalized sense of the second dual X^{**} are discussed by means of these sets. Moreover, using these results, we shall discuss the structure of abstract (L) spaces and abstract (M) spaces. There are many open problems concerning orthogonality in second dual spaces, and it is expected that significant development will take place in this theory.

The present paper is organized as follows:

Section 1. Fundamental Results.
Section 2. Generalizations of Birkhoff's Orthogonality.
Section 3. Nonreflexive Banach Spaces and (BP)-Orthogonality.
Section 4. Incomplete Normed Spaces and (R)-Orthogonality.
Section 5. Orthogonality in Second Dual Spaces.

1. Fundamental results

We begin by preparing the following lemma that is fundamental in our subsequent discussions.

LEMMA 1.1. *Let X be a normed space, $x \in X$, and let $f \in X^*$. Then we have*

$$|f(x)| = \operatorname{dist}(x, \ker(f)) \cdot \|f\|.$$

PROOF. It is sufficient to consider the case in which $f(x) \neq 0$. Let $v \in \ker(f)$. Then $\|x-v\| \cdot \|f\| \geqq |f(x-v)| = |f(x)|$. This means that $\operatorname{dist}(x, \ker(f)) \cdot \|f\| \geqq |f(x)|$. To see the converse inequality, let $y \in X - \ker(f)$. Then $f(y) \neq 0$ and

$$\|y\| = |f(y)/f(x)|\, \|x - (x-(f(x)/f(y))y\| \geqq |f(y)/f(x)| \operatorname{dist}(x, \ker(f)) \tag{1.1}$$

since $x-(f(x)/f(y))y \in \ker(f)$. Therefore $\operatorname{dist}(x, \ker(f))|f(y)| \leqq \|y\| \cdot |f(x)|$ for any $y \in X$, and we have $\operatorname{dist}(x, \ker(f))\|f\| \leqq |f(x)|$. q.e.d.

LEMMA 1.2. *Let X be a normed space, M a proper closed subspace of X, and let $x \in X$. Then there is $f \in X^*$ such that $\|f\| = 1$, $M \subset \ker(f)$ and* $\operatorname{dist}(x, M) = \operatorname{dist}(x, \ker(f)) = f(x)$.

PROOF. If $x \in M$, then by the Hahn-Banach theorem there is $f \in X^*$ such that $\|f\| = 1$ and $M \subset \ker(f)$; hence $\operatorname{dist}(x, M) = \operatorname{dist}(x, \ker(f)) = f(x) = 0$. Let $x \in X - M$. Then the Hahn-Banach theorem implies that there is $g \in X^*$ such that $M \subset \ker(g)$, $g(x) = 1$ and $\operatorname{dist}(x, M) = \|g\|^{-1}$. Put $f = \|g\|^{-1}g$. Then $\|f\| = 1$, $M \subset \ker(f)$, and $\operatorname{dist}(x, \ker(f)) = |f(x)| = f(x) = \|g\|^{-1} = \operatorname{dist}(x, M)$ by Lemma 1.1. q.e.d.

Lemma 1.2 is a geometric consequence of the Hahn-Banach theorem. Riesz's lemma is obtained independently of the duality theory, although it is of some interest to give a combined form of the lemma and Lemma 1.2:

PROPOSITION 1.3. *Let X be a normed space, M a proper closed subspace of X, $M \neq \{0\}$, and let $\varepsilon \in (0, 1)$. Then there is a pair (x, f) in $X \times X^*$ such that $\|x\| = \|f\| = 1$, $M \subset \ker(f)$, and $f(x) = \mathrm{dist}(x, M) = 1 - \varepsilon$.*

PROOF. Let $y \in X - M$. Then $d \equiv \mathrm{dist}(y, M) > 0$ and there is $z \in M$ such that $d(1 + \varepsilon/(1-\varepsilon)) > \|y - z\| \geqq d$. Set $x = (y-z)/\|y-z\|$. Then $\|x\| = 1$ and $\|x - w\| \geqq 1 - \varepsilon$ for all $w \in M$, and this means that $\mathrm{dist}(x, M) \geqq 1 - \varepsilon$. Hence Lemma 1.2 ensures that there is $f \in X^*$ with $\|f\| = 1$, $M \subset \ker(f)$ and $f(x) = \mathrm{dist}(x, M) \geqq 1 - \varepsilon$. If $\mathrm{dist}(x, M) = 1 - \varepsilon$ then (x, f) is the desired pair. In case $\mathrm{dist}(x, M) > 1 - \varepsilon$, we take $x' \in M - \{0\}$ and define $x_t = [tx + (1-t)x']/\|tx + (1-t)x'\|$ for $t \in [0, 1]$. Since $f(x_0) = 0$ and $f(x_1) > 1 - \varepsilon$, $f(x_t) = \mathrm{dist}(x_t, M) = 1 - \varepsilon$ for some t. q.e.d.

If $M = \{0\}$, then $\varepsilon = 0$ can be taken in Proposition 1.3. We here recall the notion of orthogonality in the sense of Birkhoff. An element x of a normed space X is said to be orthogonal to an element $y \in X$ (denoted by $x \perp y$) if $\|x + \alpha y\| \geqq \|x\|$ for every scalar α. Likewise, x is said to be orthogonal to a subset $M \in X$, and we write $x \perp M$ if $x \perp y$ for every $y \in M$. The above notion is related to the problem of best approximation. Given an element $x \in X$ and a subset $M \subset X$, an element $m \in M$ is said to be an element of best approximation of x (by elements of M), if $\|x - m\| = \mathrm{dist}(x, M)$.

In particular, orthogonality to hyperplanes can be discussed via the theory of bounded linear functionals on normed spaces which do not achieve their norms. Such functionals may be studied through the duality mapping of X. The duality mapping of X is a possibly multi-valued mapping F from X into X^* which assings to each $x \in X$ a closed convex subset of X^* defined by $F(x) = \{f \in X^* : f(x) = \|x\|^2 = \|f\|^2\}$. $F(0) = \{0\}$; and $F(x)$ is non-empty for any $x \in X$ by the Hahn-Banach theorem. A geometric interpretation of this fact is that for every point x on the unit surface in X there exists at least one hyperplane which supports the unit ball at x.

PROPOSITION 1.4. *Let X be a normed space, M a proper closed subspace of X, and let $x \in X - M$. Then we have:*

(a) *An element $m \in M$ is an element of best approximation of x iff $(x - m) \perp M$.*

(b) *Let $f \in X^*$. Then $x \perp \ker(f)$ iff $f \neq 0$ and $|f(x)| = \|f\| \cdot \|x\|$.*

(c) *The element x is orthogonal to M iff there is $g \in X^*$ such that $g \neq 0$, $g(x) = \|g\| \cdot \|x\|$ and $M \subset \ker(g)$.*

(d) *Let* $(x, f) \in X \times X^*$. *Then* $f \in F(x)$ *iff* $f(x) \geqq 0$, $\|f\| = \|x\|$ *and* $x \perp \ker(f)$.

Assertion (a) is readily seen from the definition of element of best approximation. See [16, Lemma 1.14]. The proof of (b) is found in [10, Theorem 2.1] and [6, Theorem 3 on page 25]. We here prove (b) and (c) via Lemmas 1.1 and 1.2.

PROOF. Let $f \in X^*$ and $f \neq 0$. Then it follows from Assertion (a) and Lemma 1.1 that $x \perp \ker(f)$ iff $\|x\| = \operatorname{dist}(x, \ker(f)) = |f(x)|/\|f\|$. This proves (b) since $x \perp \ker(f)$ implies $f \neq 0$. To show (c), assume $x \perp M$. Then Lemma 1.2 ensures the existence of $g \in X^*$ such that $M \subset \ker(g)$, $\|g\| = 1$ and $g(x) = \operatorname{dist}(x, M) = \|x\|$. Conversely, suppose that there is $g \in X^*$ with the properties mentioned in (c). Then $x \perp \ker(g)$ by (b), and so $x \perp M$ since $M \subset \ker(g)$. To show (d) it is sufficient to consider the case in which $x \neq 0$. If $f \in F(x)$, then $f(x) > 0$, $\|f\| = \|x\|$, and $\|x + \alpha y\| \geqq \|x\|^{-1} f(x + \alpha y) = \|x\|$ for every scalar α and every $y \in \ker(f)$. Conversely, if $f(x) \geqq 0$ and $x \perp \ker(f)$ then $f(x) = \|f\| \cdot \|x\|$ by (b). Hence, if in addition $\|f\| = \|x\|$ then $f \in F(x)$. q. e. d.

COROLLARY 1.5. *Let X be a normed space. If* $f \in X^*$ *and f does not achieve its norm, then no nonzero elements of X are orthogonal to* $\ker(f)$. *Conversely, assume that there exists a proper closed subspace M such that none of the elements of* $X - M$ *is orthogonal to M. Let g be any nonzero element of* X^* *with* $M \subset \ker(g)$. *Then g does not attain its norm.*

PROOF. First note that $f \neq 0$ if f does not attain $\|f\|$. Suppose $x \perp \ker(f)$ for some $x \neq 0$. Then it would follow from Proposition 1.4 (b) that $|f(x)| = \|f\| \cdot \|x\|$ and f achieves its norm. This is a contradiction, and $\ker(f)$ admits no nonzero elements orthogonal to it. Next, assume that there is a proper closed subspace M such that none of the elements of $X - M$ is orthtogoanl to M; and let $g \in X^*$, $g \neq 0$ and $M \subset \ker(g)$. Then Proposition 1.4 (a) implies that $\operatorname{dist}(y, M) < 1$ for $y \in X$ with $\|y\| = 1$. Hence for each $y \in X$ with $\|y\| = 1$ there exists $v \in M$ with $\|y - v\| < 1$, and so $|g(y)| = |g(y - v)| \leqq \|g\| \cdot \|y - v\| < \|g\|$. This shows that g does not attain its norm. q. e. d.

We now state a geometrical version of the James theorem.

THEOREM 1.6. *Let X be a Banach space. Then the following are equivalent:*

(i) *X is reflexive.*

(ii) *For every proper closed subspace M of X there is an element* $x \in X - M$ *such that* $\operatorname{dist}(x, M) = \|x\|$.

(iii) *For every proper closed subspace M of X there is an element* $x \in X - M$ *such that* $x \perp M$.

(iv) *For every closed subspace M with codimension* 1 *there is a non-zero element $x \in X$ with* $\mathrm{dist}(x, M) = \|x\|$ (*or equivalently*, $x \perp M$).

The fact that (i) and (ii) are equivalent is stated in Diestel [7, p. 6]. We here give the proof by use of Lemma 1.1.

