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LCM-stableness in ring extensions

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Introduction

In his paper [4], R. Gilmer introduced the concept of LCM-*stableness*, relating to GCD-properties of a commutative group ring. The main purpose of this paper is to point out that, in some cases, the necessary and sufficient conditions for a ring extension to be LCM-*stable* can be given in terms of polynomial grade, originally due to M. Hochster and developed by D. G. Northcott. For this purpose, we shall introduce two further notions, R_2-*stableness* and G_2-*stableness*, and investigate the relationship between LCM-stableness and them. In these discussions it is important to know when '$\mathrm{Gr}(I) \geqq 2$' implies '$\mathrm{gr}(I) \geqq 2$'. We shall give in the last section an example of a finitely generated ideal I in an integral domain, with $\mathrm{gr}(I)=1$ and $\mathrm{Gr}(I) \geqq 2$.

In §2, we shall show that flatness, INC and LCM-stableness are all equivalent notions for a simple extension which satisfies some conditions (cf. Th. 2.7). In §3, we shall examine a relation between R_2-stableness and G_2-stableness, and study universality of LCM-stableness; namely, in Th. 3.5 we shall prove that $A \subset B$ is G_2-stable if and only if $A[X] \subset B[X]$ is G_2-stable, and also if and only if $A[X] \subset B[X]$ is R_2-stable. As a corollary to this theorem, we can see that, in case A is locally a GCD-domain, $A \subset B$ is LCM-stable if and only if so is $A[X] \subset B[X]$.

In §4, we shall examine LCM-stableness of a simple extension $A \subset A[\alpha]$. Let I be the kernel of the canonical homomorphism of $A[X]$ onto $A[\alpha]$. We shall first show in Th. 4.3 that if $I=(f(X))$ $(f(X) \in A[X])$, then $A[Y] \subset A[\alpha][Y]$ is R_2-stable if and only if $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$. Moreover, we shall show in Th. 4.5 that, under some conditions, $A \subset A[\alpha]$ is R_2-stable if and only if $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$. In particular, we can show that if A is locally a GCD-domain, then $A \subset A[\alpha]$ is LCM-stable if and only if $\mathrm{Gr}(\mathrm{c}(I)) \geqq 3$ (cf. Cor. 4.6).

In §5 and §6, we shall deal with the case of doubly generated extension $A \subset A[\alpha, \beta]$. In §5, we shall study a special case (cf. Th. 5.5). In §6, we shall consider the case where $K(\alpha)$, $K(\beta)$ are linearly disjoint over the quotient field K of A. Firstly we shall treat the case when $A \subset A[\alpha]$ is (faithfully) flat (cf. Prop. 6.1, Th. 6.4), and secondly we shall examine the kernel $K_{\alpha,\beta}$ of the canonical homomorphism of $A[X, Y]$ onto $A[\alpha, \beta]$ by means of polynomial grade (cf. Prop. 6.6, Cor. 6.7, Prop. 6.8). Moreover, in case A is locally a GCD-domain, we shall give a characterization of LCM-stableness of $A \subset A[\alpha, \beta]$ (cf. Th. 6.10).

Finally, in §7, we shall give an example such that R_2-stableness does not necessarily imply G_2-stableness.

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Notation and terminology

Throughout this paper, rings will be all integral domains unless otherwise specified and X will be an indeterminate. Moreover, A will be an integral domain with the quotient field K and Ω will be the algebraic closure of K. We let $\mathrm{Spec}(A)$ and $\mathrm{Max}(A)$ stand for the set of all prime ideals of A and that of all maximal ideals of A respectively. An *overring* of A is a subring of K containing A. Let I be an ideal of A. We denote by $\mathrm{Gr}(I)$ and $\mathrm{gr}(I)$ the polynomial grade of I and the classical grade of I respectively. Let J be an ideal of $A[X]$. We denote by $\mathrm{c}(J)$ the ideal of A generated by all coefficients of all polynomials in J and we call it the *content* of J.

§1. Basic properties of LCM-stableness

Let A and B be integral domains. We say that $A\subset B$ is LCM-*stable* if $(aA\cap bA)B=aB\cap bB$ for all $a, b\in A$ (cf. [4]). It follows easily from the definition that $A\subset B$ is LCM-stable if and only if $(a:_A b)B=a:_B b$ for all $a, b\in A-\{0\}$. In this section, we examine basic properties of LCM-stableness. The following proposition is a well-known result on flatness.

PROPOSITION 1.1. *If $A\subset B$ is flat, then $A\subset B$ is LCM-stable. In particular, $A\subset A_S$ is LCM-stable for each multiplicatively closed set S in A.*

As for transitivity, the following proposition is important. However it can be proved easily, and so the proof is omitted.

PROPOSITION 1.2. *Let $A_1\subset A_2\subset A_3$ be integral domains. Then we have the following statements.*

(1) *If both $A_1\subset A_2$ and $A_2\subset A_3$ are LCM-stable, then so is $A_1\subset A_3$.*
(2) *Assume that $IA_3\cap A_2=I$ for any ideal I of A_2. If $A_1\subset A_3$ is LCM-stable, then so is $A_1\subset A_2$.*

REMARK 1.3. LCM-stableness of both $A_1\subset A_2$ and $A_1\subset A_3$ does not necessarily imply that of $A_2\subset A_3$. Moreover, LCM-stableness of both $A_1\subset A_3$ and $A_2\subset A_3$ does not necessarily imply that of $A_1\subset A_2$. For example, the former case is $\boldsymbol{Z}\subset\boldsymbol{Z}[\sqrt{5}]\subset\boldsymbol{Z}[(1+\sqrt{5})/2]$ and the latter case is $\boldsymbol{Z}[\sqrt{5}]\subset$

$\boldsymbol{Z}[(1+\sqrt{5})/2]\subset\boldsymbol{Q}[\sqrt{5}]$, where $\boldsymbol{Z}$ *is the ring* of *integers* and $\boldsymbol{Q}$ *is the rational number field.*

PROPOSITION 1.4 (cf. [12], Lemma 2). *Let $A\subset T\subset B$ be integral domains with $T\subset K$. If $A\subset B$ is LCM-stable, then so is $T\subset B$.*

PROOF. Let $x, y\in T-\{0\}$. Put $x=a/c$ and $y=b/c$, where $a, b, c\in A-\{0\}$. Then we have $x:_B y=a:_B b=(a:_A b)B\subset(a:_T b)B=(x:_T y)B$. Thus, $x:_B y=(x:_T y)B$. This shows that $T\subset B$ is LCM-stable.

COROLLARY 1.5. *Let $A\subset B$ be LCM-stable. Then the following statements hold.*

(1) *For each multiplicatively closed set S in A with $A_S\subset B$, $A_S\subset B$ is LCM-stable.*
(2) *Suppose that S and T are multiplicatively closed sets of A and B respectively and that $S\subset T$. Then $A_S\subset B_T$ is LCM-stable.*

As for A-algebras, we give some characterizations of LCM-stableness.

PROPOSITION 1.6. *For $A\subset B\subset C$, the following statements are equivalent.*

(1) *$B\subset C$ is LCM-stable.*
(2) *For each $P\in\mathrm{Spec}(A)$, $B_P\subset C_P$ is LCM-stable.*
(3) *For each $M\in\mathrm{Max}(A)$, $B_M\subset C_M$ is LCM-stable.*
(4) *For each $Q\in\mathrm{Max}(C)$ with $Q\cap B=P$, $B_P\subset C_Q$ is LCM-stable.*

PROOF. We first prove (3)⇒(1). Let $a, b\in B$ and $M\in\mathrm{Max}(A)$. We have obviously $(a:_B b)C\subset a:_C b$. Since $B_M\subset C_M$ is LCM-stable, $(a:_B b)C_M=(a:_B b)C_M=a:_{C_M} b=(a:_C b)C_M$. Therefore, $(a:_B b)C=a:_C b$. That is, $B\subset C$ is LCM-stable.

(4)⇒(1) can be proved similarly. Moreover, the assertions (1)⇒(2)⇒(3) and (1)⇒(4) follow immediately from Cor. 1.5.

PROPOSITION 1.7 (cf. [3], Lemma 6.5). *Let B be an overring of A. Then the following statements are equivalent.*

(1) *$A\subset B$ is LCM-stable.*
(2) *$(y:_A x)B=B$ for each $x/y\in B$.*
(3) *$A\subset B$ is flat.*

PROOF. The equivalence of (2) and (3) follows from Lemma 1 and Th. 1 in [12]. The implication (3)⇒(1) is obvious (cf. Prop. 1.1).

(1)⇒(2). Let $x/y\in B$, where $x, y\in A$ and $y\neq 0$. Then since $A\subset B$ is LCM-stable, we have $(y:_A x)B=y:_B x=B$.

F. Richman and D. E. Dobbs gave some characterizations of a Prüfer domain in terms of flatness (cf. [12], Th. 4 and [2], Prop. 3.1). By virtue of Prop. 1.7, we have a new characterization of a Prüfer domain.

COROLLARY 1.8. *The following statements are equivalent.*
(1) *A is a Prüfer domain.*
(2) *For any integral domain B containing A, $A \subset B$ is LCM-stable.*
(3) *For each $u \in K$, $A \subset A[u]$ is LCM-stable.*

Next, we give a sufficient condition for $A \subset B$ to be LCM-stable.

PROPOSITION 1.9. *Let $A \subset B$ be integral domains. If $A \subset A[x, y]$ is LCM-stable for any $x, y \in B$, then $A \subset B$ is LCM-stable.*

PROOF. Let $a, b \in A$ and assume that $ax = by \in aB \cap bB$, where $x, y \in B$. Then since $A \subset A[x, y]$ is LCM-stable, we have $ax = by \in aA[x, y] \cap bA[x, y] = (aA \cap bA)A[x, y] \subset (aA \cap bA)B$. Therefore, $aB \cap bB = (aA \cap bA)B$. Thus, $A \subset B$ is LCM-stable.

REMARK 1.10. In the above proposition, we can not replace two elements x and y by a single element x. In fact, let $A = \boldsymbol{Q}[s, t]_{(s,t)}$, where s, t are indeterminates over $\boldsymbol{Q}$. We can take $x, y \in \Omega$ with the properties that $x^2 + sx + s^2 = 0$, $y^2 + ty + t^2 = 0$ and $tx = sy$. Then since A is integrally closed and $A[x, y]$ is integral over A, $A[z]$ is a free A-module for each $z \in A[x, y]$. In particular, $A \subset A[z]$ is LCM-stable for each $z \in A[x, y]$. On the other hand, since $(s, t) \neq A$, $A \subset A[x, y]$ is not LCM-stable (cf. Prop. 5.3).