PROOF. It is clear from Proposition 1.4 (a) that (ii) and (iii) are equivalent. Assume that X is reflexive, and let M be any proper closed subspace of X. Then there is $f \in X^*$ with $\|f\| = 1$ and $M \subset \ker(f)$ and, by the reflexivity of X, there exists $x \in X$ such that $\|x\| = 1$ and $f(x) = 1$. Hence $1 = f(x) = f(x - v) \leqq \|x - v\|$ for $v \in M$, and so $\mathrm{dist}(x, M) = 1$. This shows that (i) implies (ii). Conversely, suppose that (ii) holds. Let f be any element of X^* with $\|f\| = 1$ and put $M = \ker(f)$. Then M is a closed subspace of X with codimension 1, and so by condition (ii) there exists an $x \in X - M$ such that $\|x\| = 1$ and $\mathrm{dist}(x, M) = 1$. Therefore Lemma 1.1 implies that $|f(x)| = 1 = \|f\|$. This means that every $f \in X^*$ achieves its norm. The James theorem can now be applied to conclude that X is reflexive. q.e.d.

Structure and topological properties of the duality mapping of a nonreflexive Banach space are extremely complicated in general. For a typical example of such duality mapping we refer to the work of Hada, Hashimoto and Oharu [9].

We finally give an example which illustrates Proposition 1.3 and Theorem 1.6.

EXAMPLE. Let X be the subspace of $C[0, 1]$ consisting of all continuous functions x on $[0, 1]$ such that $x(0) = 0$. We then define a linear functional f on X by $f(x) = \int_0^1 x(s)ds$ for $x \in X$, where the integral is taken in the sense of Riemann. Then $\|f\| = 1$ and $\ker(f) = \{x \in X : \int_0^1 x(s)ds = 0\}$, although f does not achieve its norm 1. See [17, p. 96]. Therefore, it follows from Theorem 1.6 that there is no point on the unit surface in X at unit distance from $\ker(f)$. In other words, for every point x on the unit surface in X the origin 0 can not be an element of best approximation of x. The closed subspace $M \equiv \ker(f)$ is a well-known example which shows that $\varepsilon = 0$ can not be taken in Proposition 1.3.

2. Generalizations of Birkhoff's orthogonality

If a Banach space X is nonreflexive, there is a proper closed subspace M of X such that none of the elements of X is orthogonal to M. In this section we make an attempt to generalize the notion of orthogonality in the sense of Birkhoff in order to formulate a natural orthogonality theorem for arbitrary proper closed subspace. Namely, we introduce two notions of asymptotically orthogonal sequences to proper closed subspaces of a normed space in connection with the Bishop-Phelps theorem and Riesz's Lemma and we investigate the relationship between the two basic theorems in terms of the generalized notions of orthogonality.

DEFINITION 2.1. Let M be a proper closed subspace of a normed space X. A sequence (x_n) in X with $\|x_n\| \equiv 1$ is said to be *(BP)-orthogonal* to M and we write $(x_n) \perp_{BP} M$, if it satisfies the following conditions:

(i) There is a sequence (f_n) in X^* with $f_n \in F(x_n)$ for each n.

(ii) There exists $f \in X^*$ such that $\|f\|=1$, $M \subset \ker(f)$, and $\lim \|f_n - f\| = 0$.

If M is a proper closed subspace of X and $x \perp M$ for some x with $\|x\|=1$, then the constant sequence (x_n) with $x_n \equiv x$ is orthogonal to M in the above sense. In fact, Proposition 1.4 (c) implies that there is $f \in X^*$ such that $\|f\|=1$, $M \subset \ker(f)$ and $f(x)=1$. Since $\operatorname{dist}(x, M) = \|x\| = 1$, both conditions (i) and (ii) are satisfied for the constant sequence (f_n) in X^* with $f_n \equiv f$. Moreover, we see from Lemma 1.1 that conditions (i) and (ii) together imply $\lim \operatorname{dist}(x_n, \ker(f)) = 1$ and $\lim f(x_n) = 1$. In this sense the (BP)-orthogonality is regarded as a generalization of the notion of Birkhoff orthogonality.

We then show that a strightforward extension of Theorem 1.6 to the case of general Banach space is obtained in terms of (BP)-orthogonality. First we observe the following fact which is an easy consequence of the definition of subreflexivity and the Hahn-Banach theorem.

PROPOSITION 2.1. *Let X be a normed space. Then X is subreflexive iff for every proper closed subspace M there is a sequence (x_n) in X with $\|x_n\| \equiv 1$ which is (BP)-orthogonal to M.*

Thus the Bishiop-Phelps theorem implies the following orthogonality theorem.

THEOREM 2.2. *Let X be an arbitrary Banach space. Then for every proper closed subspace M there is a sequence (x_n) in X with $\|x_n\| \equiv 1$ which is (BP)-orthogonal to M.*

Let (x_n) be a sequence in X. In what follows, we denote by $\omega^*(x_n)$ the set of all weak-star cluster points of (x_n) which is viewed as a net in X^{**}. The following fact is useful in the subsequent arguments:

LEMMA 2.3. *Let $f \in X^*$, $\|f\|=1$, and let (x_n) be a sequence in X with $\|x_n\| \equiv 1$. Then $\lim f(x_n) = 1$ iff $\omega^*(x_n) \subset F^*(f)$, where F^* denotes the duality mapping of X^*.*

PROOF. First assume that $\lim f(x_n) = 1$, and let $\lambda \in \omega^*(x_n)$. Then there is a subsequence $(x_{n(k)})$ such that $1 = \lim f(x_{n(k)}) = \lambda(f)$. But $\|\lambda\| \leqq 1$, so $1 = \lambda(f) \leqq \|\lambda\| \|f\| \leqq 1$. This means that $\lambda \in F^*(f)$. Conversely, suppose that $\omega^*(x_n) \subset F^*(f)$, and that there exists a subsequence $(x_{n(k)})$ and $\varepsilon_0 \in (0, 1)$ such that $f(x_{n(k)}) \leqq 1 - \varepsilon_0$ for $k \geqq 1$. Let $\lambda \in \omega^*(x_{n(k)})$. Then $\lambda(f) \leqq 1 - \varepsilon_0$. However, $\lambda \in \omega^*(x_n) \subset F^*(f)$ by the hypothesis; hence $\lambda(f) = 1$. This is a contradiction, and it is concluded that $\lim f(x_n) = 1$. q.e.d.

REMARK 2.1. In connection with Lemma 2.3, we see from Lemma 1.1 that $\lim \operatorname{dist}(x_n, \ker(f))=1$ iff $\lim |f(x_n)|=1$. This suggests that the Hahn-Banach theorem and Lemma 1.1 together prove Riesz's Lemma in a different way. In fact, let M be a proper closed subspace of X. Then there is $f \in X^*$ with $\|f\|=1$ and $M \subset \ker(f)$ by the Hahn-Banach theorem. Hence there is a sequence (x_n) in X with $1=\|f\|=\lim f(x_n)$, and so it follows from Lemmas 1.1 and 1.2 that that $\lim \operatorname{dist}(x_n, \ker(f))=1$. This gives another proof of Riesz's Lemma.

The next proposition gives a variant of Proposition 1.4 (d).

LEMMA 2.4. *Let $f \in X^*$, $\|f\|=1$, and let F^* be the duality mapping of X^*. If $\lambda \in F^*(f)$, then $\lambda \perp \ker(f)$ in X^{**}, where* $\ker(f)$ *is understood to be a subspace of X^{**} via the natural embedding of X into X^{**}.*

PROOF. Let $\tilde{f}=\kappa^* f$, where κ^* is the natural embedding of X^* into X^{***}, and let F^{**} denote the duality mapping of X^{**}. Then $\lambda \in F^*(f)$ iff $\tilde{f} \in F^{**}(\lambda)$. Hence, if $\lambda \in F^*(f)$ then $\lambda \perp \ker(\tilde{f})$ in X^{**} by Proposition 1.4 (d). Therefore, $\lambda \in F^*(f)$ implies $\lambda \perp \ker(f)$ in X^{**} since $\ker(f) \subset \ker(\tilde{f})$. q.e.d.

The above observations lead us to another notion of orthogonality in normed spaces.

DEFINITION 2.2. Let M be a proper closed subspace of a normed space X. A sequence in X with $\|x_n\|=1$ is said to be (R)-orthogonal to M if there is an $f \in X^*$ with $\|f\|=1$ such that $M \subset \ker(f)$ and $\omega^*(x_n) \subset F^*(f)$; and we write $(x_n) \perp_R M$.

By virtue of Lemma 2.4, (R)-orthogonality can be regarded as a generalization of the Birkhoff orthogonality. The next result illustrates the asymptotic orthogonality in the sense of Definition 2.2.

PROPOSITION 2.5. *Let X be a Banach space, M a proper closed subspace of X, and let (x_n) be a sequence in X with $\|x_n\| \equiv 1$. If X is uniformly convex, then the following are equivalent:*

(i) $(x_n) \perp_R M$.

(ii) $\lim \|x_n - x\|=0$ for some $x \in X$; and $x \perp M$.

(iii) $\lim \operatorname{dist}(\operatorname{co}\{x_k : k \geqq n\}, M)=1$.

PROOF. Suppose (i). Then, by Lemma 2.3, there is $f \in X^*$ such that $\|f\|=1$, $M \subset \ker(f)$ and $\lim f(x_n)=1$. Since X is reflexive, there is $x \in X$ with $\|x\|=1$ and $f(x)=1$. Hence $x \perp M$ by Proposition 1.4 (d). Now $\lim f(2^{-1}(x_n+x))=1$, and so $\liminf \|2^{-1}(x_n+x)\| \geqq 1$. This means that $\lim \|2^{-1}(x_n+x)\|=1$. The uniform convexity of X then implies $\lim \|x_n - x\|=0$. Thus (ii) is obtained. Conversely, let (ii) hold. Then one finds $f \in X^*$ with $\|f\|=1$, $f(x)=1$ and $M \subset \ker(f)$. Hence $\lim f(x_n)=1$, and $(x_n) \perp_R M$. If (ii) holds, then we have

$\lim \operatorname{dist}(\operatorname{co}\{x_k: k \geqq n\}, M) = \operatorname{dist}(x, M) = 1$, which yields (iii). Finally, assume that (iii) holds. Since $\|2^{-1}(x_m + x_n)\| \geqq \operatorname{dist}(2^{-1}(x_m + x_n), M) \geqq \operatorname{dist}(\operatorname{co}\{x_k: k \geqq m\}, M)$ for $1 \leqq m \leqq n$, we have $\lim_{m,n} \|2^{-1}(x_m + x_n)\| = 1$. The uniform convexity of X can then be applied to get $\lim_{m,n} \|x_m - x_n\| = 0$. So, the limit $x = \lim x_n$ exists and $\operatorname{dist}(x, M) = \lim \operatorname{dist}(\operatorname{co}\{x_k: k \geqq n\}, M) = 1$; hence $x \perp M$ by Proposition 1.4 (a). q. e. d.

The following result is directly derived form Definition 2.2 and Lemma 2.4 and it also justifies the notion of (R)-orthogonality.