It is well-known that for an overring B of A, if $A \subset B$ is flat and B is integral over A, then $A = B$ (see [12]). This fact suggests to us the following propositions on LCM-stableness.

PROPOSITION 1.11. *Let A be a quasi-local domain with the unique maximal ideal M and B be an integral domain containing A. Assume that $MB \neq B$. If $A \subset B$ is LCM-stable, then we have $B \cap K = A$.*

PROOF. Let $x = a/b \in B \cap K$, where $a, b \in A - \{0\}$. Since $A \subset B$ is LCM-stable, we have $a = bx \in (aA \cap bA)B$. Therefore, there exist $x_i \in aA \cap bA$ and $\beta_i \in B$ such that $a = bx = \sum_{i=1}^{r} x_i \beta_i$. We can put $x_i = ay_i = bz_i$ for $1 \leqq i \leqq r$, where $y_i, z_i \in A$. Then we have $1 = \sum_{i=1}^{r} y_i \beta_i$. Since $MB \neq B$, there exists i such that $y_i \notin M$. Therefore, $a \in bA$. Thus, $x \in A$. That is, we have $B \cap K = A$.

From Prop. 1.11 and Prop. 1.6, the following corollaries follow easily.

COROLLARY 1.12. *Let $A \subset B$ be integral domains. Assume that for each $P \in \operatorname{Spec}(A)$ there exists $Q \in \operatorname{Spec}(B)$ such that $Q \cap A = P$. If $A \subset B$ is LCM-stable, then we have $B \cap K = A$.*

COROLLARY 1.13. *Let B be an overring of A with $B\neq A$. Assume that $A\subset B$ is LCM-stable. Then there exists $M\in\mathrm{Max}(A)$ such that $MB=B$. In particular, B is not integral over A.*

Finally, we give a property of LCM-stableness in terms of prime ideals. For $P\in\mathrm{Spec}(A)$, we denote by $\mathrm{ht}(P)$ *the height of P.*

PROPOSITION 1.14 (cf. [3], Prop. 6.4). *Assume that $A\subset B$ is LCM-stable. Let $P\in\mathrm{Spec}(B)$ with $\mathrm{ht}(P)\leqq 1$. Then we have $\mathrm{ht}(P\cap A)\leqq 1$.*

PROOF. By Cor. 1.5, $A_{P\cap A}\subset B_P$ is LCM-stable. Therefore, we may assume that A and B are quasi-local domains with the maximal ideals P and M, respectively, and that $M\cap A=P\neq 0$ and $\mathrm{ht}(M)\leqq 1$. Let $a\in P-\{0\}$. Since $\mathrm{ht}(M)=1$ and B is a quasi-local domain, we have $M=\mathrm{rad}(aB)$. On the other hand, since $A\subset B$ is LCM-stable and $PB\neq B$, $aB\cap A=aA$ by Prop. 1.11. Therefore, $P=M\cap A=\mathrm{rad}(aB)\cap A=\mathrm{rad}(aB\cap A)=\mathrm{rad}(aA)$. This implies that $\mathrm{ht}(P)=1$.

§2. LCM-stableness of $A\subset A[\alpha]$ with $\alpha^m\in K$

Let $\alpha\in\Omega$ with $\alpha^m\in K$ for some $m>0$. In this section, we shall give some characterizations for $A\subset A[\alpha]$ to be LCM-stable.

PROPOSITION 2.1. *Let A be a quasi-local domain and $\alpha\in\Omega$. Assume that $\alpha^m=u\in K-A$ and that $A\subset A[\alpha]$ is LCM-stable. Then we have $\alpha^{-1}\in A[\alpha]$. Therefore, α^{-1} is integral over A and also so is u^{-1}.*

PROOF. Put $u=a/b$, where a, $b\in A-\{0\}$. Since $A\subset A[\alpha]$ is LCM-stable, we have $a=b\alpha^m\in(aA\cap bA)A[\alpha]$. Therefore, there exist $r>0$ and x_i, y_i, $z_i\in A$ such that $a=b\alpha^m=\sum_{i=0}^r x_i\alpha^i$ and $x_i=ay_i=bz_i$ for $0\leqq i\leqq r$. Now since $u\notin A$, y_i is a non-unit for every i. Thus, $1-y_i$ is a unit in A for each i. Therefore, we have $\alpha^{-1}=(1-y_0)^{-1}\sum_{i=1}^r y_i\alpha^{i-1}\in A[\alpha]$. This completes the proof.

Let $A\subset B$ be integral domains. We say that $A\subset B$ is INC if two different prime ideals of B with the same contraction in A can not be comparable (see [7], [16]).

COROLLARY 2.2. *Let $\alpha\in\Omega$ with $\alpha^m\in K$ for some $m>0$. If $A\subset A[\alpha]$ is LCM-stable, then $A\subset A[\alpha]$ is INC.*

PROOF. By virtue of §1 and [16], we may assume that A is a quasi-local domain. Then $A\subset A[\alpha]$ is INC by Prop. 2.1 and Cor. 3.2 in [16].

REMARK 2.3. The converse of Cor. 2.2 is false as is seen in $\boldsymbol{Z}[\sqrt{5}]\subset$

$\boldsymbol{Z}[(1+\sqrt{5})/2]$.

Let $\alpha \in \Omega$. Hereafter, by K_α we shall denote the kernel of the canonical homomorphism of $A[X]$ onto $A[\alpha]$. From now on, we examine some conditions for the converse of Cor. 2.2 to be true.

COROLLARY 2.4. *Let $\alpha \in \Omega$ with $\alpha^m \in K$ for some $m>0$. Assume that K_α is invertible. Then $A \subset A[\alpha]$ is LCM-stable if and only if $A \subset A[\alpha]$ is INC; and when that is so, $A \subset A[\alpha]$ is flat.*

PROOF. The assertions follow immediately from Prop. 1.1, Cor. 2.2, Cor. 3.2 in [16] and Cor. 2.20 in [10].

Here, we need two lemmas relating to a linear base. It is well-known that A is integrally closed if and only if K_u has a linear base for each $u \in K$ (cf. (11.13) in [8] and [11]). The following lemma is a generalization of Th. 1 in [11] which can be proved in the same manner.

LEMMA 2.5. *Let $\alpha \in \Omega$ with $\alpha^m = u \in K - \{0\}$ for some $m>0$ and put $u = a/b$ where $a, b \in A - \{0\}$. Put $B_u = \{dx - e \mid d, e \in A$ and $be = ad\}$ and $B_\alpha = \{dX^m - e \mid d, e \in A$ and $be = ad\}$. Then the following statements are equivalent.*

(1) *$K_u = B_u A[X]$; that is, K_u has a linear base.*

(2) *If $bX^m - a$ is irreducible over K, then $K_\alpha = B_\alpha A[X]$.*

(3) *$(a, b)^n \cap (b^{n+1} :_A a) \subset b^n A$ for each $n>0$.*

Generally, it is easily shown that for $u \in K$ if A is integrally closed in $A[u]$, then K_u has a linear base (cf. (11.13) in [8]). On the other hand, the converse is false as is seen in $A \subset A[u]$, where $A = \boldsymbol{Z} + \boldsymbol{Z}2\sqrt{-1}$ and $u = 1/2\sqrt{-1}$. Therefore, the following lemma is a slight generalization of the $u - u^{-1}$ Lemma which are essentially proved in Th. 67 in [7].

LEMMA 2.6. *Let A be a quasi-local domain with the unique maximal ideal M and take $u \in K$. Assume that K_u has a linear base. If $K_u \not\subset MA[X]$, then either $u \in A$ or $u^{-1} \in A$.*

THEOREM 2.7. *Let $\alpha \in \Omega - \{0\}$ with $\alpha^m = u \in K$ for some $m>0$. Put $u = a/b$, where $a, b \in A - \{0\}$. Assume that K_u has a linear base and that $bX^m - a$ is irreducible over K. Then the following statements are equivalent.*

(1) *$A \subset A[\alpha]$ is LCM-stable.*

(2) *$A \subset A[\alpha]$ is INC.*

(3) *$A \subset A[\alpha]$ is flat.*

(4) *(a, b) is invertible.*

PROOF. Since incomparability, LCM-stableness, flatness and the property that K_u has a linear base, where $u \in K$, are local properties (see [16] and §1), we may assume that A is a quasi-local domain with the unique maximal ideal M. Then we have only to show the implications (2)⇒(4) and (4)⇒(3), since the others are obvious (cf. Cor. 2.2).

(2)⇒(4). Assume that $A \subset A[\alpha]$ is INC. Then we have $c(K_\alpha)=A$ by Cor. 3.2 in [16]. On the other hand, it follows easily from Lemma 2.5 that $c(K_\alpha)=c(K_u)$. Therefore, $K_uA[X] \not\subset MA[X]$. Thus, by Lemma 2.6, we have either $u \in A$ or $u^{-1} \in A$. That is, (a, b) is principal.

(4)⇒(3). Assume that (a, b) is invertible. Since A is a quasi-local domain, we have easily either $u \in A$ or $u^{-1} \in A$. Suppose that $u \in A$. Then we have $K_\alpha=(X^m-u)A[X]$ by the assumption. Therefore, $A \subset A[\alpha]$ is obviously flat. We now proceed to the case $u^{-1} \in A$. Similarly, we have $K_\alpha=(u^{-1}X^m-1)A[X]$. Therefore, $A \subset A[\alpha]$ is flat by Cor. 2.20 in [10].

COROLLARY 2.8 (cf. Cor. 4.4). *Let aX^m-b be a prime element of $A[X]$, where $m>0$ and $a, b \in A-\{0\}$. Then the following statements are equivalent.*

(1) *$A \subset A[X]/(aX^m-b)$ is LCM-stable.*

(2) *$A \subset A[X]/(aX^m-b)$ is flat.*

(3) *$(a, b)=A$.*

§3. Universality

In this section, we shall examine the universality of LCM-stableness. For this purpose, we prepare two notions, *R_2-stableness* and *G_2-stableness*, related to LCM-stableness. Let $A \subset B$ be integral domains. We say that $A \subset B$ is *G_2-stable* if $\mathrm{Gr}\,(IB) \geqq 2$ for each non-zero finitely generated ideal I of A with $\mathrm{Gr}\,(I) \geqq 2$. Moreover, we say that $A \subset B$ is *R_2-stable* if $a :_B b=a$ for any $a, b \in A-\{0\}$ with $a :_A b=a$. Obviously, if $A \subset B$ is LCM-stable, then $A \subset B$ is R_2-stable and if A is a GCD-domain, then the converse holds. Let I be an ideal of A. If $\mathrm{Gr}\,(I) \leqq 2$, then we have $A :_K I=A$. But the converse is false as is seen in Remark 2.4 in [6]. On the other hand, in case I is finitely generated, $\mathrm{Gr}\,(I) \geqq 2$ if and only if $A :_K I=A$ by virtue of Th. 7 of Chap. 5 in [9]. Therefore, by Ex. 1 and Ex. 2 (p. 102) in [7], if $A \subset B$ is G_2-stable, then $A \subset B$ is R_2-stable and moreover, if A is a Noetherian domain, then the converse is true. However, neither G_2-stableness nor R_2-stableness does necessarily imply LCM-stableness as is seen in $\boldsymbol{Z}[\sqrt{5}] \subset \boldsymbol{Z}[(1+\sqrt{5})/2]$. So we first study a regular sequence of length 2 in a polynomial ring. We denote by $\mathrm{Z}(R)$ the set of all zero-divisors of a ring R.