PROPOSITION 2.6. *Let M be a proper closed subspace of a normed space X and (x_n) a sequence in X with $\|x_n\| \equiv 1$. If $(x_n) \perp_R M$, then every weakstar cluster point λ of (x_n), viewed as a net in X^{**}, has norm 1 and is orthogonal to M in X^{**}. Moreover $\lambda(f) = 1$.*

We have seen that conditions (i), (ii) imply $\lim f(x_n) = 1$. Hence we infer from Lemma 2.3 that (BP)-orthogonality implies (R)-orthogonality. Moreover, as is seen from Proposition 2.1 and Theorem 4.1 below, Theorem 2.2 need not be valid for incomplete normed spaces. However, we obtain the following orthogonality theorem in terms of (R)-orthogonality.

THEOREM 2.7. *Let X be a normed space. Then for any proper closed subspace M of X there is a sequence (x_n) in X with $\|x_n\| = 1$ such that $(x_n) \perp_R M$.*

PROOF. Let M be a proper closed subspace of X. Then, as mentioned in Remark 2.1, there exists a sequence (x_n) in X and $f \in X^*$ such that $\|f\| = 1$, $M \subset \ker(f)$ and $\lim f(x_n) = 1$. Hence $\omega^*(x_n) \subset F^*(f)$. This means that $(x_n) \perp_R M$. q. e. d.

Combining Proposition 2.6 and Theorem 2.7, we obtain the following result that is parallel to Theorem 1.6, (iii).

COROLLARY 2.8. *Let X be a normed space. Then for every proper closed subspace M of X there is an element $\lambda \in X^{**}$ such that $\lambda \perp_B M$ in X^{**}.*

We have thus introduced two generalized notions of orthogonality in normed spaces. We expect that these notions will be applied to nonlinear functional analysis.

Finally, in the remainder of this section, we make some observations which suggest that problems on (R)-orthogonality to proper closed subspaces in normed sapces can be passed to those on Birkhoff's orthogonality in certain sequence spaces.

Let X be a normed space and M a subspace of X. Let $m(X)$ (resp. $m(M)$) be the space of all bounded sequences in X (resp. M). We now consider the mapping φ which assigns to each $x \in X$ a sequence (x_n) with $x_n \equiv x$. For $(x_n) \in m(X)$ we define

$$p((x_n)) = \limsup_{n\to\infty} \|x_n\|.$$

p defines a seminorm on the linear space $m(X)$ and $p^{-1}(\{0\})$ is a subspace of $m(X)$ which consists of sequences converging to 0. We denote the quotient space $m(X)/p^{-1}(\{0\})$ by $\hat{X}$ and define the subspace $\hat{M} = \{\hat{x} \in \hat{X} : \hat{x} \ni (x_n), (x_n) \in m(M)\}$. On $\hat{X}$ one can define a norm by

$$\|\hat{x}\| = p((x_n)), \qquad (x_n) \in \hat{x},\ \hat{x} \in \hat{X}.$$

Also, let $\hat{\varphi}$ be the mapping which assigns to each $x \in X$ the element $\hat{x}$ with $\varphi(x) \in \hat{x}$. Then $\|\hat{\varphi}(x)\| = p(\varphi(x)) = \|x\|$ for $x \in X$, and so X is isometrically embedded into $\hat{X}$.

On the other hand, $m(X)$ becomes a normed space under the supremum norm

$$\|(x_n)\| \equiv \sup_n \|x_n\|, \qquad (x_n) \in m(X).$$

The space $m(X)$ equipped with the supremum norm is sometimes denoted by $\ell^\infty(X)$ and M is isometrically embedded into $m(M)$ with respect to the above norm since $\|\varphi(x)\|_\infty = \|x\|$ for $x \in M$.

Employing the above spaces, we obtain the following result.

PROPOSITION 2.9. *Let M be a proper closed subspace of a normed space X and (x_n) a sequence with $\|x_n\| \equiv 1$. Then the following are equivalent*:

(i) $(x_n) \perp_R M$.

(ii) *There exists $f \in X^*$ such that $\|f\| = 1$, $M \subset \ker(f)$, $f(x_n) \geqq 0$ for n sufficiently large, and $(x_{n(k)}) \perp m(\ker(f))$ for any subsequence $(x_{n(k)})$ of (x_n).*

(iii) *There is $f \in X^*$ such that $\|f\| = 1$, $M \subset \ker(f)$, $f(x_n) \geqq 0$ for n sufficiently large, and $\hat{v} \perp (\ker(f))^\wedge$ for any $\hat{v} \in \hat{X}$ containing a subsequence of (x_n), where $(\ker(f))^\wedge = \{\hat{x} \in \hat{X} : \hat{x} \ni (x_n), (x_n) \in m(\ker(f))\}$.*

To prove the above proposition, we need the following lemma.

LEMMA 2.10. *We have*:

$$\inf_{(y_n)\in m(M)} \liminf_{n\to\infty} \|x_n + y_n\| = \limsup_{n\to\infty} \inf_{y\in M} \|x_n + y\|. \tag{2.1}$$

$$\inf_{(y_n)\in m(M)} \sup_n \|x_n + y_n\| = \sup_n \inf_{y\in M} \|x_n + y\|. \tag{2.2}$$

PROOF. We here give the proof of (2.2). The relation (2.1) is similarly verified. First, given $\varepsilon > 0$ and n, there is $y_n^\varepsilon \in M$ such that $\inf_{y\in M} \|x_n + y\| + \varepsilon > \|x_n + y_n^\varepsilon\|$. Then $(y_n^\varepsilon)_{n=1}^\infty \in m(M)$ and

$$\sup_n \inf_{y\in M} \|x_n + y\| + \varepsilon \geqq \sup_n \|x_n + y_n^\varepsilon\| \geqq \inf_{(y_n)\in m(M)} \sup_n \|x_n + y_n\|.$$

Since $\varepsilon > 0$ is arbitrary, we infer that the left side of (2.2) is not greater than the right side. To show the converse inequality, we observe that $\inf_{y\in M}\|x_n+y\| \leqq \|x_n+y_n\|$ for every $(y_n)\in m(M)$. Hence $\sup_n \inf_{y\in M}\|x_n+y\| \leqq \sup_n \|x_n+y_n\|$ for $(y_n)\in m(M)$, from which it follows that the left side of (2.2) does not exceed the right side. Thus, we obtain the relation (2.2). q.e.d.

PROOF OF PROPOSITION 2.9. Suppose that $(x_n)\perp_R M$. Then there is $f\in X^*$ such that $\|f\|=1$, $M\subset \ker(f)$ and $\lim_n f(x_n)=1$. Let $(x_{n(k)})$ be any subsequence of (x_n). Since $\|f\|=1$, we have $\|(x_{n(k)})+(y_k)\|_\infty \geqq \sup_k |f(x_{n(k)})+f(y_k)| = \sup_k |f(x_{n(k)})|=1$ for every $(y_k)\in m(\ker(f))$. This means that $(x_{n(k)})\perp m(\ker(f))$ and (ii) is obtained. Conversely, assume that (ii) holds. In view of Lemma 2.3, it is sufficient to show that $\lim_n f(x_n)=1$. puppose there is a subsequence $(x_{n(k)})$ and a constant $c\in(0,1)$ such that $0\leqq f(x_{n(k)})\leqq c$ for all k. Since $(x_{n(k)})\perp m(\ker(f))$ in $\ell^\infty(X)$, Lemmas 2.10 and 1.1 imply that $1=\operatorname{dist}((x_{n(k)}), m(\ker(f))=\sup_k \operatorname{dist}(x_{n(k)}, \ker(f))=\sup_k f(x_{n(k)})$. This is a contradiction, and we must have $\lim_n f(x_n)=1$. The equivalence between (i) and (iii) is similarly proved. q.e.d.

3. Nonreflexive Banach spaces and (BP)-orthogonality

We begin with the following theorem which is a restatement of Theorem 1.6.

THEOREM 3.1. *Let X be a Banach space. Then the following are equivalent*:

(i) *X is nonreflexive.*

(ii) *There exists $f\in X^*$ which does not achieve its norm.*

(iii) *There exists a proper closed subspace M such that none of the elements of X is orthogonal to M.*

In contrast with the assertions of Theorem 3.1, we showed in the preceding section that for every proper closed subspace M of X there is a sequence (x_n) such that $\|x_n\|\equiv 1$ and $(x_n)\perp_{BP} M$ even if X is a noreflexive Banach space. In this section we discuss some typical properties characteristic to nonreflexive Banach spaces.

First the following result is deduced from Theorem 2.7, Proposition 2.6 and Theorem 3.1.

THEOREM 3.2. *Let X be a nonreflexive Banach space. Then there is a proper closed subspace M of X and a sequence (x_n) in X such that $(x_n)\perp_{BP} M$, and every weak-star cluster point λ of (x_n), viewed as a net of X^{**}, is orthogonal to M in the second dual X^{**} but never lies in X.*

REMARK 3.1. It should be noted that the orthogonality theorems, Theorems 2.2 and 2.7, hold for proper closed subspaces of Banach or normed spaces, and that Corollary 2.8 as well as Theorem 3.2 are fomulated in the second dual spaces even though the proper subspaces are contained in the original normed or Banach spaces. It is also interesting to investigate elements of X^{**} which is orthogonal to the canonical embedding of X, and this problem will be discussed in Section 5.

THEOREM 3.3. *A Banach space X is nonreflexive iff there is a proper closed subspace M of codimension 1 such that any (BP)-orthogonal sequence to M does not converge weakly to an element of X.*

PROOF. Let X be reflexive and M an arbitrary subspace of X. Let (x_n) be any sequence in X which is (BP)-orthogonal to M and let a sequence (f_n) in X^* and an element $f \in X^*$ satisfy conditions (i) and (ii) stated in Definition 2.1. Then (x_n) contains a subsequence $(x_{n(i)})$ converging weakly to some element $x \in X$. Hence we have $|f(x)-1| = |f(x)-f_{n(i)}(x_{n(i)})| \leqq |f(x-x_{n(i)})| + \|f-f_{n(i)}\|$ for each i. Since the right side goes to 0 as $i \to \infty$ and $\|x\| \leqq 1$, it follows that $f(x)=1$ and $\|x\|=1$. This proves the "if" part. In order to prove the "only if" part assume that X is nonreflexive and every proper closed subspace M of codimension 1 admits a (BP)-orthogonal sequence (x_n) which is weakly convergent in X. Now let M be such a closed subspace, $(x_n) \perp_{BP} M$, and assume that the sequence (x_n) converges weakly to some $x \in X$. Further, let $f \in X^*$ and a sequence (f_n) in X^* satisfy conditions (i) and (ii) in Definition 2.1. Then $M = \ker(f)$ since $M \subset \ker(f)$ and $\operatorname{codim} M = 1$. Hence Lemma 1.1, together with conditions (i) and (ii), implies that $1 \geqq f(x_n) \geqq \operatorname{dist}(x_n, M)$ for n sufficiently large. Thus $f(x) = \lim_n f(x_n) = 1$, and Proposition 1.4 ensures that $x \perp M$. Since M is an arbitrary closed subspace of X with $\operatorname{codim} M = 1$, Theorem 1.6 can be applied to conclude that X is reflexive. This is a contradiction, and the "only if" part is proved.

q. e. d.

Finally we give an example which illustrates the assertion of Theorem 3.2.

EXAMPLE. Let X be the real sequence space ℓ^1. Then X^* and X^{**} are respectively identified with the sequence space ℓ^∞ and the space ba of bounded, finitely additive measures on the power set Σ of the set of all positive integers N. We now take any monotone increasing sequence (ξ_k) with $0 < \xi_k < 1$ for $k \in N$ and $\lim_k \xi_k = 1$, and define a functional $f \in X^*$ by $f(x) = \Sigma \xi_k \eta_k$ for $x = (\eta_k) \in \ell^1$. The functional f does not achieve its norm since $|f(x)| \leqq \Sigma \xi_k |\eta_k| < \Sigma |\eta_k| = \|x\|_1$ for $x = (\eta_k) \in X - \{0\}$ and $\|f\|_\infty = \|(\xi_k)\|_\infty = 1$. Hence, by Corollary 1.5, $\ker(f) \equiv \{x \in X \colon x = (\eta_k), \Sigma \xi_k \eta_k = 0\}$ admits no nonzero elements orthogonal to it. We then find a sequence $((x_n, f_n))$ in $X \times X^*$ with properties (i) and (ii) as mentioned in Definition 2.1 and show that all of the weak-star cluster points in ba of the net (x_n) are purely finitely additive measures on Σ. For each $n \in N$, we define $x_n \in X$ by $x_n = (\delta_{n,k})$, $\delta_{n,k}$ being Kronecker's delta, and define $f_n = (\xi_{n,k})$ by setting $\xi_{n,k} = \xi_k$ for $1 \leqq k \leqq n-1$ and $\xi_{n,k} = 1$ for $k \geqq n$. Then $\lim \|f_n - f\|_\infty = \lim (1 - \xi_n) = 0$ and

$\|x_n\|_1=\|f_n\|_\infty=f_n(x_n)=1$ for all $n\in N$. Let P be the set of all weak-star cluster points of the net (x_n) in ba. We demonstrate that P is precisely the set of all purely finitely additive 0–1 measures. First let λ be any purely finitely additive 0–1 measure. Then we infer from Proposition 2.4 in [9] that for every $g=(\xi_n)\in \ell^\infty$ the value $\lambda(g)$ is given as a cluster point of the bounded sequence (ζ_n). But $g(x_n)=\zeta_n$ for $n\in N$; hence it follows that $\lambda\in P$. In particular, $\|\lambda\|=1$ and $\lambda(f)=\lim f(x_n)=\lim \xi_n=1$. Next, let λ be any element of P. Then, by [9, Lemma 7.2], λ is also a 0–1 measure since each x_n is regarded as a point mass δ_n in the sense that $\delta_n(E)=1$ if $n\in E$, $\delta_n(E)=0$ if $n\notin E$, and $\delta_n(g)=g(x_n)=\zeta_n$ for $g=(\zeta_n)\in \ell^\infty$. See [9, page 77]. Hence it suffices to show that λ is purely finitely additive. Let E be any finite set in Σ and let χ_E denote the sequence (γ_k) such that $\gamma_k=1$ for $k\in E$ and $\gamma_k=0$ for $k\in N-E$. Then $\chi_E\in\ell^\infty$ and $(\chi_E(x_n))$ forms a sequence such that $\lim\chi_E(x_n)=0$. But $\lambda(E)$ is a cluster point of the sequence $(\chi_E(x_n))$, so $\lambda(E)=0$. Thus it follows from Propositions 2.1 and 2.2 of [9] that λ is purely finitely additive. Finally, Theorem 3.2 implies that $\lambda\perp\ker(f)$ in ba.

4. Incomplete normed spaces and (R)-orthogonality

In this section we discuss various orthogonality problems in incomplete normed spaces. In case a normed space X is incomplete, orthogonality theorems such as Theorems 1.6, 2.2, 3.1 and 3.3 are no longer obtained. We here make an attempt to characterize reflexivity as well as nonreflexivity of the completions of incomplete normed spaces in terms of (R)-orthogonality. Moreover, if a normed space X is incomplete then Theorem 1.6 is not applicable; and it is a new problem to investigate as to whether or not there exists a bounded linear functional on an incomplete normed space which does not achieve its norm. In the latter half of this section we shall find an important class of incomplete normed spaces X such that there exist elements in X^* which do not attain their norms.

The following result together with Proposition 2.1 shows that for an incomplete normed space X, Theorem 2.2 need not be valid and in such a case one can find $f\in X^*$ which does not achieve its norm on X even though the completion of X is reflexive.

THEOREM 4.1. *Let X be a real normed space. Then X is incomplete iff there is a normed space Y that is isomorphic to X but not subreflexive.*

PROOF. The "if" part is obvious from the Bishop-Phelps theorem. In order to prove the "only if" part, we employ the idea of Bishop and Phelps [4, p. 31]. Let X be incomplete. Since X is a proper subspace of the completion $\hat{X}$, there exists $\hat{x}$ in $\hat{X}-X$ such that $\|\hat{x}\|=1$. By the Hahn-Banach theorem, one can find f in $\hat{X}^*$ $(=X^*)$ such that $\|f\|=1=f(\hat{x})$. Let $\hat{D}=\{\hat{y}\colon \hat{y}\in\hat{X},\ \|\hat{y}\|\leqq 1$ and $f(\hat{y})=0\}$, $\hat{C}=\{\hat{z}\colon \hat{z}=\lambda\hat{x}+(1-\lambda)\hat{y},\ \hat{y}\in\hat{D}$ and $\lambda\in[0,1]\}$, and define $\hat{B}=\hat{C}\cup$

$(-\hat{C})=\{\alpha\hat{x}+\beta\hat{y}\colon |\alpha|+|\beta|\leqq 1,\ \hat{y}\in\hat{D}\}$. Then it is easily seen that $\hat{B}$ is absolutely convex, bounded, closed and the interior of $\hat{B}$ contains 0. Let $B=\hat{B}\cap X$. Then B is also absolutely convex, bounded, closed and 0 is an interior point of B relative to the subspace X. Hence the gauge function $|||\cdot|||$ on $\hat{X}$ induced by $\hat{B}$ gives a norm on $\hat{X}$ that is equivalent to the original norm $\|\cdot\|$. Hence there is $\alpha>1$ such that $\alpha\|\hat{z}\|\geqq|||\hat{z}|||\geqq\|\hat{z}\|$ for all $\hat{z}\in\hat{X}$. We then denote by Y the space X with the norm $|||\cdot|||$. It is clear that X is isomorphic to Y. Regarding the above-mentioned functional f as an element of $\hat{Y}^*$ $(=Y^*)$, we will show that if $|||f-g|||\equiv \sup(f-g)(\hat{B})<1/2\alpha$ and $g\in Y^*$, then $|||g|||=g(\hat{x})>g(x)$ for every $x\in B$. To this end, we first note that $g(\hat{x})>0$ and $g(\hat{x}-\hat{y})>0$ for every $\hat{y}\in\hat{D}$. In fact, $|f(\hat{x}-\hat{y})-g(\hat{x}-\hat{y})|\leqq 2\|f-g\|\leqq 2\alpha|||f-g|||<1$ for $\hat{y}\in\hat{D}$. Let $\hat{z}$ be any element of $\hat{B}$. If $\hat{z}$ is represented as $\hat{z}=\lambda\hat{x}+(1-\lambda)\hat{y}$ for some $\hat{y}\in\hat{D}$ and $\lambda\in[0,1]$, then we have $g(\hat{z}-\hat{x})=(1-\lambda)g(\hat{y}-\hat{x})\leqq 0$ and $g(\hat{z})\leqq g(\hat{x})$. Thus $g(\hat{z})\leqq g(\hat{x})$ for every $\hat{z}\in\hat{B}$ and $g(\hat{x})=\sup g(\hat{B})=\sup g(B)=|||g|||$. Let x be any element of B. Suppose first that x is represented as $x=\lambda\hat{x}+(1-\lambda)\hat{y}$ for some $\hat{y}\in\hat{D}$ and $\lambda\in[0,1]$. Since $\hat{x}\notin B$, $x\neq\hat{x}$ and we have $\lambda<1$. Hence we have $g(x)=(1-\lambda)g(\hat{y}-\hat{x})+g(\hat{x})<g(\hat{x})$ since $g(\hat{y}-\hat{x})<0$. Next, if x is represented as $x=\lambda(-\hat{x})+(1-\lambda)\hat{y}$ for some $\hat{y}\in\hat{D}$ and $\lambda\in[0,1]$, then $\lambda<1$ by the same reason as above and we have $g(x)=(1-\lambda)g(\hat{y}-\hat{x})-2\lambda g(\hat{x})+g(\hat{x})<g(\hat{x})$. Thus $g(\hat{x})=\sup g(B)>g(x)$ for every $x\in B$, and it is concluded that if $|||g-f|||<1/2\alpha$ and $g\in Y^*$, then $|||g|||>g(x)$ for every $x\in B$. This shows that Y is not subreflexive. q.e.d.

REMARK 4.1. Every pre-Hilbert space is subreflevive. Hence the above theorem does not assert that no incomplete normed spaces are subreflexive. However the theorem implies the following interesting fact: There are "many" normed spaces such that the completions are reflexive but the normed spaces themselves are not even subreflexive.

Theorem 4.1 suggests that (R)-orthogonality might be more adequate than (BP)-orthogonality to treat orthogonality problems in incomplete normed spaces. In fact, we have the following result.

THEOREM 4.2. *Let X be an incomplete normed space and let $\hat{X}$ be the completion of X. Then the following are equivalent:*

(i)′ *$\hat{X}$ is reflexive.*

(ii)′ *For every proper closed subspace M of X, there exists a Cauchy sequence (x_n) in X such that $\|x_n\|\equiv 1$ and* $\lim \operatorname{dist}(x_n, M)=1$.