LEMMA 3.1. *Let R be a commutative ring with identity and Q be the total quotient ring of R. Let $f(X)=a_0+a_1X+\cdots+a_kX^k \in R[X]$. Assume that $\mathrm{c}(f)$ contains a non-zero-divisor. Then the following statements are equivalent.*

(1) *$a :_{R[X]} f(X)=a$ for each $a \in R-\mathrm{Z}(R)$.*

(2) $a:_{R[X]}f(X)=a$ *for each* $a\in \mathrm{c}(f)-\mathrm{Z}(R)$.
(3) $a:_R \mathrm{c}(f)=a$ *for each* $a\in \mathrm{c}(f)-\mathrm{Z}(R)$.
(4) $a:_R \mathrm{c}(f)=a$ *for some* $a\in \mathrm{c}(f)-\mathrm{Z}(R)$.
(5) $R:_Q \mathrm{c}(f)=R$.

PROOF. The equivalences (3)⇔(4)⇔(5) are easy and (2)⇔(3) follows from Th. 7 of Chap. 5 in [9]. Moreover, (1)⇒(2) is obvious.

(2)⇒(1). Let $a\in R-\mathrm{Z}(R)$. By the assumption, there exists $b\in \mathrm{c}(f)-\mathrm{Z}(R)$. Since $ab\in \mathrm{c}(f)-\mathrm{Z}(R)$, we have $ab:_{R[X]}f(X)=ab$, Thus, $a:_{R[X]}f(X)=a$.

THEOREM 3.2. *Let R be a commutative ring with identity and Q be the total quotient ring of R. Let* $f(X), g(X)\in R[X]$. *Assume that* $\mathrm{c}(f)$ *contains a non-zero-divisor. Then* $f(X):_{R[X]}g(X)=f(X)$ *if and only if* (i) $f(X):_{Q[X]}g(X)=f(X)$ *and* (ii) $R:_Q(\mathrm{c}(f)+\mathrm{c}(g))=R$.

PROOF. Suppose first that $f(X):_{R[X]}g(X)=f(X)$. Since $R[X]\subset Q[X]$ is flat, we have obviously $f(X):_{Q[X]}g(X)=f(X)$. Let $a/b\in R:_Q(\mathrm{c}(f)+\mathrm{c}(g))$, where $a\in R$ and $b\in R-\mathrm{Z}(R)$. Then there exist $\phi(X)$, $\psi(X)\in R[X]$ such that $af(X)=b\phi(X)$, $ag(X)=b\psi(X)$. Since $b\notin \mathrm{Z}(R)$, we have $f(X)\psi(X)=g(X)\phi(X)$. Therefore, $\phi(X)\in f(X):_{R[X]}g(X)=f(X)$. That is, we can take $c(X)\in R[X]$ so that $\phi(X)=c(X)f(X)$. Since $f(X)\notin \mathrm{Z}(R[X])$, we have $a=bc(X)\in bR[X]\cap R=bR$. Thus, $a/b\in R$. This implies that $R:_Q(\mathrm{c}(f)+\mathrm{c}(g))=R$.

Conversely, let $h(X)\in f(X):_{R[X]}g(X)$ and take $\phi(X)\in R[X]$ so that $h(X)g(X)=f(X)\phi(X)$. Since $h(X)\in f(X)Q[X]$ by (i), there exist $a\in R-\mathrm{Z}(R)$ and $\psi(X)\in R[X]$ such that $ah(X)=f(X)\psi(X)$. Then since $f(X)\notin \mathrm{Z}(R[X])$, we have $a\phi(X)=g(X)\psi(X)$. Put $F(X)=X^nf(X)+g(X)$, where $n>\deg g$. Then $\mathrm{c}(F)=c(f)+\mathrm{c}(g)$ and by (ii) $R:_Q \mathrm{c}(F)=R$. Since $F(X)\psi(X)=a(X^nh(X)+\phi(X))$, we have $\psi(X)\in aR[X]$ by Lemma 3.1. Therefore, $h(X)\in f(X)R[X]$ by noting $a\notin \mathrm{Z}(R)$. That is, $f(X):_{R[X]}g(X)=f(X)$.

COROLLARY 3.3. *With the notation of Th.* 3.2, *let* $a\in R-\mathrm{Z}(R)$. *Then* $a:_{R[X]}f(X)=a$ *if and only if* $R:_Q(a, \mathrm{c}(f))=R$. *Moreover, assume that* $R:_Q \mathrm{c}(f)=R$. *Then for each* $b\in R-\{0\}$, $a:_{R[X]}bf(X)=a$ *if and only if* $a:_R b=a$.

PROPOSITION 3.4. *Let I be a non-zero proper ideal of* $A[X]$. *If* $\mathrm{Gr}(I)\geqq 2$, *then* $\mathrm{gr}(I)\geqq 2$.

PROOF. Suppose that $I\cap A=0$. Then we have $IK[X]\neq K[X]$. Therefore, $\mathrm{Gr}(IK[X])\leqq 1$. On the other hand, $\mathrm{Gr}(I)\leqq \mathrm{Gr}(IK[X])\leqq 1$ by Ex. 10 of Chap. 5 in [9], a contradiction. Thus, $I\cap A\neq 0$. Take $a\in I\cap A-\{0\}$. Since $\mathrm{Gr}(I/(a))\geqq 1$ by Th. 15 of Chap. 5 in [9], we have obviously $\mathrm{gr}(I/(a))\geqq 1$. Thus, $\mathrm{gr}(I)\geqq 2$.

With these preparations, we study universality.

THEOREM 3.5. *For $A \subset B$, the following statements are equivalent.*
(1) *$A \subset B$ is G_2-stable.*
(2) *$A[X] \subset B[X]$ is G_2-stable.*
(3) *$A[X] \subset B[X]$ is R_2-stable.*

PROOF. (1)$\Rightarrow$(3). Let $f(X), g(X) \in A[X] - \{0\}$ and assume that $f(X):_{A[X]} g(X) = f(X)$. Then by Th. 3.2, we have (i) $f(X):_{K[X]} g(X) = f(X)$ and (ii) $A:_K (c(f)+c(g)) = A$. Let L be the quotient field of B. By (i), we have immediately $f(X):_{L[X]} g(X) = f(X)$. Since $\mathrm{Gr}(c(f)+c(g)) \geqq 2$ by (ii) and $A \subset B$ is G_2-stable, $\mathrm{Gr}((c(f)+c(g))B) \geqq 2$. Therefore, $B:_L (c(f)+c(g)) = B$. Thus, $f(X):_{B[X]} g(X) = f(X)$ by Th. 3.2. That is, $A[X] \subset B[X]$ is R_2-stable.

(3)$\Rightarrow$(2). Let I be a finitely generated ideal of $A[X]$ with $\mathrm{Gr}(I) \geqq 2$. We may assume that $I \neq A[X]$. Then by Prop. 3.4 we have $\mathrm{gr}(I) \geqq 2$. Since $A[X] \subset B[X]$ is R_2-stable, $\mathrm{gr}(IB) \geqq 2$. Therefore, $\mathrm{Gr}(IB) \geqq 2$. That is, $A[X] \subset B[X]$ is G_2-stable.

The implication (2)$\Rightarrow$(1) follows easily from the definition.

If $A[X] \subset B[X]$ is R_2-stable, then obviously so is $A \subset B$. The converse is false as is seen in §7. As for the converse, we consider the following condition. We say that A satisfies *the condition* (*) if A_P is a valuation ring for any $P \in \mathrm{Spec}(A)$ with $\mathrm{gr}(P) = 1$. By Th. 2.2 in [14], if A is a GCD-domain, then A satsisfies (*). Moreover, if A satisfies (*), A is integrally closed by Cor. 2.16 in [1].

THEOREM 3.6. *Assume that A satisfies the condition (*). Then for $A \subset B$, $A \subset B$ is G_2-stable if and only if $A \subset B$ is R_2-stable.*

PROOF. Suppose that $A \subset B$ is R_2-stable. Let I be a finitely generated ideal of A with $\mathrm{Gr}(I) \geqq 2$. We may assume that $IB \neq B$. Then there exists $Q \in \mathrm{Spec}(B)$ such that $\mathrm{Gr}(IB) = \mathrm{Gr}(Q)$ by Th. 16 of Chap. 5 in [9]. Put $Q \cap A = P$. Then we have $I \subset P$. Assume that $\mathrm{gr}(P) = 1$. By the assumption, A_P is a valuation ring. Therefore, IA_P is a proper principal ideal of A_P. On the other hand, since $A:_K I = A$, $A_P:_K IA_P = A_P$. This is a contradiction. Thus, $\mathrm{gr}(P) \geqq 2$. Since $A \subset B$ is R_2-stable, $\mathrm{gr}(PB) \geqq 2$. Therefore, $\mathrm{Gr}(IB) = \mathrm{Gr}(Q) \geqq \mathrm{Gr}(PB) \geqq 2$. That is, $A \subset B$ is G_2-stable.

COROLLARY 3.7. *Let A be a GCD-domain. Then the following statements are equivalent.*
(1) *$A \subset B$ is LCM-stable.*
(2) *$A \subset B$ is R_2-stable.*
(3) *$A \subset B$ is G_2-stable.*
(4) *$A[X] \subset B[X]$ is LCM-stable.*

(5) $A[X]\subset B[X]$ *is* R_2*-stable.*
(6) $A[X]\subset B[X]$ *is* G_2*-stable.*

COROLLARY 3.8. *Let A be locally a GCD-domain. Then $A\subset B$ is LCM-stable if and only if $A[X]\subset B[X]$ is LCM-stable.*

Hereafter, we shall fix $A\subset B$ and let L be the quotient field of B. Assume that A is integrally closed. With this assumption, we examine LCM-stableness of $A[X]\subset B[X]$.

LEMMA 3.9. *Let $f(X)$, $g(X)\in A[X]-\{0\}$. If $f(X):_{K[X]}g(X)=f(X)$, then we have $f(X):_{A[X]}g(X)=(A:_K(c(f)+c(g))f(X)A[X]$.*

PROOF. Let $x\in A:_K(c(f)+c(g))$. Then $xf(X)$, $xg(X)\in A[X]$. Therefore, we have $xf(X)\in f(X):_{A[X]}g(X)$. Thus, $(A:_K(c(f)+c(g))f(X)A[X]\subset f(X):_{A[X]}g(X)$.

Conversely, let $h(X)\in f(X):_{A[X]}g(X)$. Then there exists $\phi(X)\in A[X]$ such that $h(X)g(X)=f(X)\phi(X)$. Since $f(X):_{K[X]}g(X)=f(X)$, there exist $a\in A-\{0\}$ and $\psi(X)\in A[X]$ such that $ah(X)=f(X)\psi(X)$. Then we have $a\phi(X)=g(X)\psi(X)$. Put $F(X)=f(X)X^n+g(X)$, where $n>\deg g$. Then $c(F)=c(f)+c(g)$ and $a(h(X)X+\phi(X))=F(X)\psi(X)$. Therefore, $h(X)X^n+\psi(X)\in F(X)K[X]\cap A[X]$. On the other hand, since A is integrally closed, we have $F(X)K[X]\cap A[X]=(A:_K c(F))F(X)A[X]$ by Th. B in [15]. Thus, there exist $x_i\in A:_K c(F)$ and $g_i(X)\in A[X]$ such that $h(X)X^n+\phi(X)=\sum_{i=1}^r x_iF(X)g_i(X)$. Therefore, we have $\psi(X)=a\sum_{i=1}^r x_ig_i(X)$. Thus, $h(X)=\sum_{i=1}^r x_if(X)g_i(X)\in(A:_K c(F))f(X)A[X]$. That is, $f(X):_{A[X]}g(X)\subset(A:_K(c(f)+c(g))f(X)A[X]$. This completes the proof.

PROPOSITION 3.10. *Assume that $A[X]\subset B[X]$ is LCM-stable. Then for each non-zero finitely generated ideal I of A, $B:_L I=(A:_K I)B$.*

PROOF. Suppose that $I=(a, a_0, a_1,\dots, a_n)$ is a non-zero finitely generated ideal of A (in case I is principal, we set $n=0$ and $a_0=$a), and put $f(X)=\sum_{i=0}^n a_iX^i$. By Lemma 3.9, we have $f(X):_{A[X]}a=(A:_K I)f(X)A[X]$. On the other hand, generally $(A:_K I)f(X)B[X]\subset(B:_L I)f(X)B[X]\subset f(X):_{B[X]}a$. Since $A[X]\subset B[X]$ is LCM-stable, $(A:_K I)f(X)B[X]=(B:_L I)f(X)B[X]$. Therefore, $(A:_K I)B=B:_L I$.

THEOREM 3.11. *Assume that B is integrally closed and that L is algebraic over K. Then the following statements are equivalent.*
(1) *$A[X]\subset B[X]$ is LCM-stable.*
(2) *$B:_L I=(A:_K I)B$ for any non-zero finitely generated ideal I of A.*
(3) *$a:_B I=(a:_A I)B$ for any $a\in A-\{0\}$ and non-zero finitely generated ideal I of A.*

PROOF. (1)$\Rightarrow$(2). This follows from Prop. 3.10.

(2)$\Rightarrow$(1). Let $f(X)$, $g(X) \in A[X]-\{0\}$. Since $K[X]$ is a PID, there exist $d(X) \in K[X]$ and $f_1(X)$, $g_1(X) \in A[X]$ such that $f(X)=d(X)f_1(X)$, $g(X)=d(X)g_1(X)$ and $f_1(X):_{K[X]} g_1(X)=f_1(X)$. Then $f(X):_{A[X]} g(X)=f_1(X):_{A[X]} g_1(X)$. Therefore, we may assume that $f(X):_{K[X]} g(X)=f(X)$. Then we have obviously $f(X):_{L[X]} g(X)=f(X)$. Thus, since B is integrally closed, by Lemma 3.9 and the assumption we have

$$\begin{aligned} f(X):_{B[X]} g(X) &= (B:_L (\mathrm{c}(f)+\mathrm{c}(g))f(X)B[X] \\ &= (A:_K (\mathrm{c}(f)+\mathrm{c}(g))f(X)B[X] \\ &= (f(X):_{A[X]} g(X))f(X)B[X]. \end{aligned}$$

Therefore, $A[X] \subset B[X]$ is LCM-stable.

(2)$\Leftrightarrow$(3). Since L is algebraic over K, $L=B\otimes_A K$ and the assertion follows easily.

§4. Simple extensions

In this section, we shall give a necessary and sufficient condition for a simple extension over A, which is locally a GCD-domain, to be LCM-stable and discuss a difference between LCM-stableness and flatness. Let I be a finitely generated proper ideal of A. It is well-known that if $\mathrm{gr}(I) \geqq 2$, then $\mathrm{Gr}(I) \geqq 2$, or equivalently $A:_K I=A$, and if A is a Noetherian domain, then the converse is true. Moreover, the converse holds for a polynomial ring as is seen in Prop. 3.4. More generally we can show that this is true for a wider class of domains, containing Noetherian domains and Krull domains. We say that I has a primary decomposition if $I=\bigcap_{i=1}^r Q_i$ for some primary ideals $Q_1, Q_2, \dots, Q_r$.

LEMMA 4.1. *Assume that each proper principal ideal of A has a primary decomposition. Let I be a finitely generated proper ideal of A. If* $\mathrm{Gr}(I) \geqq 2$, *then we have* $\mathrm{gr}(I) \geqq 2$.

PROOF. Suppose that $\mathrm{Gr}(I) \geqq 2$. In particular, $I \neq 0$. Let $a \in A-\{0\}$. Then we have $a:_A I=a$. Let $aA=\bigcap_{i=1}^r Q_i$ be an irredundant primary decomposition of aA. We put $P_i=\mathrm{rad}(Q_i)$. Then $\mathrm{Z}(A/aA)=\bigcup_{i=1}^r P_i$. Assume that $I \subset \mathrm{Z}(A/aA)$. There exists i such that $I \subset P_i$. Since I is finitely generated, $I^n \subset Q_i$ for some $n>0$. Take $b \in \bigcap_{j\neq i} Q_j - Q_i$. Then $b \notin aA$ and $bI^n \subset aA$. Since $a:_A I=a$, we have $a:_A I^n=a$. This is a contradiction. Therefore, $I \not\subset \mathrm{Z}(A/aA)$, by which we have easily $\mathrm{gr}(I) \geqq 2$.

The following Lemma follows immediately from Ex. 10 of Chap. 5, Th. 5 of Chap. 6 in [9] and Th. 3.5 in [13].

LEMMA 4.2. *Let I be an ideal of $A[X]$ generated by an $A[X]$-sequence of length n $(n \geqq 0)$ and let $a(X) \in A[X]$ with $a(X) \notin I$. Let Q be a minimal prime ideal of $I:_{A[X]} a(X)$. Put $Q \cap A = P$. Then $\mathrm{Gr}(Q) = \mathrm{Gr}(QA[X]_Q) = n$ and if $\mathrm{Gr}(P) \geqq n$, then $Q = PA[X]$.*

Throughout the following Th. 4.3, Cor. 4.4 and Th. 4.5, let $f(X)$ be *a prime element of* $A[X]$ with $\deg f \geqq 1$ and let $B = A[X]/(f(X))$.

THEOREM 4.3. *$A[Y] \subset B[Y]$ is R_2-stable if and only if $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$, where Y is an indeterminate. In particular, if $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$, then $A \subset B$ is R_2-stable.*

PROOF. Suppose that $A[Y] \subset B[Y]$ is R_2-stable. We may assume that $\mathrm{c}(f) \neq A$. Let $a \in \mathrm{c}(f) - \{0\}$. Since $f(X)$ is a prime element of $A[X]$, $a:_{A[Y]} f(Y) = a$. Also, since $A[Y] \subset B[Y]$ is R_2-stable, $a:_{B[Y]} f(Y) = a$. Therefore, $\{f(X), a, f(Y)\}$ is an $A[X, Y]$-sequence in $\mathrm{c}(f)A[X, Y]$. Thus, $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$.

Conversely, suppose that $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$. Let $a(Y), b(Y) \in A[Y] - \{0\}$ and assume that $a(Y):_{A[Y]} b(Y) = a(Y)$. Since $f(X)$ is a prime element of $A[X]$, we have either $f(X):_{A[X,Y]} a(Y) = f(X)$ or $f(X):_{A[X,Y]} b(Y) = f(X)$. Say $f(X):_{A[X,Y]} a(Y) = f(X)$. If $(f(X), a(Y), b(Y)) = A[X, Y]$, then $(a(Y), b(Y))B[Y] = B[Y]$ and therefore, we have $a(Y):_{B[Y]} b(Y) = a(Y)$. So suppose that $(f(X), a(Y), b(Y)) \neq A[X, Y]$. Assume that $\{f(X), a(Y), b(Y)\}$ is not an $A[X, Y]$-sequence. Then there exists $h(X, Y) \in A[X, Y]$ such that $b(Y)h(X, Y) \in (f(X), a(Y))$ and $h(X, Y) \notin (f(X), a(Y))$. Let Q be a minimal prime ideal of $(f(X), a(Y)):_{A[X,Y]} h(X, Y)$ and put $Q \cap A[Y] = P$. Then $Q \supset (f(X), a(Y), b(Y))$ and therefore, $P \supset (a(Y), b(Y))$. Thus, $\mathrm{Gr}(P) \geqq 2$. By Lemma 4.2, we have $\mathrm{Gr}(Q) = 2$ and $Q = PA[X, Y]$. Then since $f(X) \in Q$, $\mathrm{c}(f) \subset P \cap A$. Therefore, $\mathrm{Gr}(Q) = \mathrm{Gr}(P) \geqq \mathrm{Gr}(\mathrm{c}(f)) \geqq 3$. This is a contradiction. Thus, $\{f(X), a(Y), b(Y)\}$ is an $A[X, Y]$-sequence. That is, $a(Y):_{B[Y]} b(Y) = a(Y)$. This implies that $A[Y] \subset B[Y]$ is R_2-stable.

COROLLARY 4.4. *Let A be a GCD-domain. Then $A \subset B$ is* LCM*-stable if and only if* $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$.

THEOREM 4.5. *Assume that each principal proper ideal of A has a primary decomposition. Then the following statements are equivalent.*

(1) *$A \subset B$ is R_2-stable.*

(2) *$A[X] \subset B[X]$ is R_2-stable.*

(3) $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$.