PROOF. First assume that X is reflexive. Let M be any closed and proper subspace M of X; hence the closure $\hat{M}$ in $\hat{X}$ of M is a proper subspace of $\hat{X}$. Then, by Theorem 1.6, there is an $\hat{x}\in\hat{X}-\hat{M}$ satisfying $\|\hat{x}\|=1$ and $\operatorname{dist}(\hat{x},\hat{M})=1$. Hence one can choose a sequence (x_n) in X so that $\|x_n\|\equiv 1$ and $\lim\|x_n-\hat{x}\|=0$. Since M is dense in $\hat{M}$, we have $\lim\operatorname{dist}(x_n, M)=\lim\operatorname{dist}(x_n,\hat{M})=1$. Thus (i)′

implies (ii)$'$. Conversely, suppose that (ii)$'$ holds and let $\hat{f}$ be any element of $\hat{X}^*$ with $\|\hat{f}\|=1$. Let f be the restriction of $\hat{f}$ to X. Then $f\in X^*$, $\|f\|=1$, and $M\equiv\ker(f)\subsetneqq X$. Hence condition (ii)$'$ implies that there is a Cauchy sequence (x_n) in X with $\|x_n\|\equiv 1$ and $\lim \operatorname{dist}(x_n, M)=1$. Thus the Cauchy sequence (x_n) converges in $\hat{X}$ to some $\hat{x}$ and, by Lemma 1.1, we have $1=\lim\operatorname{dist}(x_n, M)=\lim|f(x_n)|=\lim|\hat{f}(x_n)|=|\hat{f}(\hat{x})|$. This means that every element $\hat{f}\in\hat{X}^*$ attains its norm, and so Theorem 1.6 can be applied to conclude that $\hat{X}$ is reflexive.

q. e. d.

REMARK 4.2. If $\hat{X}$ is weakly sequentially complete, then condition (ii)$'$ can be replaced by the following weaker condition:

(iii)$'$ For every proper closed subspace M of X, there is a weak Cauchy sequence (x_n) in X which is (R)-orthogonal to M.

REMARK 4.3. In an incomplete normed space condition (ii) stated in Theorem 1.6 implies condition (ii)$'$. Hence if conition (ii) holds, then its completion is reflexive. However the converse is not always true even if the incomplete normed space is an inner product space. In fact, we infer from Proposition 4.6 below that the converse does not hold whenever X is a pre-Hilbert space.

EXAMPLE. Let $X=\{x\in C[0, 1]: x(0)=0\}$ and define an inner product on X by $(x, y)=\int_0^1 x(s)y(s)ds$ for $x, y\in X$, where the integral is taken in the sense of Lebesque. Then X becomes an incomplete inner product space and the real Hilbert space $X=L^2(0, 1)$ is understood to be the completion of X. Let $M=\left\{z\in C[0, 1]: \int_0^1 z(s)ds=0\right\}$. Then there is no element $x\in X$ such that $x\perp M$. In fact, suppose that $x\in X$, $\|x\|=(x, x)^{1/2}=1$, and $(x, z)=0$ for all $z\in M$. Then x is orthogonal to the closure $\hat{M}$ of M, and $\hat{M}$ is regarded as the set of all Lebesgue measurable functions y on $[0, 1]$ such that y^2 is Lebesgue integrable over $[0, 1]$ and $\int_0^1 y(s)ds=0$. We shall demonstrate that $x(t)\equiv\alpha$ on $(0, 1)$ for some $\alpha\in\boldsymbol{R}$ with $|\alpha|=1$, which contradicts the assumption that $x\in X$. Suppose that $x(t_1)<x(t_2)$ for some $t_1, t_2\in(0, 1)$, where we may assume that $t_1<t_2$. Let $0<\varepsilon<(x(t_2)-x(t_1))/3$. Since $x\in C[0, 1]$, there is $\delta=\delta(\varepsilon)\in(0, \min\{t_1, t_2-t_1, 1-t_2\})$ such that $|t-t'|<\delta$ implies $|x(t)-x(t')|<\varepsilon$. We then define a step function y on $[0, 1]$ by setting $y(s)=1/2\sqrt{\delta}$ for $s\in[t_1-\delta, t_1+\delta]$, $y(s)=-1/2\sqrt{\delta}$ for $s\in[t_2-\delta, t_2+\delta]$ and $y(s)=0$ otherwise. Then $y\in\hat{M}$, and so $0=(x, y)=\int_{-\delta}^{\delta}(x(t_1+\xi)-x(t_2+\xi))d\xi<-2\delta\xi<0$, a contradiction. This shows that x is constant on $(0, 1)$. Let $x(t)\equiv\alpha$ on $(0, 1)$. Then $1=\|x\|=|\alpha|$. However this is impossible because $x\in C[0, 1]$ and $x(0)=0$.

The next result is a restatement of Theorem 4.2 and corresponds to Theorem

3.3. It may be of some interest to combine the assertion with Theorem 3.2.

COROLLARY 4.2. *Let X be an incomplete normed space. Then $\hat{X}$ is nonreflexive iff there is a proper closed subspace M of X such that any sequence (x_n) with $(x_n)\perp_R M$ is not Cauchy in norm.*

As mentioned in Corollary 1.5, the Birkhoff orthogonality to proper closed subspaces of normed spaces can be studied through norm-attaining functionals on the normed spaces. In the rest of this section we discuss important and typical classes of incomplete normed spaces X such that there exist functionals in X^* which are not norm-attaining.

In what follows, let X be an arbitrary but fixed incomplete normed space and let $\hat{X}$ denote the completion of X. First suppose that $\hat{X}$ is nonreflexive. Then Theorem 3.1 states that there is $\hat{f}\in\hat{X}^*$ which does not achieve its norm. Hence the restriction f of $\hat{f}$ to X belongs to X^* and does not achieve its norm. It thus suffices to consider the case in which $\hat{X}$ is reflexive. However it is extremely difficult at this moment to study the reflexive case in a general setting. We here treat two typical classes of incomplete normed spaces.

PROPOSITION 4.4. *Let X be a real normed space. If there exists a strictly increasing sequence (x_n) of non-trivial closed subspaces of X such that $X=\bigcup_n X_n$, then there exists an $f\in X^*$ which does not attain its norm.*

By Baire's category theorem a normed space as mentioned in the above proposition is necessarily incomplete and is known as a typical example of incomplete normed spaces. To show this proposition, we need the following lemma.

LEMMA 4.5. *Let X be a real normed space and Y a proper closed subspace of X. Then for each $\eta>0$ and each $g\in X^*$ there is $f_\eta\in X^*$ such that $f_\eta(y)=g(y)$* for $y\in Y$ and $\|f_\eta\|_X=\|g\|_Y+\eta$.

Proof. Let $g\in X^*$ and $\eta>0$. Choose any $x_0\in X-Y$ and for every $t\in\boldsymbol{R}$ define a linear functional g_t on the direct sum $Z=Y\oplus[x_0]$ by $g_t(\alpha x_0+y)=\alpha t+g(y)$ for $\alpha\in\boldsymbol{R}$ and $y\in Y$. Then each g_t is continuous on Z. Moreover $\operatorname{dist}(x_0, Y)>0$ and $|g_t(\alpha x_0+y)-g_{t'}(\alpha x_0+y)|=|\alpha(t-t')|\leqq(\|\alpha x_0+y\|/\operatorname{dist}(x_0, Y))|t-t'|$ for every $t, t', \alpha\in\boldsymbol{R}$ and $y\in Y$; so that $\|g_t-g_{t'}\|_Z\leqq|t-t'|/\operatorname{dist}(x_0, Y)$. Therefore $\|g_t\|_Z$ is continuous on $\boldsymbol{R}$ with respect to t. First we see in the same way as in the proof of the Hahn-Banach theorem that there exists a $t_0\in\boldsymbol{R}$ such that $\|g_{t_0}\|_Z=\|g\|_Y$. Next, we have $\lim_{t\to\infty}\|g_t\|_Z=\infty$. Hence the application of the mean-value theorem implies that there exists $t(\eta)\in\boldsymbol{R}$ such that $\|g_{t(\eta)}\|_Z=\|g\|_Y+\eta$. Hence by the Hahn-Banach theorem one finds $f_\eta\in X^*$ such that $f_\eta(x)=g_{t(\eta)}(x)$ for all $x\in Z$ and $\|f_\eta\|_X=\|g\|_Y+\eta$. q.e.d.

PROOF OF PROPOSITION 4.4: Take any $f_1\in X_1^*$. Then by Lemma 4.5 there exists $f_2 \in X_2^*$ such that $\|f_2\|_{X_2} = \|f_1\|_{X_1} + 1/2$ and $f_2(x) = f_1(x)$ for $x \in X_1$. Inductively we can choose for every $n \geqq 1$ an $f_n \in X_n^*$ such that $f_{n+1}(x) = f_n(x)$ for $x \in X_n$ and $\|f_{n+1}\|_{X_{n+1}} = \|f_n\|_{X_n} + 1/2^n$. We then define a functional f on X by putting $f(x) = f_n(x)$ if $x \in X_n$. The functional f is well-defined and we have $\sup\{|f(x)|: x \in X, \|x\| = 1\} = \sup \|f_n\|_{X_n} = \|f_1\|_{X_1} + 1$. Then $f \in X^*$ and it remains to show that f does not attain its norm. From the way of construction of the functionals f_n, $n \geqq 1$, we see that $\|f\|_X = \|f\|_{X_1} + 1 > \|f_n\|_{X_n}$ for $n \geqq 1$. Hence, if $x \in X$ and $\|x\| = 1$ then $x \in X_n$ for some n and $|f(x)| = |f_n(x)| < \|f\|_X$, which means that f is not norm-attaining. q. e. d.

PROPOSITION 4.6. *Let X be an incomplete normed space such that the completion $\hat{X}$ of X is strictly convex. Then there exists $f \in X^*$ which does not achieve its norm.*

PROOF. Take any element $\hat{x} \in \hat{X} - X$ with $\|\hat{x}\| = 1$. Then by the Hahn-Banach theorem one finds $\hat{f} \in \hat{X}^* - \{0\}$ such that $\hat{f}(\hat{x}) = \|\hat{f}\| \, \|\hat{x}\| = \|\hat{f}\|$ Suppose then that $\hat{f}(x) = \|\hat{f}\|$ for some $x \in X$ with $\|x\| = 1$. Then, since $x \neq \hat{x}$ and $\hat{X}$ is strictly convex, we would have $\|\hat{f}\| = 2^{-1}\|\hat{f}\| + 2^{-1}\|\hat{f}\| = f^{-1}(2^{-1}(x + \hat{x})) \leqq \|\hat{f}\| \, \|2^{-1}(x + \hat{x})\| < \|\hat{f}\|$. But this is impossible, and it is concluded that $\hat{f}(x) < \|\hat{f}\|$ for every $x \in X$ with $\|x\| = 1$. Let f be the restriction of $\hat{f}$ to X. Then $\|f\| = \|\hat{f}\|$ and $f(x) < \|f\|$ for $x \in X$ with $\|x\| = 1$. q. e. d.

REMARK 4.3. The above proof tells us more than the statement of Proposition 4.6, namely: To each point $\hat{x}$ in $\hat{X} - X$ there corresponds a funciotnal $f \in X^*$ that is not norm-attaining. This fact is interesting since many of well-known function spaces are strictly convex. Finally, we note that there exists an incomplete normed space X such that each $f \in X^*$ achieves its norm. See James [12].