PROOF. We have only to prove (1)⇒(3). Suppose that $A \subset B$ is R_2-stable. We may assume that $\mathrm{c}(f) \neq A$. By Lemma 3.1, $\mathrm{Gr}(\mathrm{c}(f)) \geqq 2$. Therefore, by Lemma 4.1, there exist $a, b \in \mathrm{c}(f)$ such that $\{a, b\}$ is an A-sequence. Since $A \subset B$ is R_2-stable, we have $a:_B b = a$. Thus, $\{f(X), a, b\}$ is an $A[X]$-sequence in $\mathrm{c}(f)A[X]$. Therefore, $\mathrm{Gr}(\mathrm{c}(f)) \geqq 3$.

LEMMA 4.6. *Let I be a finitely generated ideal of A. Then we have* $\mathrm{Gr}(I)=\inf\{\mathrm{Gr}(IA_M);\ M\in \mathrm{Max}(A)\}$.

PROOF. Let $A(Y)$ be a localization of $A[Y]$ by a multiplicatively closed set consisting of all polynomials $f(Y)$ of $A[Y]$ with $\mathrm{c}(f)=A$, where Y is a finite set of variables. By Cor. 1 of Prop. 2 in [5], we have $\mathrm{Gr}(I)=\mathrm{Gr}(IA(Y))$. Therefore, $\inf\{IA_M\}=\inf\{IA_M(Y)\}=\inf\{IA(Y)_{MA(Y)}\}$, $M\in \mathrm{Max}(A)$. Since there exists a bijection between $\mathrm{Max}(A)$ and $\mathrm{Max}(A(Y))$, we may assume that $\mathrm{Gr}(I)=n$ and $\{a_1, a_2,\dots, a_n\}$ is an A-sequence in I. Then $\mathrm{Gr}(I/(a_1, a_2,\dots, a_n))=\mathrm{Gr}(I)-n=0$ and $\inf\{\mathrm{Gr}(IA_M/(a_1, a_2,\dots, a_n))\}=\inf\{\mathrm{Gr}(IA_M)\}-n$, $M\in \mathrm{Max}(A)$. Therefore, we may assume that $\mathrm{Gr}(I)=0$. Then since I is finitely generated, there exists $x\in A-\{0\}$ such that $xI=0$ by Th. 8 of Chap. 5 in [9]. Take $M\in \mathrm{Max}(A)$ so that $x/1\neq 0$ in A_M. Then we have $\mathrm{Gr}(IA_M)=0$ by Th. 8 of Chap. 5 in [9]. This completes the proof.

THEOREM 4.7. *Let A be locally a GCD-domain and $\alpha\in\Omega-\{0\}$. Let I be the kernel of the canonical homomorphism of $A[X]$ onto $A[\alpha]$. Then $A\subset A[\alpha]$ is LCM-stable if and only if* $\mathrm{Gr}(\mathrm{c}(I))\geqq 3$.

PROOF. Suppose that $\mathrm{Gr}(\mathrm{c}(I))\geqq 3$. Let $M\in \mathrm{Max}(A)$. Since A_M is a GCD-domain, there exists $f_M(X)\in A_M[X]$ such that $IA_M[X]=f_M(X)A_M[X]$. Therefore, we have $\mathrm{c}(IA_M[X])=\mathrm{c}(f_M)$. Thus, $\mathrm{Gr}(\mathrm{c}(f_M))\geqq 3$. By Cor. 4.4, $A_M\subset A_M[\alpha]$ is LCM-stable. Therefore, $A\subset A[\alpha]$ is LCM-stable by Prop. 1.6.

Conversely, suppose that $A\subset A[\alpha]$ is LCM-stable. Let $M\in \mathrm{Max}(A)$. Take $f_M(X)\in A_M[X]$ so that $IA_M[X]=f_M(X)A_M[X]$. Since $A_M\subset A_M[\alpha]$ is LCM-stable by Cor. 1.5, we have $\mathrm{Gr}(\mathrm{c}(IA_M[X]))=\mathrm{Gr}(\mathrm{c}(f_M))\geqq 3$ by Cor. 4.4. That is, $\mathrm{Gr}(\mathrm{c}(I)A_M)\geqq 3$ for each $M\in \mathrm{Max}(A)$. Therefore, $\mathrm{Gr}(\mathrm{c}(I))\geqq 3$ by Lemma 4.6.

Finally, we give an example of $A\subset B$ which is not flat but LCM-stable.

Example 4.8. Let $A=k[s, t, u]$ where k is a field and s, t and u are indeterminates. Let $B=A[X]/(sX^2+tX+u)$. Then $A\subset B$ is LCM-stable but is not flat.

§5. LCM-stableness of $A\subset A[\alpha, \beta]$

Let $\alpha, \beta\in\Omega-\{0\}$. Even if both $A\subset A[\alpha]$ and $A\subset A[\beta]$ are LCM-stable, $A\subset A[\alpha, \beta]$ is not necessarily LCM-stable as is seen in Remark 1.10. So we shall examine LCM-stableness of $A\subset A[\alpha, \beta]$ under the condition $\alpha/\beta\in K$ in §5 and under the condition that $K(\alpha)$, $K(\beta)$ are linearly disjoint over K in §6. The following lemma follows easily from Prop. 1.2, Cor. 1.5 and Prop. 1.6.

LEMMA 5.1. *Let* $A \subset B$ *be integral domains and* $a_1, a_2, \ldots, a_n \in A$. *Assume that* $(a_1, a_2, \ldots, a_n)B = B$. *Then* $A \subset B$ *is LCM-stable if and only if* $A \subset B_{a_i}$ *is LCM-stable for every* i *with* $1 \leqq i \leqq n$.

Throughout this section, we assume that A is *integrally closed* and that $a\alpha = b\beta$ for some $a, b \in A - \{0\}$ with $a :_A b = a$.

LEMMA 5.2. *If* $A \subset A[\alpha, \beta]$ *is LCM-stable, then there exists* $\gamma \in A[\alpha, \beta]$ *such that* $\alpha = b\gamma$, $\beta = a\gamma$ *and* $A[\alpha, \beta] = A[\gamma]$.

PROPOSITION 5.3. *Assume that both* α *and* β *are integral over* A. *Then* $A \subset A[\alpha, \beta]$ *is LCM-stable if and only if* $(a, b) = A$.

PROOF. Suppose that $(a, b) = A$. Since $a\alpha = b\beta$, we have $A_a[\alpha, \beta] = A_a[\beta]$ and $A_b[\alpha, \beta] = A_b[\alpha]$. Since both $A \subset A_a[\beta]$ and $A \subset A_b[\alpha]$ are LCM-stable, so is $A \subset A[\alpha, \beta]$ by Lemma 5.1.

Conversely, suppose that $A \subset A[\alpha, \beta]$ is LCM-stable. By Lemma 5.2, we can take $\gamma \in A[\alpha, \beta]$ so that $\alpha = b\gamma$, $\beta = a\gamma$ and $A[\alpha, \beta] = A[\gamma]$. Put $\gamma = f(\alpha, \beta) \in A[\alpha, \beta]$. Since both α and β are integral over A, so is γ. Therefore, $A[\gamma]$ is a free A-module. Since $\gamma = f(\alpha, \beta) = f(b\gamma, a\gamma)$, we have $1 \in (a, b)$. Thus, $(a, b) = A$.

In order to generalize Prop. 5.3, we need a lemma.

LEMMA 5.4. *Let* $f_\alpha(X) = \sum_{i=0}^{k} s_i X^i$ *and* $f_\beta(X) = \sum_{i=0}^{k} t_i X^i$ *be irreducible polynomials of* α *and* β *over* K *with coefficients in* A, *respectively. Then we have* $t_i \in a^{k-i} :_A s_k$ *and* $s_i \in b^{k-i} :_A t_k$ *for* $0 \leqq i \leqq k-1$.

PROOF. Put $g(X) = \sum_{i=0}^{k} t_i b^{k-i} a^i X^i$. Then since $g(\alpha) = b^k f_\beta(\beta) = 0$, $f_\alpha(X)$ devides $g(X)$ in $K[X]$. Since $\deg f_\alpha = \deg g$, there exist $c, d \in A - \{0\}$ such that $cf_\alpha(X) = dg(X)$. Then we have $cs_i = dt_i b^{k-i} a^i$ for $0 \leqq i \leqq k$. Therefore, $s_k t_i b^{k-i} = t_k s_i a^{k-i}$ for $0 \leqq i \leqq k-1$. Since $a :_A b = a$, $a^{k-i} :_A b^{k-i} = a^{k-i}$ for $0 \leqq i \leqq k-1$. Thus, for $1 \leqq i \leqq k-1$, there exists $x_i \in A$ such that $s_k t_i = a^{k-i} x_i$ and $t_k s_i = b^{k-i} x_i$. This completes the proof.

THEOREM 5.5. *Let* α *be integral over* A. *Then* $A \subset A[\alpha, \beta]$ *is LCM-stable if and only if* $A \subset A[\beta]$ *is LCM-stable and* $(a, b) = A$.

PROOF. Since $A_a[\alpha, \beta] = A_a[\beta]$ and $A_b[\alpha, \beta] = A_b[\alpha]$, it suffices to prove the 'only if' part by Lemma 5.1. Suppose that $A \subset A[\alpha, \beta]$ is LCM-stable. We first show that $(a, b) = A$. Let $1, \alpha, \ldots, \alpha^{k-1}$ be a free basis of $A[\alpha]$ over A. Since $a^{k-1}\alpha^{k-1} = b^{k-1}\beta^{k-1}$, $a^{k-1} :_A b^{k-1} = a^{k-1}$ and $A \subset A[\alpha, \beta]$ is LCM-stable, there exist $f_i(\beta) \in A[\beta]$ $(0 \leqq i \leqq k-1)$ such that $\beta^{k-1} = a^{k-1} \sum_{i=0}^{k-1} f_i(\beta)\alpha^i$. Thus, we have

$$(\sharp) \qquad \beta^{k-1} = \sum_{i=0}^{k-1} a^{k-i-1} b^i \beta^i f_i(\beta).$$

Let $f_\beta(X)=\sum_{i=0}^{k-1} t_i X^i \in A[X]$ be an irreducible polynomial of β over K. Since A is integrally closed, the kernel of the canonical homomorphism of $A[X]$ onto $A[\beta]$ equals $(A:_K c(f_\beta))f_\beta(X)A[X]$ by Th. B in [15]. By (#), there exist $x_i \in A:_K c(f_\beta)$ and $g_i(X)\in A[X]$ such that

$$X^{k-1}-\sum_{i=0}^{k-1} a^{k-i-1}b^i f_i(X)X^i$$
$$=\sum_{i=1}^{r} x_i f_\beta(X) g_i(X).$$

Therefore, $1\in(a, b)+\sum_{i=1}^{r} c(x_i f_\beta)$. Put $x_i t_k = t_{ik}$ for $1\leqq i\leqq r$. Then $t_{ik}\in A$. Since α is integral over A, $1\in(a, b)+\sum_{i=1}^{r} t_{ik}A$ by Lemma 5.4. For each i with $1\leqq i\leqq r$, $A_{t_{ik}}[\beta]$ is integral over $A_{t_{ik}}$ and $A_{t_{ik}}\subset A_{t_{ik}}[\alpha, \beta]$ is LCM-stable. Therefore, we have $(a, b)A_{t_{ik}}=A_{t_{ik}}$ by Prop. 5.3. Thus, $t_{ik}\in \operatorname{rad}(a, b)$ for each i. That is, $(a, b)=A$.

We now prove that $A\subset A[\beta]$ is LCM-stable. Since $A_a[\alpha, \beta]=A_a[\beta]$, $A\subset A_a[\beta]$ is LCM-stable by Cor. 1.5. Moreover, since $A_b[\beta]\subset A_b[\alpha, \beta]=A_b[\alpha]$ and since α is integral over A, $A\subset A_b[\beta]$ is obviously LCM-stable. Thus, $A\subset A[\beta]$ is LCM-stable by Lemma 5.1.

REMARK 5.6. Let k be a field and s, t, u and b be indeterminates.

(1) Let $\alpha, \beta\in\Omega$. Even if both $A\subset A[\alpha]$ and $A\subset A[\alpha, \beta]$ are LCM-stable, $A\subset A[\beta]$ is not necessarily so. In fact, let $A=k[s, t, u, b]$ and take $\gamma\in\Omega$ which satisfies $s\gamma^2+t\gamma+u=0$. Put $a=1-sb$, $\alpha=a\gamma$ and $\beta=b\gamma$. Then we have $A[\gamma]=A[\alpha, \beta]$. Both $A\subset A[\alpha]$ and $A\subset A[\alpha, \beta]$ are LCM-stable. But $A\subset A[\beta]$ is not LCM-stable.

(2) Let $\alpha, \beta\in\Omega$. LCM-stableness of $A\subset A[\alpha, \beta]$ does not necessarily imply $(a, b)=A$. In fact, let $A=k[s, t, u]$ and take $\gamma\in\Omega$ which satisfies $s^2u^2\gamma^2+stu\gamma+(1-su)=0$. Put $\alpha=u\gamma$ and $\beta=s\gamma$. Then we have $A[\alpha, \beta]=A[\gamma]$. Moreover, $A\subset A[\alpha]$, $A\subset A[\beta]$ and $A\subset A[\alpha, \beta]$ are all LCM-stable. But, obviously $(u, s)\neq A$.

§6. LCM-stableness of $A\subset A[\alpha, \beta]$ (continued)

Throughout this section, let $\alpha, \beta\in\Omega-\{0\}$ and assume that $K(\alpha)$, $K(\beta)$ *are linearly disjoint over* K.

PROPOSITION 6.1. *If* $A\subset A[\alpha]$ *is flat and if* $A\subset A[\beta]$ *is LCM-stable, then* $A\subset A[\alpha, \beta]$ *is LCM-stable. Moreover, if* $A\subset A[\alpha]$ *is faithfully flat, then* $A\subset A[\alpha, \beta]$ *is LCM-stable if and only if so is* $A\subset A[\beta]$.

PROOF. Since $A\subset A[\alpha]$ is flat and $K(\alpha)$, $K(\beta)$ are linearly disjoint over K, we have $A[\alpha, \beta]=A[\alpha]\otimes_A A[\beta]$. Therefore, $A\subset A[\alpha, \beta]$ is LCM-stable by Prop. 1.2, (1).

Suppose that $A \subset A[\alpha]$ is faithfully flat and $A \subset A[\alpha, \beta]$ is LCM-stable. Then $A[\beta] \subset A[\alpha, \beta]$ is faithfully flat and therefore, $A \subset A[\beta]$ is LCM-stable by Prop. 1.2, (2).

COROLLARY 6.2. *Assume that A is integrally closed and α is integral over A. Then $A \subset A[\alpha, \beta]$ is LCM-stable if and only if so is $A \subset A[\beta]$.*

LEMMA 6.3. *Assume that $(\sum_{i=0}^{k} a_i X^i)$ is the kernel of the canonical homomorphism of $A[X]$ onto $A[\alpha]$. Then $A \subset A[\alpha]$ is faithfully flat if and only if $(a_1, a_2,..., a_k) = A$.*

PROOF. Let $M \in \mathrm{Max}(A)$. Put $f(X) = \sum_{i=0}^{k} a_i X^i$ and $\bar{A} = A/M$. We denote by $\bar{f}(X)$ the reduction of $f(X)$ modulo M. Then we have $A[\alpha]/MA[\alpha] = \bar{A}[X]/(\bar{f}(X))$. Therefore, this lemma follows immediately from Cor. 2.20 in [10].

THEOREM 6.4. *In addition to the assumption of Lemma 6.3, we assume that $A \subset A[\alpha]$ is flat. Then $A \subset A[\alpha, \beta]$ is LCM-stable if and only if $A \subset A_{a_i}[\beta]$ is LCM-stable for every i, $1 \leqq i \leqq k$.*

PROOF. Since $A \subset A[\alpha]$ is flat, $(a_0, a_1,..., a_k) = A$ by Cor. 2.20 in [10]. Therefore, we have $(a_1, a_2,..., a_k)A[\alpha] = A[\alpha]$. By Lemma 5.1, $A \subset A[\alpha, \beta]$ is LCM-stable if and only if $A \subset A_{a_i}[\alpha, \beta]$ is LCM-stable for every i. Fix i with $1 \leqq i \leqq k$. By Prop. 1.2 and Cor. 1.5, $A \subset A_{a_i}[\alpha, \beta]$ is LCM-stable if and only if $A_{a_i} \subset A_{a_i}[\alpha, \beta]$ is LCM-stable. Moreover, since $A_{a_i} \subset A_{a_i}[\alpha]$ is faithfully flat by Lemma 6.3, $A_{a_i} \subset A_{a_i}[\alpha, \beta]$ is LCM-stable if and only if $A_{a_i} \subset A_{a_i}[\beta]$ is LCM-stable by Prop. 6.1. Also, $A_{a_i} \subset A_{a_i}[\beta]$ is LCM-stable if and only if $A \subset A_{a_i}[\beta]$ is LCM-stable. Thus, this theorem holds.

REMARK 6.5. In Th. 6.4, $A \subset A[\beta]$ is not necessarily LCM-stable and therefore, the converse of the first half of Prop. 6.1 is false. In fact, let $A = k[s, t]$ where k is a field and s, t are indeterminates. Take $\alpha, \beta \in \Omega$ so that $s\alpha^2 + t\alpha + 1 = 0$ and $s\beta + t = 0$, respectively. Then $A \subset A[\alpha]$ is flat, but $A \subset A[\beta]$ is not LCM-stable by Cor. 2.8. Since $K(\beta) = K$, $K(\alpha)$, $K(\beta)$ are obviously linearly disjoint over K. On the other hand, $A \subset A[\alpha, \beta]$ is LCM-stable by Th. 6.4.

In what follows, let Y be an indeterminate and we denote by K_α (resp. K_β) the kernel of the canonical homomorphism of $A[X]$ (resp. $A[Y]$) onto $A[\alpha]$ (resp. $A[\beta]$). Moreover, we denote by $K_{\alpha,\beta}$ the kerenel of the canonical homomorphism of $A[X, Y]$ onto $A[\alpha, \beta]$. We now examine $K_{\alpha,\beta}$. In the following Prop. 6.6 and Cor. 6.7, we assume that $K_\alpha = (f_\alpha(X))$ and $K_\beta = (f_\beta(Y))$, where $f_\alpha(X), f_\beta(X) \in A[X]$.

PROPOSITION 6.6. $K_{\alpha,\beta} = (f_\alpha(X), f_\beta(Y))$ *if and only if* $\mathrm{Gr}(c(f_\alpha) + c(f_\beta)) \geqq 3$.

PROOF. Suppose that $K_{\alpha,\beta}=(f_\alpha(X), f_\beta(Y))$. We may assume that $c(f_\alpha)+c(f_\beta)\neq A$. Let $a\in c(f_\alpha)-\{0\}$. Then $\{a, f_\alpha(X)\}$ is an $A[X]$-sequence since $f_\alpha(X)$ is a prime element of $A[X]$. Let $f(X, Y)\in(a, f_\alpha(X)):_{A[X,Y]}f_\beta(Y)$. Then we can take $g(X, Y), h(X, Y)\in A[X, Y]$ so that $f(X, Y)f_\beta(Y)=ag(X, Y)+f_\alpha(X)h(X, Y)$. We have $g(X, Y)\in K_{\alpha,\beta}$. By the assumption, there exist $\phi_\alpha(X, Y)$, $\phi_\beta(X, Y)\in A[X, Y]$ such that $g(X, Y)=f_\alpha(X)\phi_\alpha(X, Y)+f_\beta(Y)\phi_\beta(X, Y)$. Therefore, $f_\alpha(X)\cdot(h(X, Y)+a\phi_\alpha(X, Y))=f_\beta(Y)(f(X, Y)-a\phi_\beta(X, Y))$. Since $f_\alpha(X):_{A[X,Y]}f_\beta(Y)=f_\alpha(X)$, $f(X, Y)-a\phi_\beta(X, Y)\in f_\alpha(X)A[X, Y]$, and therefore $f(X, Y)\in(a, f_\alpha(X))$. Thus, $(a, f_\alpha(X)):_{A[X,Y]}f_\beta(Y)=(a, f_\alpha(X))$. That is, $\{a, f_\alpha(X), f_\beta(Y)\}$ is an $A[X, Y]$-sequence in $c(f_\alpha)+c(f_\beta)$, which shows that $\mathrm{Gr}(c(f_\alpha)+c(f_\beta))\geqq 3$.