5. Orthogonality in second dual spaces

In this section we are concerned with the orthogonality in second dual spaces. As mentioned in Remark 3.1, the orthogonality theorems we have established so far are all formulated for proper closed subspaces of a given normed or Banach space. However, if a Banach space X is nonreflexive, X itself is regarded as a proper closed subspace of its second dual X^{**} and it turns out to be a new interesting problem to make a detailed study of elements of X^{**} orthogonal to X. In fact, if we wish to apply the orthogonality theorems established in the preceding sections, we need to take account of the fourth dual X^{****} in order to find elements orthogonal to X in the sense of Birkhoff. In this sense the above problem is important in connection with the study of the geometric structure of second dual spaces.

In the first half of this section we introduce the set of all element $x^{**} \in X^{**}$

with $X \perp x^{**}$ as well as the set of all elements $x^{**} \in X^{**}$ with $x^{**} \perp X$, and we investigate the structure of X^{**} via the two sets. In the latter half of this section we discuss the orthogonality in the second duals of abstract (L) and abstract (M) spaces which are known as typical nonreflexive Banach spaces. It will be shown that there is a remarkable difference between abstract (L) and (M) spaces.

First we observe that Birkhoff orthogonality to splitting subspaces of normed spaces is characterized in terms of projections to the subspaces with norm 1.

PROPOSITION 5.1. *Let X be a Banach space. Suppose that X is represented as the direct sum of two closed subspaces L and R, and that P_L is the projection of X onto L. Then $L \perp R$ iff $\|P_L\| = 1$.*

Let X be a Banach space and let M be a proper closed subspace of X. For the subspace M we define two subsets $L(M)$ and $R(M)$ by

$$L(M) = \{x \in X : x \perp v \quad \text{for} \quad v \in M\},$$
$$R(M) = \{x \in X : v \perp x \quad \text{for} \quad v \in M\},$$

respectively. In what follows, we say that M is *(L)-complemented* in X if $L(M)$ is a closed linear subspace of X and $X = L(M) \oplus M$. Likewise, M is said to be *(R)-complemented* in X if $R(M)$ is a closed linear subspace of X and $X = M \oplus R(M)$.

Proposition 5.1 states that if M is (L)-complemented, then the norm of the projection of X onto $L(M)$ is 1. If M is (R)-complemented, then the norm of the projection of X onto M is 1. Also, both $L(M)$ and $R(M)$ are closed under scalar multiplication and norm closed in X. Finally, it is easy to see that if $X = Y \oplus Z$ and $\|P_Y\| = 1$, then $R(Y) \supset Z$.

We then give two general results in conjunction with the above-mentioned sets $L(M)$ and $R(M)$. Let X be a Banach space which is represented as the direct sum of two closed subspaces Y and Z, i.e., $X = Y \oplus Z$. We say that condition (YH) holds for the decomposition $X = Y \oplus Z$, if

$$(YH) \qquad \|y + z\| = \|y\| + \|z\| \qquad \text{for} \quad y \in Y \quad \text{and} \quad z \in Z.$$

PROPOSITION 5.2. *Suppose that (YH) holds for $X = Y \oplus Z$. Then:*
(a) *$z \perp Y$ iff $z \in Z$; and $y \perp Z$ iff $y \in Y$.*
(b) *Let $x \in X$ and $x = y + z$, $y \in Y$, $z \in Z$. Then*

$$Y \perp x \text{ iff } \|y\| \leqq \|z\|; \quad \text{and} \quad Z \perp x \text{ iff } \|z\| \leqq \|y\|.$$

PROOF. (a): It is clear that $z \in Z$ implies $z \perp Y$, and that $y \in Y$ implies $y \perp Z$. Let $z \perp Y$ and $z = u + v$ be its decomposition. Then $\|v\| = \|z - u\| \geqq \|z\| = \|u + v\| = \|u\| + \|v\|$ by condition (YH). Hence $u = 0$ and $z = v \in Z$. Similarly, $y \perp Z$ implies $y \in Y$.

(b): We here give the proof of the first half since the latter half is similarly proved. Let $Y \perp x$. Then $\|z\| = \|y+(-x)\| \geqq \|y\|$. Conversely, if $\|y\| \leqq \|z\|$, then for every $w \in Y$ and every scalar α we have $\|w+\alpha x\| = \|(w+\alpha y)+\alpha z\| = \|w+\alpha y\| + |\alpha|\,\|z\|$ by (YH); and hence $\|w+\alpha x\| \geqq \|w\| - |\alpha|\,\|y\| + |\alpha|\,\|z\| = \|w\| + |\alpha|(\|z\| - \|y\|) \geqq \|w\|$. This means that $\|y\| \leqq \|z\|$ implies $Y \perp x$. q. e. d.

The above results may be applied to our problem in the following way:

COROLLARY 5.3. *Let X be a Banach space, and assume that there exist linear subspaces Y and Z of X such that $X = Y \oplus Z$ and $\|y+z\| = \|y\| + \|z\|$ holds for $y \in Y$ and $z \in Z$. Then we have:*

(a)′ $L(Y) = \check{Z}$ *and* $L(Z) = Y$.

(b)′ *If $Y \neq \{0\}$ and $Z \neq \{0\}$, then $Z \subsetneqq R(Y)$ and $Y \subsetneqq R(Z)$.*

PROOF. First, we observe that Y and Z are necessarily closed. Hence (a)′ is obtained by applying Proposition 5.2 (a) to the decomposition $X = Y \oplus Z$. Next, $\|P_Y\| = 1$ and $\|P_Z\| = 1$ by assumption. As mentioned before, this implies that $R(Y) \supset Z$ and $R(Z) \supset Y$. The second assertion (b)′ is now obvious from Proposition 5.2 (b). q. e. d.

The next result is important in the subsequent discussions.

THEOREM 5.4. *Suppose that X is represented as the direct sum of two closed subspaces Y and Z, i.e., $X = Y \oplus Z$. Then we have:*

(i) *If $\|P_Y\| = 1$ and $R(Y)$ is convex, then $R(Y) = Z$.*

(ii) *If $\|P_Z\| = 1$ and $R(Z)$ is convex, then $R(Z) = Y$.*

Assume in addition that condition (YH) holds for the decomposition $X = Y \oplus Z$. Then we have:

(i)′ *If $R(Y)$ is convex, then $R(Y) = \{0\}$ and $X = Y$.*

(ii)′ *If $R(Z)$ is convex, then $R(Z) = \{0\}$ and $X = Z$.*

PROOF. Since $R(Y)$ and $R(Z)$ are closed under scalar multiplication, $R(Y)$ (resp. $R(Z)$) becomes a linear subspace if $R(Y)$ (resp. $R(Z)$) is convex. We here give only the proof of the first assertion (i). Assume that $\|P_Y\| = 1$, and let $R(Y)$ be convex and $y \in R(Y)$. Then $P_Y y - y \in \ker(P_Y) = Z \subset R(Y)$ by Corollary 5.3. Hence $P_Y y \in R(Y)$. But $P_Y y \in Y$, and so $P_Y y = 0$. Thus $y \in \ker(R_Y) = Z$, and it is concluded that $R(Y) \subset Z$ and $R(Y) = Z$. We next prove (i)′ in the second assertion since (ii)′ is obtained in the same way. Suppose that (YH) holds and $R(Y)$ is convex. Then $\|P_Y\| = 1$, and $R(Y) = Z$ by the assertion (i). Hence it follows from Corollary 5.3 (b)′ that either $Y = \{0\}$ or $Z = \{0\}$. But $Y \neq \{0\}$ since $\|P_Y\| = 1$. Hence Z must be $\{0\}$ and $R(Y) = \{0\}$. q. e. d.

Let κ and κ^* be the canonical embedding of X into X^{**} and that of X^* into

X^{***}, respectively. In what follows, we simply denote $\kappa(X)$ and $\kappa^*(X)$ by X and X^*, respectively. In the subsequent arguments we often employ the Dixmier decomposition of the third dual space X^{***}: Let X be a nonreflexive Banach space and define $X^{\perp}=\{f\in X^{***}: X\subset\ker(f)\}$. Then Dixmier's decomposition theorem states that $X^{***}=X^*\oplus X^{\perp}$. The Dixmier decomposition of X^{***} implies $X^*\perp X$ since the projection of X^{***} onto X^* has norm 1 and Proposition 5.1 can be applied. Hence $L(X^{\perp})\supset X^*$ and $R(X^*)\supset X^{\perp}$. If in addition (YH) holds for the decomposition $X^{***}=X^*\oplus X^{\perp}$, then $X^*\perp X^{\perp}$, $X^{\perp}\perp X^*$ and we have the relations $L(X^{\perp})=X^*$, $R(X^*)\supsetneqq X^{\perp}$, $L(X^*)=X^{\perp}$ and $R(X^{\perp})\supsetneqq X^*$ by Corollary 5.3. These facts might be applicable to the second duals of dual Banach spaces, See also Theorem 5.9 below.

To discuss when X is (L)-complemented or (R)-complemented in X^{**}, it is important to find useful characterizations of the elements of the sets $L(X)$ and $R(X)$ in X^{**}.

PROPOSITION 5.5. *Let X be a nonreflexive Banach space. Let $L(X)=\{\mu\in X^{**}: \mu\perp x$ for $x\in X\}$ and let $\lambda\in X^{**}$. Then the following two conditions are equivalent:*

(i) $\lambda\in L(X)$, *i.e.*, $\|\lambda+x\|\geqq\|\lambda\|$ *for* $x\in X$.

(ii) $\|\lambda\|=\sup\{|f(\lambda)|: f\in X^{\perp}$ *and* $\|f\|=1\}$.

Moreover condition (ii) implies the following condition:

(iii) $F^{**}(\lambda)\not\subset X^*$, *where F^{**} is the duality mapping of X^{**}.*

If in addition, (YH) holds for the Dixmier decomposition of X^{***}, then (iii) implies (ii).

PROOF. Let $\lambda\in L(X)$. We may assume that $\lambda\neq 0$ since (ii) is trivial for $\lambda=0$. Then $\lambda\notin X$, and so the application of Lemma 1.1 and the Hahn-Banach theorem implies that there is $f\in X^{***}$ such that $\|f\|=1$, $X\subset\ker(f)$ and $f(\lambda)=\|\lambda\|$, from which (ii) follows. Conversely, assume that (ii) holds. Then in view of the Dixmier decomposition of X^{***}, we have $\|\lambda+x\|\geqq\sup\{|f(\lambda+x)|: f\in X^{\perp}, \|f\|\leqq 1\}=\|\lambda\|$ and we obtain (i). To show the second assertion, we observe that (ii) holds iff $F^{**}(\lambda)\cap X^{\perp}\neq\emptyset$. Hence it is obvious that (ii) implies (iii). Conversely, assume that (iii) and (YH) hold for the decomposition $X^{***}=X^*\oplus X^{\perp}$. Let $f\in F^{**}(\lambda)-X^*$ and consider the decomposition $f=f_1+f_2$, $f_1\in X^*$, $f_2\in X^{\perp}$. Then $f_2\neq 0$ and we have $\|\lambda\|^2=f(\lambda)=f_1(\lambda)+f_2(\lambda)\leqq(\|f_1\|+\|f_2\|)\|\lambda\|=\|f\|\cdot\|\lambda\|=\|f\|^2$ by (YH). Hence $f_2(\lambda)=\|f_2\|\cdot\|\lambda\|$. Put $g=\|f_2\|^{-1}\|\lambda\|f_2$. Then $g(\lambda)=\|\lambda\|^2$, $\|g\|=\|\lambda\|$ and $g\in X^{\perp}$, from which it is concluded that $F^{**}(\lambda)\cap X^{\perp}\neq\emptyset$ and (ii) is obtained. q.e.d.