Conversely, suppose that $\mathrm{Gr}(c(f_\alpha)+c(f_\beta))\geqq 3$. Let $a\in(c(f_\alpha)+c(f_\beta))-\{0\}$. Assume that $(f_\alpha(X), f_\beta(Y)):_{A[X,Y]}a\neq(f_\alpha(X), f_\beta(Y))$. Then we can take $h(X, Y)\in A[X, Y]$ so that $ah(X, Y)\in(f_\alpha(X), f_\beta(Y))$ and $h(X, Y)\notin(f_\alpha(X), f_\beta(Y))$. Let Q be a minimal prime ideal of $(f_\alpha(X), f_\beta(Y)):_{A[X,Y]}h(X, Y)$. Then $a, f_\alpha(X), f_\beta(Y)\in Q$. Put $Q\cap A[X]=P$. Since $\{f_\alpha(X), f_\beta(Y)\}$ is an $A[X, Y]$-sequence and $a, f_\alpha(X)\in P$, we have $\mathrm{Gr}(Q)=\mathrm{Gr}(QA[X, Y]_Q)=2$ and $Q=PA[X, Y]$ by Lemma 4.2. Since $f_\beta(Y)\in Q$, $c(f_\beta)\subset Q\cap A$. On the other hand, $\{a, f_\beta(Y)\}$ is an $A[X]$-sequence. Thus, $\mathrm{Gr}(Q\cap A)=\mathrm{Gr}(P\cap A)\geqq 2$. Since $\mathrm{Gr}(PA[X]_P)=\mathrm{Gr}(QA[X, Y]_Q)=2$, $P=(P\cap A)A[X]$ by Th. 3.5 in [13]. That is, $Q=(Q\cap A)A[X, Y]$. Thus, $c(f_\alpha)+c(f_\beta)\subset Q\cap A$. By the assumption, we have $\mathrm{Gr}(Q)=\mathrm{Gr}(Q\cap A)\geqq\mathrm{Gr}(c(f_\alpha)+c(f_\beta))\geqq 3$. This is a contradiction. Therefore, $(f_\alpha(X), f_\beta(Y)):_{A[X,Y]}a=(f_\alpha(X), f_\beta(Y))$. Let S be the multiplicatively closed set of A generated by the leading coefficients of $f_\alpha(X)$ and $f_\beta(Y)$. Since $K(\alpha)$, $K(\beta)$ are linearly disjoint over K, we have $K_{\alpha,\beta}A_S[X, Y]=(f_\alpha(X), f_\beta(Y))A_S[X, Y]$. Therefore, $K_{\alpha,\beta}=(f_\alpha(X), f_\beta(Y))$ by the relation obtained above.

COROLLARY 6.7. *If $A\subset A[\alpha]$ is G_2-stable, then we have $K_{\alpha,\beta}=(f_\alpha(X), f_\beta(Y))$.*

PROOF. Let $a\in c(f_\alpha)-\{0\}$. Since $f_\beta(Y):_{A[Y]}a=f_\beta(Y)$ and since $A[Y]\subset A[\alpha][Y]$ is R_2-stable by Th. 3.5, we have $f_\beta(Y):_{A[\alpha,Y]}a=f_\beta(Y)$. Therefore, $(f_\alpha(X), f_\beta(Y)):_{A[X,Y]}a=(f_\alpha(X), f_\beta(Y))$. Thus, $\mathrm{Gr}(c(f_\alpha)+c(f_\beta))\geqq 3$, and $K_{\alpha,\beta}=(f_\alpha(X), f_\beta(Y))$ by Prop. 6.6.

COROLLARY 6.8. *Let A be locally a GCD-domain. If $A\subset A[\alpha]$ is LCM-stable, then $K_{\alpha,\beta}=(K_\alpha, K_\beta)A[X, Y]$.*

PROOF. Let $M\in\mathrm{Max}(A)$. Since A_M is a GCD-domain, both $K_\alpha A_M[X]$ and $K_\beta A_M[Y]$ are principal and $A_M\subset A_M[\alpha]$ is G_2-stable by Cor. 1.5 and Cor. 3.7. Therefore, we have $K_{\alpha,\beta}A_M[X, Y]=(K_\alpha, K_\beta)A_M[X, Y]$ by Cor. 6.7. Thus, $K_{\alpha,\beta}=(K_\alpha, K_\beta)A[X, Y]$.

Let $a_1, a_2,\dots, a_n\in A$. Hereafter, we say that $\{a_1, a_2,\dots, a_n\}$ is an *A-sequence* even if $(a_1, a_2,\dots, a_n)=A$.

PROPOSITION 6.9. *Let A be locally a GCD-domain. If both $A \subset A[\alpha]$ and $A \subset A[\alpha, \beta]$ are LCM-stable, then we have* $\mathrm{Gr}(\mathrm{c}(f_\alpha)+\mathrm{c}(f_\beta)) \geqq 4$.

PROOF. By virtue of Lemma 4.6, we may assume that A is a local domain. Then A is a GCD-domain by the assumption. Therefore, both K_α and K_β are principal. Put $K_\alpha=(f_\alpha(X))$ and $K_\beta=(f_\beta(Y))$, where $f_\alpha(X), f_\beta(X) \in A[X]$. Moreover, $A \subset A[\alpha]$ is G_2-stable by Cor. 3.7. Let Z be an indeterminate. We can take a positive integer n so that $\mathrm{c}(f_\alpha(Z)+f_\beta(Z)Z^n)=\mathrm{c}(f_\alpha)+\mathrm{c}(f_\beta)$. Put $F(Z)=f_\alpha(Z)+f_\beta(Z)Z^n$. Since $f_\alpha(Z)$ is a prime element of $A[Z]$, we have $A:_K \mathrm{c}(F)=A$. Let $a \in \mathrm{c}(f_\alpha)-\{0\}$. Then $a:_{A[Z]} F(Z)=a$ by Lemma 3.1. Since $A[Z] \subset A[\alpha, \beta][Z]$ is R_2-stable by Th. 3.5, $a:_{A[\alpha,\beta,Z]} F(Z)=a$. Therefore, we have $(f_\alpha(X), f_\beta(Y), a):_{A[X,Y,Z]} F(Z)=(f_\alpha(X), f_\beta(Y), a)$ by Cor. 6.7. On the other hand, it is easily shown by Cor. 6.7 that $\{f_\alpha(X), f_\beta(Y), a\}$ is an $A[X, Y]$-sequence. Thus, $\{f_\alpha(X), f_\beta(Y), a, F(Z)\}$ is an $A[X, Y, Z]$-sequence in $(\mathrm{c}(f_\alpha)+\mathrm{c}(f_\beta))A[X, Y, Z]$. Thus, $\mathrm{Gr}(\mathrm{c}(f_\alpha)+c(f_\beta)) \geqq 4$. This implies that $\mathrm{Gr}(\mathrm{c}(K_\alpha)+c(K_\beta)) \geqq 4$.

THEOREM 6.10. *Let A be locally a GCD-domain. Assume that both $A \subset A[\alpha]$ and $A \subset A[\beta]$ are LCM-stable. Then $A \subset A[\alpha, \beta]$ is LCM-stable if and only if* $\mathrm{Gr}(\mathrm{c}(K_\alpha)+\mathrm{c}(K_\beta)) \geqq 4$.

PROOF. By virtue of Prop. 6.9, it is sufficient to prove the 'if' part. By Prop. 1.6 and Ex. 10 of Chap. 5 in [9], we may assume that A is a local domain. Then A is a GCD-domain. Therefore, it is sufficient to show that $A \subset A[\alpha, \beta]$ is R_2-stable. Moreover, we can put $K_\alpha=(f_\alpha(X))$ and $K_\beta=(f_\beta(Y))$, where $f_\alpha(X), f_\beta(X) \in A[X]$. Suppose that $\mathrm{Gr}(\mathrm{c}(f_\alpha)+\mathrm{c}(f_\beta)) \geqq 4$. Let $a, b \in A-\{0\}$ and assume that $a:_A b=a$. Since $A \subset A[\alpha]$ is G_2-stable by Cor. 3.7, it is easily shown by Cor. 6.7 that $\{f_\alpha(X), f_\beta(Y), a\}$ is an $A[X, Y]$-sequence. Assume that $\{f_\alpha(X), f_\beta(Y), a, b\}$ is not an $A[X, Y]$-sequence. Then there exists $h(X) \in A[X, Y]$ such that $bh(X, Y) \in (f_\alpha(X), f_\beta(Y), a)$ and $h(X, Y) \notin (f_\alpha(X), f_\beta(Y), a)$. Let Q be a minimal prime ideal of $(f_\alpha(X), f_\beta(Y), a):_{A[X,Y]} h(X, Y)$. Then we have $f_\alpha(X), f_\beta(Y), a, b \in Q$. Put $Q \cap A[X]=P$ and $Q \cap A=P_0$. Since $A \subset A[\alpha]$ is LCM-stable, $\{f_\alpha(X), a, b\}$ is an $A[X]$-sequence in P. Thus, $\mathrm{Gr}(P) \geqq 3$. Therefore, $\mathrm{Gr}(Q)=\mathrm{Gr}(QA[X, Y]_Q)=3$ and $Q=PA[X, Y]$ by Lemma 4.2. Hence, $\mathrm{Gr}(PA[X]_P)=\mathrm{Gr}(QA[X, Y]_Q)=3$ and $\mathrm{c}(f_\beta) \subset P_0$. Since $A \subset A[\beta]$ is LCM-stable, $\mathrm{Gr}(P_0) \geqq \mathrm{Gr}(\mathrm{c}(f_\beta)) \geqq 3$ by Th. 4.7. Therefore, $P=P_0A[X]$ by Th. 3.5 in [13]. Thus, $Q=P_0A[X, Y]$. Then we have $\mathrm{c}(f_\alpha), \mathrm{c}(f_\beta) \subset P_0$. By the assumption, $\mathrm{Gr}(Q)=\mathrm{Gr}(P_0) \geqq \mathrm{Gr}(\mathrm{c}(f_\alpha)+\mathrm{c}(f_\beta)) \geqq 4$. This is a contradiction. That is, $(f_\alpha(X), f_\beta(Y), a):_{A[X,Y]} b=(f_\alpha(X), f_\beta(Y), a)$. By Cor. 6.7, we have $a:_{A[\alpha,\beta]} b=a$. Thus, $A \subset A[\alpha, \beta]$ is R_2-stable. This completes the proof.

REMARK 6.11. In Th. 6.10, the condition that $A \subset A[\beta]$ is LCM-stable can not be omitted. In fact, let $A=\boldsymbol{Q}[s, t, u, v]$, where s, t, u, v are indeterminates and take $\alpha, \beta \in \Omega$ so that $s\alpha^2+t\alpha+u=0$ and $v\beta^2+t\beta+t=0$ respectively. Then $A \subset A[\alpha]$ is LCM-stable, but $A \subset A[\beta]$ is not LCM-stable. By prop. 6.6, we see that the kernel of the canonical homomorphism of $A[X, Y]$ onto $A[\alpha, \beta]$ is equal to (sX^2+tX+u, vY^2+tY+t), and therefore it is easily shown that $A \subset A[\alpha, \beta]$ is not LCM-stable.

§7. Examples

In §4 we have seen that, if $\mathrm{Gr}(I) \geqq 2$, then $\mathrm{gr}(I) \geqq 2$ under some conditions on the ideal I. It seems plausible to the author that '$\mathrm{Gr}(I) \geqq 2$' does not necessarily imply '$\mathrm{gr}(I) \geqq 2$'; however such an example can be found nowhere in the literature. So, in this section we give an example and by making use of it, we show that R_2-stableness does not necessarily imply G_2-stableness.

Let I be a non-zero proper ideal of A. We first construct a ring B so that $\mathrm{gr}(IB)=1$. For the ideal I, we consider a set of indeterminates $\{X_{\lambda\mu}\}_{\lambda,\mu \in I}$. Let $R=A[\{X_{\lambda\mu}\}_{\lambda,\mu \in I}]$ and $J=(X_{\lambda\mu}X_{\alpha\beta} \mid \lambda, \mu, \alpha, \beta \in I)R$. Put $I_{\lambda\mu}=(\lambda, \mu)$ for any $\lambda, \mu \in I$. We denote by B a subdomain $A+\sum I_{\lambda\mu}X_{\lambda\mu}+J$ $(\lambda, \mu \in I)$ of R. Let $f \in B$. Then there exist uniquely $f_0 \in A$, $f_{\lambda\mu} \in I_{\lambda\mu}$ $(\lambda, \mu \in I)$ and $f_1 \in J$ such that $f=f_0+\sum f_{\lambda\mu}X_{\lambda\mu}+f_1$ $(\lambda, \mu \in I)$, where $f_{\lambda\mu}=0$ for almost all $\lambda, \mu \in I$. We say that $f=f_0+\sum f_{\lambda\mu}X_{\lambda\mu}+f_1$ $(\lambda, \mu \in I)$ is the decomposition of f.

LEMMA 7.1. *Let $f \in B$ and $f_0+\sum f_{\lambda\mu}X_{\lambda\mu}+f_1$ $(\lambda, \mu \in I)$ be the decomposition of f. Then we have*

(1) *for $\lambda, \mu \in I$, $X_{\lambda\mu}f \in B$ if and only if $f_0 \in I_{\lambda\mu}$,*

(2) *if $X_{\lambda\mu}f \in B$, then $X_{\lambda\mu}f \notin fB$.*

COROLLARY 7.2. $\mathrm{gr}(IB)=1$.

PROOF. Let $f, g \in IB$ and let $f_0+\sum f_{\lambda\mu}X_{\lambda\mu}+f_1$, $g_0+\sum g_{\lambda\mu}X_{\lambda\mu}+g_1$ be the decompositions of f, g respectively. Since $f, g \in IB$, we have $f_0, g_0 \in I$. Therefore, $X_{f_0g_0}f \in f:_B g$ and $X_{f_0g_0}f \notin fB$ by Lemma 7.1. Thus, $f:_B g \neq f$. This implies that $\mathrm{gr}(IB)=1$.

Next, we consider the following condition (**) to make $\mathrm{Gr}(IB) \geqq 2$.

(**) $$(\alpha, \beta):_A I = (\alpha, \beta) \qquad \text{for any} \quad \alpha, \beta \in I.$$

For example, let $A=k[s, t, u]$ where k is a field and s, t, u are all indeterminates. Put $I=(s, t, u)$. Then I satisfies the condition (**).

PROPOSITION 7.3. *Assume that I satisfies the condition (**). Then we have $\lambda:_B I=\lambda$ for each $\lambda \in I$. In particular, if I is finitely generated, then $\mathrm{Gr}(IB) \geqq 2$.*

PROOF. Let $\lambda \in I$. We assume that $\lambda \neq 0$. Let $f \in \lambda:_B I$ and let f_0+

$\sum f_{\alpha\beta}X_{\alpha\beta}+f_1$ be the decomposition of f. Then for each $\mu\in I$, there exists $g_\mu\in B$ such that $\mu f=\lambda g_\mu$. Let $g_0^\mu+\sum g_{\alpha\beta}^\mu X_{\alpha\beta}+g_1^\mu$ be the decomposition of g_μ for each $\mu\in I$. Then the following (i), (ii) and (iii) hold for each $\mu\in I$: (i) $\mu f_0=\lambda g_0^\mu$, (ii) $\mu f_{\alpha\beta}=\lambda g_{\alpha\beta}^\mu$ for any $\alpha, \beta\in I$, (iii) $\mu f_1=\lambda g_1^\mu$. By (i) and the condition (**), $f_0\in\lambda:_A I=\lambda$. Therefore, we can take $h_0\in A$ so that $f_0=\lambda h_0$. Next, by (iii) and the condition (**), $f_1\in\lambda:_R I=(\lambda:_A I)R=\lambda R$. Therefore, we can take $h_1\in R$ so that $f_1=\lambda h_1$. Then since $f_1\in J$, we have $h_1\in J$. Moreover, by (ii) and the condition (**), $f_{\alpha\beta}\in\lambda:_A I=\lambda$ for any $\alpha, \beta\in I$. Therefore, we can take $h_{\alpha\beta}\in A$ so that $f_{\alpha\beta}=\lambda h_{\alpha\beta}$ for any $\alpha, \beta\in I$. Put $h=h_0+\sum h_{\alpha\beta}X_{\alpha\beta}+h_1$ $(\alpha, \beta\in I)$. Then we have $h\in R$ and $f=\lambda h$. Since $\mu h=g_\mu$, $\mu h_{\alpha\beta}=g_{\alpha\beta}^\mu\in I_{\alpha\beta}$ for any $\mu, \alpha, \beta\in I$. Thus, we have $h_{\alpha\beta}\in(\alpha, \beta):_A I=(\alpha, \beta)$ by the condition (**). That is, $h\in B$. Therefore, $f\in\lambda B$. This implies that $\lambda:_B I=\lambda$.

LEMMA 7.4. *Let $A[\{X_\lambda\}_{\lambda\in\Lambda}]$ be a polynomial ring in variables $\{X_\lambda\}_{\lambda\in\Lambda}$ over A. Let $f\in A[\{X_\lambda\}_{\lambda\in\Lambda}]$ with $f(0)=1$. Then we have $a:_{A[\{X_\lambda\}_{\lambda\in\Lambda}]}f=a$ for each $a\in A$.*

Here, let $A=k[s, t, u]_{(s,t,u)}$, where k is a field and s, t, u are all indeterminates. Put $M=(s, t, u)A$ and let $R=A[\{X_{\alpha\beta}\}_{\alpha,\beta\in M}]$, where $\{X_{\alpha\beta}\}_{\alpha,\beta\in M}$ is a set of variables. Moreover, put $M_{\alpha\beta}=(\alpha, \beta)$ for any $\alpha, \beta\in M$ and put $J=(X_{\alpha\beta}X_{\lambda\mu}|\alpha, \beta, \lambda, \mu\in M)R$. Let $B=A+\sum M_{\alpha\beta}X_{\alpha\beta}+J$ $(\alpha, \beta\in M)$ and $T=A+\sum MX_{\alpha\beta}+J$ $(\alpha, \beta\in M, \alpha\neq 0$ or $\beta\neq 0)$. Then we have $A\subset B\subset T\subset R$.

PROPOSITION 7.5. *With the above notation, we have* $\mathrm{Gr}(MT)=1$. *In particular, $B\subset T$ is not G_2-stable.*

PROOF. Let $a, \alpha, \beta\in M-\{0\}$. Then we have $aX_{\alpha\beta}\in T$. Since $m(aX_{\alpha\beta})=a(mX_{\alpha\beta})$ for each $m\in M$, $aX_{\alpha\beta}\in a:_T M$. On the other hand, since $X_{\alpha\beta}\notin T$, $aX_{\alpha\beta}\notin aT$. Therefore, $a:_T M\neq a$. Thus, $\mathrm{Gr}(MT)=1$. Furthermore, we have $\mathrm{Gr}(MB)\geqq 2$ by Prop. 7.3. That is, $B\subset T$ is not G_2-stable.

PROPOSITION 7.6. *With the notation of Prop. 7.5, $B\subset T$ is R_2-stable.*

PROOF. Let $f, g\in B$ and assume that $f:_B g=f$. Let $f_0+\sum f_{\alpha\beta}X_{\alpha\beta}+f_1$, $g_0+\sum g_{\alpha\beta}X_{\alpha\beta}+g_1$ $(\alpha, \beta\in M)$ be the decompositions of f, g respectively. By the proof of Cor. 7.2, it is easy to see that either $f_0\notin M$ or $g_0\notin M$. Say $f_0\notin M$. Since A is a local domain, we may assume that $f_0=1$. Let $h\in f:_T g$ and take $\phi\in T$ so that $hg=f\phi$. Put $h=h_0+\sum h_{\alpha\beta}X_{\alpha\beta}+h_1$ $(\alpha, \beta\in M, \alpha\neq 0$ or $\beta\neq 0)$ and $\phi=\phi_0+\sum\phi_{\alpha\beta}X_{\alpha\beta}+\phi_1$ $(\alpha, \beta\in M, \alpha\neq 0$ or $\beta\neq 0)$, where $h_0, \phi_0\in A$, $h_{\alpha\beta}, \phi_{\alpha\beta}\in M$ for any $\alpha, \beta\in M$ and $h_1, \phi_1\in J$. If $h_{\alpha\beta}=\phi_{\alpha\beta}=0$ for any $\alpha, \beta\in M$, then $h, \phi\in B$. Therefore, $h\in f:_B g=f$. That is, $h\in fB\subset fT$. Now, suppose that there exist $\alpha, \beta\in M$ such that $h_{\alpha\beta}\neq 0$ and $\phi_{\alpha\beta}\neq 0$. Then we can take $a\in\cap M_{\alpha\beta}-\{0\}$, the interesection ranging over all $\alpha, \beta\in M$ with $h_{\alpha\beta}\neq 0$ and $\phi_{\alpha\beta}\neq 0$. Then we have

ah, $a\phi \in B$. Since $f:_B g=f$ and $g(ah)=f(a\phi)$, there exists $\psi \in B$ such that $ah=f\psi$ and $a\phi=g\psi$. Moreover, there exists $\xi \in R$ such that $h=f\xi$ and $\psi=a\xi$ by Lemma 7.4. Put $\xi=\xi_0+\sum \xi_{\alpha\beta}X_{\alpha\beta}+\xi_1$ (α, $\beta \in M$), where ξ_0, $\xi_{\alpha\beta} \in A$ for any α, $\beta \in M$ and $\xi_1 \in J$. Then we have $h_{\alpha\beta}=\xi_{\alpha\beta}+\xi_0 f$ for any α, $\beta \in M$. (In particular, $\xi_{00}=0$). Therefore, $\xi_{\alpha\beta} \in M$ for any α, $\beta \in M$. Thus, $\xi \in T$. That is, $h=f\xi \in fT$. This implies that $f:_T g=f$. Thus, $B \subset T$ is R_2-stable.

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