EXAMPLE. It might be of some interest to illustrate the above mentioned results in the case where $X=c_0$, $X^*=\ell^1$ and $X^{**}=\ell^{\infty}$. Let $\lambda=(\zeta_n)_{n=1}^{\infty}\in X^{**}$.

Then we see from [9, Proposition 5.5] that $\lambda \in L(X)$ iff $\|\lambda\| = \limsup_{n\to\infty} |\zeta_n|$. In fact, in this particular case, conditions (i), (ii), (iii) and (iv): $\|\lambda\| = \limsup |\zeta_n|$ are equivalent. First it follows from [9, Proposition 5.5] that (iii) and (iv) are equivalent. Next, assume that $\|\lambda\| = \limsup |\zeta_n|$. Then there is a subsequence $(|\zeta_{n(k)}|)$ converging to $\|\lambda\|$. We then consider the infinite set $\sigma = \{n(k) : k \in N\}$ and choose a free ultrafilter $\mathscr{U}$ with $\sigma \in \mathscr{U}$. Then the associated 0–1 measures, say f, on 2^N belongs to X and $\|f\| = 1$. Moreover $|f(\lambda)| = \lim |\zeta_{n(k)}| = \|\lambda\|$, and (ii) is obtained. Thus, in view of Proposition 5.5, it is concluded that the above four conditions are equivalent.

The next result gives a characterization of $R(X)$.

PROPOSITION 5.6. *Let X be a nonreflexive Banach space and $B(X^*)$ the closed unit ball with center* 0. *Let $R(X) = \{\mu \in X^{**} : x \perp \mu \text{ for } x \in X\}$ and $\lambda \in X^{**}$. Then the following are equivalent*:

(i)′ *$\lambda \in R(X)$, i.e., $\|\lambda + x\| \geqq \|x\|$ for $x \in X$.*

(ii)′ $\ker(\lambda) \cap B(X^*)$ *is weakly-star dense in* $B(X^*)$.

(iii)′ $\ker(\lambda) \cap B(X^*)$ *is a norming set.*

PROOF. The equivalence between (i)′ and (ii)′ is proved in [8, Lemma 2]. The equivalence between (ii)′ and (iii)′ is easily obtained. q.e.d.

Using the above characterization, Godefroy showed that $R(X)$ is a weak-star closed linear subspace of X^{**} if any one of the following conditions is satsisfied:

(a) X^* has the Radon-Nikodym property.

(b) X is separable and does not contain an isomorphic copy of ℓ^1.

(c) There is an equivalent norm on X and a dense subset M of X such that the norm is Fréchet differentiable at each point of M.

The next interesting result is given in [8, Théorème 7].

PROPOSITION 5.7. *Suppose that $R(X)$ is a weak-star closed linear subspace of X^{**}. Then the following are equivalent*:

(1°) *X has a predual.*

(2°) *X is (R)-complemented in X^{**}.*

(3°) *There is a projection of norm 1 of X^{**} onto $\kappa(X)$.*

Moreover, in this case, any pair of preduals are isometric.

In the rest of this section we consider abstract (L) spaces and abstract (M) spaces and investigate the sets $L(X)$ and $R(X)$ in their second dual. We shall see that both of the sets $L(X)$ and $R(X)$ are considerably complicated even if X is such a particular space.

Let S be a set, and Σ be a σ-field of subsets of S. Let $\mathscr{N}$ be a proper subfamily of Σ which is closed under the formation of countable unions and has the property that $N \in \mathcal{N}$, $A \subset S$, and $A \subset N$ imply $A \in \mathcal{N}$. For a Σ-measurable real-valued function $f(s)$ on S, we define the essential supremum of $|f(\cdot)|$ (denoted by $\|f\|_\infty$) be the infimum of the set of numbers α with $\{s \in S: |f(s)| > \alpha\} \in \mathcal{N}$ if the set is non-void. Othwrwise, we write $\|f\|_\infty = \infty$. Let $L^\infty(S, \Sigma, \mathcal{N})$ be the usual space consisting of all measurable real-valued functions on S with finite norm $\|\cdot\|_\infty$, where two functions f and f' are identified if $\|f - f'\|_\infty = 0$. Under the usual addition, scalar multiplication, and the norm $\|\cdot\|_\infty$, $L^\infty(S, \Sigma, \mathcal{N})$ becomes a real Banach space. Let $ba(S, \Sigma, \mathcal{N})$ be the space of all bounded and finitely additive set functions on Σ which vanish on $\mathcal{N}$. The norm of an element λ of $ba(S, \Sigma, \mathcal{N})$ is defined by $\|\lambda\| = \sup_{E \in \Sigma} \lambda(E) - \inf_{E \in \Sigma} \lambda(E)$, and $ba(S, \Sigma, \mathcal{N})$ becomes a Banach space with this norm. It is well-known that $ba(S, \Sigma, \mathcal{N})$ is identified with the dual of $L^\infty(S, \Sigma, \mathcal{N})$. For $\lambda, \nu \in ba(S, \Sigma, \mathcal{N})$, we write $\nu \geqq \lambda$ if $\nu - \lambda \geqq 0$. This is a partial ordering in $ba(S, \Sigma, \mathcal{N})$ and the space $ba(S, \Sigma, \mathcal{N})$ becomes a Banach lattice. For any pair λ, ν in $ba(S, \Sigma, \mathcal{N})$, we define new measures $\lambda \wedge \nu$ and $\lambda \vee \nu$ by

$$(\lambda \wedge \nu)(E) = \inf_{F \subset E, F \in \Sigma} (\lambda(F) + \lambda(E \cap F^c)), \qquad E \in \Sigma,$$

and $\lambda \vee \nu = -((-\lambda) \wedge (-\nu))$, respectively. A bounded real-valued measure μ on Σ is said to be purely finitely additive if every countably additive measure λ satisfying $0 \leqq \lambda(E) \leqq |\mu|(E)$ for $E \in \Sigma$ is identically zero. In what follows we permit ourselves the common abbreviations, *c.a.* measure and *p.f.a.* measure, in referring respectively to the countably additive and purely finitely additive measures.

Basic to the study of the space $ba(S, \Sigma, \mathcal{N})$ is the following

THEOREM 5.8. *Every measure λ in $ba(S, \Sigma, \mathcal{N})$ is uniquely decomposed as the sum of a c.a. measure λ_c and a p.f.a. measure λ_p in $ba(S, \Sigma, \mathcal{N})$. In this case, we have $\|\lambda\| = \|\lambda_c\| + \|\lambda_p\|$. If in particular $\lambda \geqq 0$, then $\lambda_c \geqq 0$ and $\lambda_p \geqq 0$.*

We call the decomposition $\lambda = \lambda_c + \lambda_p$ the Yosida-Hewitt decomposition of λ. See [18]. For the proof of the second assertion we refer to [9, Proposition 1.3]. Also, we denote by $ca(S, \Sigma, \mathcal{N})$ and $pfa(S, \Sigma, \mathcal{N})$ the set all *c.a.* measures in $ba(S, \Sigma, \mathcal{N})$ and that of all *p.f.a.* measures in $ba(S, \Sigma, \mathcal{N})$, respectively. We then give some characteristic properties of the above sets.

First both of the sets are closed linear subspaces of $ba(S, \Sigma, \mathcal{N})$. Hence we can apply Theorem 5.8, Propositions 5.1 and 5.2 to get the following result.

THEOREM 5.9. *Let $ca(S, \Sigma, \mathcal{N})$, $pfa(S, \Sigma, \mathcal{N})$ be as above.*

(i) *$ca(S, \Sigma, \mathcal{N}) \perp pfa(S, \Sigma, \mathcal{N})$ and $pfa(S, \Sigma, \mathcal{N}) \perp ca(S, \Sigma, \mathcal{N})$.*

(ii) *Let $\lambda \in ba(S, \Sigma, \mathcal{N})$ and $\lambda = \lambda_c + \lambda_p$ the Yosida-Hewitt decomposition. Then:*

(a) *$\lambda \perp ca(S, \Sigma, \mathcal{N})$ iff $\lambda \in pfa(S, \Sigma, \mathcal{N})$;*
$\lambda \perp pfa(S, \Sigma, \mathcal{N})$ iff $\lambda \in ca(S, \Sigma, \mathcal{N})$.

(b) *$ca(S, \Sigma, \mathcal{N}) \perp \lambda$ iff $\|\lambda_c\| \leqq \|\lambda_p\|$;*

$pfa(S, \Sigma, \mathcal{N}) \perp \lambda$ *iff* $\|\lambda_c\| \geqq \|\lambda_p\|$.

Assertion (ii)–(a) gives an extension of the result of Bilyeu and Lewis [1, Lemma 2.1]. Assertion (ii) tells us that for $\lambda \in ba(S, \Sigma, \mathcal{N})$, $\lambda \in pfa(S, \Sigma, \mathcal{N})$ implies $ca(S, \Sigma, \mathcal{N}) \perp \lambda$ but the converse is not necessarily true, and that $\lambda \in ca(S, \Sigma, \mathcal{N})$ implies $pfa(S, \Sigma, \mathcal{N}) \perp \lambda$ but the converse is not always valid.

Now let $(S, \Sigma, \mathcal{N})$ be a finite nonnegative complete measure space and let $\mathcal{N}=\{E \in \Sigma: \mu(E)=0\}$. Then, by the Radon-Nikodym theorem, the space $L^1(S, \Sigma, \mathcal{N})$ is (L)-complemented in its second dual space $ba(S, \Sigma, \mathcal{N})$. However, if $L^1(S, \Sigma, \mathcal{N})$ is infinite-dimentional then $pfa(S, \Sigma, \mathcal{N})$ is nontrivial and $L^1(S, \Sigma, \mathcal{N})$ is never (R)-complemented by Corollary 5.3. Moreover, in this case, Theorem 5.4 implies that both $R(L^1(S, \Sigma, \mathcal{N}))$ and $R(pfa(S, \Sigma, \mathcal{N}))$ are nonconvex. Thus we have obtained fairly precise results concerning the structure of the space $ba(S, \Sigma, \mathcal{N})$.

We next treat the space of continuous functions on a compact Hausdorff space. Let S be a compact Hausdorff space and let $C(S)$ denote the space of all continuous functions on S. The space $C(S)$ is a Banach space under the supremum norm and, by the Riesz representation theorem, its dual space can be identified with the space $\mathcal{M}(S)$ of Radon measures (i.e., regular Borel measures) on S whose norm is defined by the total variation. Our first result for the space $C(S)$ is the following.

THEOREM 5.10. *Let S be a compact Hausdorff space consisting of infinitely many points. Then there exists an element λ in $C(S)^{**}-C(S)$ such that λ is orthogonal to $C(S)$. Thus $L(C(S)) \neq \{0\}$.*

PROOF. Since S is an infinite set, there is a Baire set B that is not open. We then define λ: $S \to \boldsymbol{R}$ by $\lambda(s)=1$ if $s \in B$ and $\lambda(s)=-1$ if $s \in B^c$. Then λ is not continuous, although it is Baire measurable. Therefore $\lambda \in C(S)^{**}-C(S)$ and $\|\lambda\|=\sup_{s \in S}|\lambda(s)|=1$. We then demonstrate that $\lambda \perp C(S)$. Since the boundary ∂B of B is nonempty, we can take an $s_0 \in \partial B$. Let x be any element of $C(S)$. Then there are three cases to check: (i) $x(s_0)>0$, (ii) $x(s_0)<0$, and (iii) $x(s_0)=0$. In the first case (i), the continuity of x implies that there exists a neighborhood $U(s_0)$ of s_0 such that $x(s)>0$ for all $s \in U(s_0)$. On the other hand, $U(s_0) \cap B^c \neq \emptyset$; hence we can take a point s_1 in $U(s_0) \cap B^c$. Then we have $1<x(s_0)+1=|\lambda(s_1)-x(s_1)| \leqq \|\lambda-x\|$. Similarly, we have $\|\lambda-x\| \geqq 1$ for the second case (ii). In the third case (iii), one finds for every $\varepsilon>0$ a neighborhood $V(s_0)$ of s_0 such that $|x(s)| \leqq \varepsilon$ for all $s \in V(s_0)$. Take any point $s_2 \in V(s_0) \cap B$ $(\neq \emptyset)$. Then $\|\lambda-x\| \geqq |\lambda(s_2)-x(s_2)| \geqq |\lambda(s_2)|-|x(s_2)| \geqq 1-\varepsilon$ for all $x \in C(S)$. Consequently we obtain $\|\lambda-x\| \geqq 1$ for all $x \in C(S)$. This shows that λ is orthogonal to $C(S)$. q.e.d.

The following result shows that the space $C(S)$ of real-valued continuous functions on a compact Hausdorff space is not (L)-complemented in $C(S)^{**}$ provided $\dim C(S)=\infty$.

THEOREM 5.11. *Let S be a compact Hausdorff space consisting of infinitely many points. Then there exist λ_1 and $\lambda_2\in C(S)^{**}-C(S)$ such that $\lambda_1\perp C(S)$ and $\lambda_2\perp C(S)$, but $\lambda_1+\lambda_2$ is not orthogonal to $C(S)$. Thus $L(C(S))$ is not linear and $C(S)$ is not (L)-complemented in $C(S)^{**}$.*

PROOF. We first notice that it suffices to show that there are two nonempty Baire subsets B_1 and B_2 which are not clopen (i.e., not closed or not open) and satisfy $B_1\cap B_2=\emptyset$ and $B_1\cup B_2\subsetneqq S$. In fact, assume that B_1 and B_2 are such subsets of S. Hence we can define $\lambda_i(s)=1$ if $s\in B_i$ and $\lambda_i(s)=-1$ if $s\in B_i^c$, $i=1, 2$. Then we see in the same way as in the proof of Theorem 5.10 that $\lambda_i\in C(S)^{**}-C(S)$, $\operatorname{dist}(\lambda_i, C(S))=1$, $i=1, 2$, and that $\operatorname{dist}(\lambda_1+\lambda_2, C(S))\leqq 1\leqq\|\lambda_1+\lambda_2\|=2$. Thus $\lambda_1\perp C(S)$, $\lambda_2\perp C(S)$, but $\lambda_1+\lambda_2$ is not orthogonal to $C(S)$. We then show that there are Baire subsets B_1 and B_2 as mentioned above. To this end, we consider the following two cases: (i) S is disconnected, (ii) S is connected. In the first case (i) we can take a nonempty and proper clopen subset B of S. The set B is a Baire set, and it is easy to see that B is also a compact Hausdorff space with respect to the relative topology. Also, we may assume without loss of generality that B is infinite. Hence there exists a Baire subset $B_1\subset B$ which is not open. If not, there would exist an infinite disjoint family of clopen subsets of B, which contradicts the compactness of B. We then put $B_2=B-B_1$. Then B_1 and B_2 are both Baire, not clopen, and satisfy $B_1\cap B_2=\emptyset$ and $B_1\cup B_2=B\subsetneqq S$. Finally assume that S is connected and choose any but distinct elements s_1 and s_2 of S. Then it follows from Urysohn's Theorem that there exists an $x\in C(S)$ such that $x(s_1)=0$ and $x(s_2)=1$. Since S is connected, the range of x must contain the closed interval $[0, 1]$. Put $B_1=\{s\in S: 0\leqq f(s)<1/3\}$ and $B_2=\{s\in S: 2/3\leqq f(s)<1\}$. Then B_1 and B_2 are the desired sets. q.e.d.

We now consider the set $R(C(S))$ in $C(S)^{**}$. In contrast with the set $L(C(S))$ in $C(S)^{**}$, the structure of $R(C(S))$ depends strongly upon the compact Hausdorff space S.

First we see that $R(C(S))\neq\{0\}$ if $C(S)$ is infinite-dimensional. In fact, S is not finite in this case and contains at least one point s_0 which is not isolated. Hence one can define a bounded Borel function λ by $\lambda(s)=1$ if $s=s_0$ and $\lambda(s)=0$ if $s\neq s_0$, and it is easily seen that $C(S)\perp\lambda$.

If $C(S)$ is identified with the space c of convergent sequences, then $R(c)$ is a 1-dimensional subspace of ℓ^∞. In fact, let κ be the canonical mapping from c into ℓ^∞. Then κ assings to each $x=(\xi_n)_{n=1}^\infty\in c$ with $\xi_0=\lim\xi_n$ the element $\kappa(x)=(\xi_0, \xi_1, \xi_2,\ldots)\in\ell^\infty$. Let $y=(\zeta_n)_{n=0}^\infty\in\ell^\infty$ and suppose that $\|\kappa(x)+y\|\geqq\|x\|$ for $x\in c$. Let $e_n=(\delta_{n,k})_{k=1}^\infty\in c$, where $\delta_{n,k}$ denotes Kronecker's delta. Then

$\kappa(e_n)=(\delta_{n,k})_{k=0}^{\infty}$ for $n\in N$, where $\delta_{n,0}=0$; and so $\|\kappa(2\|y\|e_n)+\theta y\|=|2\|y\|+\theta\zeta_n|\geqq 2\|y\|$ for $\theta\in C$ with $|\theta|=1$ by the hypothesis. Put $\theta_n=-\zeta_n/|\zeta_n|$ if $\zeta_n\neq 0$ and $\theta_n=1$ if $\zeta_n=0$. Then $|\theta_n|=1$, and so $2\|y\|-|\zeta_n|=|2\|y\|-|\zeta_n||=|2\|y\|+\theta_n\zeta_n|\geqq 2\|y\|$. This implies $\zeta_n=0$ for $n\in N$. Conversely, if $y=(\zeta_n)_{n=0}^{\infty}\in\ell^{\infty}$ and $\zeta_n=0$ for $n\in N$, then for every $x=(\xi_n)_{n=1}^{\infty}\in c$ with $\xi_0=\lim\xi_n$ we have $\|\kappa(x)+y\|=\max\{|\xi_0+\zeta_0|, \|x\|\}\leqq\|x\|$. This shows that $R(c)=\{y=(\zeta_n)_{n=0}^{\infty}: \zeta_n=0$ for $n\in N\}$. Therefore, $\ell^{\infty}\supsetneqq c+R(c)$ and c is not (R)-complemented in ℓ^{∞}.

In case S is the closed unit interval $[0, 1]$, $R(C(S))$ is not a linear subspace of ℓ^{∞}. In fact, let λ be a function on $[0, 1]$ which takes the value 1 at rational points and the value 0 at irrational points. Then both λ and $1-\lambda$ are regarded as elements of $R(C(S))$ in $C(S)^{**}$, but the sum of λ and $1-\lambda$, the constant function 1, does not belong to $R(C(S))$.

Finally, we give a result concerning the orthogonality in the second duals of abstract (L) spaces and abstract (M) spaces. A Banach lattice X is called an abstract (L) space, if $\|x+y\|=\|x\|+\|y\|$ whenever $x, y\in X$ and $x\wedge y=0$; and X is called an abstract (M) space, if $\|x+y\|=\max(\|x\|, \|y\|)$ whenever $x, y\in X$ and $x\wedge y=0$. An element $e\geqq 0$ of X is siad to be a *weak unit* of X if $e\wedge x=0$ for $x\in X$ implies $x=0$. An element $e\geqq 0$ of X is said to be a *strong unit* of X provided that $\|x\|\leqq 1$ iff $|x|\leqq e$. The space c_0 is an example of an abstract (M) space without a strong unit which has, however, a weak unit.

The following theorem is due to S. Kakutani ([13], [14]).

THE KAKUTANI'S REPRESENTATION THEOREM. (a) *An abstract (L) space X is order isometric to some Lebesgue space $L^1(S, \Sigma, \mu)$. If in addition X has a weak unit, then μ can be chosen to be a finite measure.*

(b) *An abstract (M) space X is order isometric to a sublattice of $C(S)$ for some compact Hausdorff space S. If in addition X has a strong unit, then X is order isometric to some $C(S)$.*

Applying the Kakutani's representation theorem to the previous results, we obtain the following:

THEOREM 5.12. *An infinite-dimensional abstract (L) space X is (L)-complemented in the second dual space X^{**}, but it is never (R)-complemented in X^{**}. An infinite-dimensional abstract (M) space X with a strong unit is never (L)-complemented in its second dual X^{**}, and it is not (R)-complemented in general. Moreover, in both cases, $L(X)\neq\{0\}$ and $R(X)\neq\{0\}$.*

Added in Proof. 1) Theorem 5.12 states that Proposition 5.7 is not applicable to Banach spaces such as abstract (L) and (M) spaces. However some of the well-known Banach spaces are (R)-complemented in their second duals. Let

J and JT denote the James space and the James tree space, respectively. Then it is seen from Propositions 5.6 and 5.7 that J^*, JT and the odd dual spaces $JT^{(2n+1)}$ $(n \geqq 0)$ are examples of such spaces. The spaces J^* and JT are known to be separable dual spaces which do not contain ℓ^1; the dual spaces of J^* and $JT^{(2n+1)}$ have the RNP.

2) In connection with Theorem 1.6 it should be noted that a Banach space X is reflexive iff for every Banach space Y containing X (isometrically) there is a nonzero $y \in Y$ such that $y \perp X$ in Y.

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